The convergent sequence to the weighted operator

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Abstract. This paper focuses on the periodic weighted operator and explains its spectrum. The resolvent of the operator sequence of the weighted operator is discussed, and proof is given to indicate that this operator sequence is restricted to the space of all real valued Lebesgue measurable functions that are square integrable on real numbers. The sequence of this operator's convergence to the resolvent of the second derivative operator with the individual boundary conditions is then demonstrated. Consequently, this convergence is used to analyze the results of weighted operator model in the image processing.
Keywords: weighted operator, spectral theory, differential operator, image processing, total variation, partial differential equations.

1. Introduction

The investigations of Sturm and Liouville on the eigenvalues of certain deferential equations of second order with given boundary conditions, now referred to as Sturm-Liouville theory mark the beginning of the history of periodic spectral theory. A solution to the inverse problem (including characterization) for the vector-valued Sturm-Liouville operator on a finite interval with Dirichlet conditions was published recently in [11]. Since the periodic case is more complicated, many works have concentrated only on the direct spectral problem for periodic systems.

Lyapunov, who proved that the spectrum of the weighted periodic operator has band structure and studied the needed properties of the Lyapunov function

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[21] was the first to investigate the spectral properties of the weighted periodic operator, with the results for the more general case including 2×2 systems being reproved later by Krein [20]. Firsova [7] and Marchenko and Ostrovski [23] were responsible for simultaneously introducing the global quasimomentum into the spectral theory of the Hill operator. Two-sided estimates for various parameters of the Hill operator and as well as for the Dirac operator were obtained in [8, 9, 12, 13, 15, 17, 19] as well as the sharp two-sided estimates for gap lengths and potential [14].

The spectral properties of the Hill operator and the weighted operator show a sizeable difference, whereby the gap length of the weighted operator can go to infinity at high energy, while the gap length of the Hill operator tends to 0 at high energy [18]. The fact of the weighted operator being raised in different situations is the physical motivation for this. Examples include the photonic crystal problem, the propagation of one-dimensional waves in periodic homogeneous media with velocity and a hydrogen atom in an external, homogeneous, time-periodic magnetic field [10]. Reference to the results on the weighted operator and its history which the authors are made in [4], albeit with no claim to completeness. In particular, mention is made of many applications to physics [4]. Likewise, the weighted operator has a lot applications in the image processing, as introduced in [25, 26].

In order to investigate the spectrum of an ordinary differential equation with periodic coefficients the construction of its Bloch solutions is required. The transform matrix corresponding to a basis in the space of the solutions is introduced. Therefore, first of all, a general tool is briefly described, in the hope that the transformation operation is very natural for the spectral analysis of difference equations with periodic coefficients. Subsequently, the spectrum of an operator is investigated by this tool.

The main result of the current paper is contained in Theorem (3), describing the resolvent convergence of the periodic weighted operator sequence. We show that the resolvent of the operator $\mathcal{H}$ is restricted to set of all Lebesgue measurable functions. Also, this resolvent coincides with another resolvent which defined on the differential operator under a certian boundary conditions.

A brief outline of the contents of the paper follows: Section (2) re-states some basic concepts which are necessary for our future work. In Section (3) a periodic weighted operator $\mathcal{W}_\omega$ is studied, the spectrum of this operator being subsequently investigated. It is shown that the spectrum of $\mathcal{W}_\omega$ is absolutely continuous and consists of intervals $\mathrm{I}_n, \mathrm{I}_{n+1}$ separated by gaps $\mathcal{G}_n = (\pi n_L, \pi n_R)$; with the lengths $\mid \mathcal{G}_n \mid$, Thorem (2). Then, Section (4) comprises an investigation of the resolvent convergence of this operator sequence. In particular, our main result (Theorem (3)) is proven. Finally, Section (5) shows some applications of the weighted operator that can be used in the future work.

2. Preliminaries

At the beginning let us briefly recall some of the results that given in the previous work. We consider here the second derivative operator $\mathscr{L} = -\frac{d^2}{dy^2}$, acting in the Hilbert space $L_2(\mathbb{R})$ defined on the functions from $W_2^2(\mathbb{R} \setminus \{n\}_{n\in\mathbb{Z}})$ satisfying the boundary conditions,

$$\begin{pmatrix} u_R(n) \\ u'_R(n) \end{pmatrix} = \exp(i\theta)\,\mathcal{A}\begin{pmatrix} u_L(n) \\ u'_L(n) \end{pmatrix},\ n \in \mathbb{Z},$$

where $\mathcal{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \mu \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R})$ (the unimodule group) such that $\alpha, \beta, \gamma, \mu \in \mathbb{R}$ and let $0 \leq \theta < 2\pi$. In addition, the adjoint of this operator coincides with a self-adjoint extension of the operator $\mathscr{L} = -\frac{d^2}{dy^2}$, restricted to the set of functions satisfying the boundary conditions at the origin, and fulfilling the condition $\alpha\mu - \beta\gamma = 1$ [2]. Furthermore, there is an increase in the widths of the forbidden gaps and the gap-to-band ratio is not small when the operator $\mathscr{L}$ is discribed [2].

Investigating the spectrum of an ordinary differential equation with periodic coefficients necessitates the construction of its Bloch solutions, introducing the transfer matrix $\mathbb{T}_{\mathscr{L}}$ corresponding to a basis in the space of the solutions. Consequently, the transfer matrix for the operator $\mathscr{L}$ is defined as follows [1]:

$$\mathbb{T}_{\mathscr{L}} = \begin{bmatrix} \cos\kappa & -\frac{1}{\kappa}\sin\kappa \\ \kappa\sin\kappa & \cos\kappa \end{bmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \mu \end{pmatrix},\ \text{where}\,\kappa = \sqrt{\lambda}.$$

And since $\det(\mathbb{T}_{\mathscr{L}}) = 1$, therefore, the specific determinant of this matrix is given by

$$\det(\mathbb{T}_{\mathscr{L}} - \lambda\mathrm{I}) = \lambda^2 - \lambda\mathrm{Tr}(\mathbb{T}_{\mathscr{L}}) + 1.$$

Thus, the operator's spectrum coincides with the set of λ where the spectrum of this operator is calculated as zeros of the following inequality [22],

$$\mid \mathrm{Tr}(\mathbb{T}_{\mathscr{L}}) \mid \leq 2,$$

then

$$\mid (\alpha+\mu)\cos k + (\frac{\beta}{\kappa} - \gamma\kappa)\sin\kappa \mid \leq 2. \tag{1}$$

Let now introducing the function f to describe the spectrum of $\mathscr{L}$,

$$\mid f(\kappa) \mid = \mid (\alpha+\mu)\cos\kappa + (\frac{\beta}{\kappa} - \gamma\kappa)\sin\kappa \mid \leq 2. \tag{2}$$

The following theorem gives an explicit description of the spectrum of this operator $\mathscr{L}$ [6].

Theorem 1 ([6]). *Assume that α, γ, β, and μ are an arbitrary, then there are infinite numbers of bands of the operator $\mathscr{L}$, which has a continuous spectrum.*

3. The operator's spectrum

First, consideration is given to the periodic weighted operator, defined as follows [16],

$$\mathcal{W}\zeta = -\frac{1}{p}\frac{d}{dy}(p\frac{d}{dy}\zeta), \tag{3}$$

where ζ is an eigenfunction, $p > 0$, in $L_2(\mathbb{R})$. This operator has a continuous spectrum which contains an infinite number of bands separated by gaps $\mathcal{G}_n = (\pi n_L, \pi n_R)$ where the lenght of gaps $|\mathcal{G}_n| \geq 0$, and located in the intervals $\mathrm{I}_n = [\pi^2(n-1)^2, \pi^2 n^2]$, [16]. The corresponding intervals I_n, I_{n+1} of the gaps are merge when $| \mathcal{G}_n |= 0$. It should be noted that there exists the following asymptotic:

$$\sqrt{\pi \pm n} = (\lambda n)(1 + \mathrm{I}(1)) \,\mathrm{as}\, n \to \infty,$$

which provides a rough asymptotic of gap lengths [5]. Now we define a periodic weighted operator as follows:

$$\mathcal{W}_\omega\zeta = -\frac{1}{p_\omega(y)}\frac{d}{dy}(p_\omega(y)\frac{d}{dy}\zeta), \tag{4}$$

where

$$p_\omega(y) = 1 + \sum_{n=-\infty}^{\infty} \eta\frac{1}{\omega}\delta_\omega(y-n),\ \eta \in \mathbb{R}^+, \tag{5}$$

is called the density function which defined by using the delta function

$$\delta_\omega(y) = \begin{cases} 1, & y \in [0,\omega) \\ 0, & y \notin [0,\omega). \end{cases} \tag{6}$$

Thus, since p is chosen, if $\omega \to 0$ we get

$$\begin{aligned} \lim_{\omega\to 0} p_\omega(y) &= \lim_{\omega\to 0}[1 + \sum_{n=-\infty}^{\infty} \eta\frac{1}{\omega}\delta_\omega(y-n)] \\ &= 1 + \lim_{\omega\to 0}\sum_{n=-\infty}^{\infty} \eta\frac{1}{\omega}\delta_\omega(y-n) \\ &= 1 + \sum_{n\in\mathbb{Z}} \frac{\eta}{\omega}\delta(y-n). \end{aligned}$$

Since $\sum \frac{\eta}{\omega}\delta(y-n)$ represents the delta function, then the density function p_ω is convergent to $\sum_{n\in\mathbb{Z}} \delta_\omega + 1$ when $\omega \to 0$.

Using the relation (2) the following Theorem can easily be shown.

Theorem 2. *Assume that $\alpha = 1,\ \gamma = h,\ \beta = 0,\ \mu = 1$. Then there are infinite numbers of bands of the periodic operator $\mathcal{W}_\omega$ as $\omega \to 0$ which has an absolutely continuous spectrum.*

Proof. From the equation (5), $p_\omega(y)$ is dis-continuous at $y = n$ and $y = n + \omega$, then each functions take from the domain of the operator $\mathcal{W}_\omega$ is satisfying the boundary conditions as follow

$$\begin{aligned}
\zeta(n_R) &= \zeta'(n_L),\\
(1+\eta\frac{1}{\omega})\zeta'(n_R) &= \zeta'(n_L),\\
\zeta((n+\omega)_R) &= \zeta((n+\omega)_L),\\
\zeta'((n+\omega)_R) &= (1+\eta\frac{1}{\omega})\zeta'((n+\omega)_L).
\end{aligned}$$

Thus, the functions ζ and $p_\omega\zeta$ are continuous under these conditions. Therefore, there are four matrices for the transfer matrix of the operator $\mathcal{W}_\omega$. The first two transfer matrices defined on the differential operator on the intervals $(0^+, \omega^-)$ and $(\omega^+, 1^-)$, and another two matrices corresponded to discontinuous at $y = 0$ and $y = \omega$, i.e.

$$\begin{aligned}
\mathbb{T}_{\mathcal{W}_\omega} &= \mathbb{T}_{\frac{d^2}{dy^2}}(\omega^+, 1^-)\begin{bmatrix}1 & 0\\ 0 & 1+\frac{\eta}{\omega}\end{bmatrix}\mathbb{T}_{\frac{d^2}{dy2}}(0^+, 1^-)\begin{bmatrix}1 & 0\\ 0 & 1-\frac{\eta}{\omega+\eta}\end{bmatrix}\\
&= \begin{bmatrix}\cos\kappa - \frac{1}{\omega}\sin(1-\omega)\kappa\sin\omega\kappa & \frac{1}{\kappa}\sin\kappa - \frac{1}{\kappa(\omega+\eta)}\cos(1-\omega)\kappa\sin\omega\kappa\\ -\kappa\sin\kappa - \frac{\kappa}{\omega}\cos(1-\omega)\kappa\sin\omega\kappa & \cos\kappa + \frac{1}{\omega+\eta}\sin(1-\omega)\kappa\sin\omega\kappa\end{bmatrix}.
\end{aligned}$$

Now, since $\det(\mathbb{T}_{\mathcal{W}_\omega}) = 1$, then the spectrum of the operator can be determined by the trace of the transfer matrix $\mid \mathrm{Tr}(\mathbb{T}_{\mathcal{W}_\omega}) \mid$[22]; i.e.:

$$\mid \mathbb{T}_{\mathcal{W}_\omega} \mid \leq 2.$$

Thus,

$$\mid \mathbb{T}_{\mathcal{W}_\omega} \mid = \mid 2\cos\kappa - \frac{\eta}{\omega(\omega+1)}\sin((1-\omega)\kappa)\sin\omega\kappa \mid \leq 2,$$

when the limit $\omega \to 0$. Implies that the last equation can be transformed into the equation;

$$\mid 2\cos\kappa - \eta\kappa\sin\kappa \mid \leq 2.$$

Hence, this equation is coincided with the equation (2) when the four parameters take the following values;

$$\alpha = 1,\ \gamma = h,\ \beta = 0,\ \mu = 1.$$

As a result, we conclude that there is an infinite number of band of the operator $\mathscr{L}$, which has an absolutely continuous spectrum in which the operator $\mathcal{W}_\omega$ is convergent when $\omega \to 0$ depending on these parameters. □

4. The resolvent of the operator sequence

The main goal in this section is to discuss the resolvent of the operator sequence of the operator $\mathcal{W}_\omega$. At first, let us extend the Hilbert space as $\mathbb{H} = L_2(\mathbb{R}) \oplus \ell_2$ and construct a self-adjoint operator $\mathcal{H}$ which acting on this space.

Now the self-adjoint operator $\mathcal{H}$ is defined by the following formula

$$\mathcal{H}\begin{bmatrix}\zeta\\ \xi\end{bmatrix} = \begin{bmatrix}-\frac{d^2}{dy^2}\zeta\\ \{[\xi']\}/\sqrt{\eta}\end{bmatrix} \tag{7}$$

on functions $(\zeta, \xi) \in W_2^2(\mathbb{R}\setminus\mathbb{C})$ satisfying the boundary conditions

$$\begin{matrix}\zeta(n_R) = \zeta(n_L)\\ \xi_n = -\sqrt{\eta}\{\zeta\}_n\end{matrix}\ ,\ n \in \mathbb{Z}, \tag{8}$$

where $-\frac{d^2}{dy^2}\zeta$ and $\{\xi'\}$ denote the vectors from ℓ_2 with coordinates $-\frac{d^2}{dy^2}\zeta(n_R) - \frac{d^2}{dy^2}\zeta(n_L)$ and $\{\xi\}$, respectively. Since the operator $\mathcal{H}$ is symmetric and the range of $\mathbb{H}$ and $\mathcal{H} - \lambda$ is coincide when λ is arbitrary, $\Im\lambda \neq 0$, therefore, the operator $\mathcal{H}$ is a self-adjoint [3].

The following theorem is the principal result of this paper, containing the resolvents of $\mathcal{W}_\omega$ which diverges form any resolvent of operator that acting on Hilbert space $L_2(\mathbb{R})$.

Theorem 3. *The operator $\mathcal{H}$ which limited to the space of $L_2(\mathbb{R})$ and the differential operator $(-\frac{d^2}{dy^2})$ have an identical resolvent under certian individual boundary conditions when y=n;*

$$\begin{bmatrix}\zeta(n_R)\\ \zeta'(n_R)\end{bmatrix} = \begin{bmatrix}1 & 0\\ -\eta\kappa^2 & 1\end{bmatrix}\begin{bmatrix}\zeta(n_L)\\ \zeta'(n_L)\end{bmatrix}. \tag{9}$$

Proof. Firstly, we calculate the restricted resolvent of the operator $\mathcal{H}$, since from (7),

$$\mathcal{H}\begin{bmatrix}\zeta\\ \xi\end{bmatrix} = \begin{bmatrix}-\frac{d^2}{dy^2}\zeta\\ \{[\xi']\}/\sqrt{\eta}\end{bmatrix},$$

then for any function $F \in L_2(\mathbb{R})$ we get

$$(\mathcal{H} - \lambda)\begin{bmatrix}\zeta\\ \xi\end{bmatrix} = \mathcal{H}\begin{bmatrix}\zeta\\ \xi\end{bmatrix} - \lambda\begin{bmatrix}\zeta\\ \xi\end{bmatrix} = \begin{bmatrix}F\\ 0\end{bmatrix}$$
$$= \begin{cases}-\dfrac{d^2}{dy^2}\zeta(y) - \lambda\zeta(y) = F(y),\ y \notin \mathbb{Z}\\ \dfrac{1}{\sqrt{\eta}}\{[\zeta']\} - \lambda\xi = 0\end{cases}$$

implies that $\{[\zeta']\} = \sqrt{\eta}\lambda\xi$, but from the second part of the equation (8); $\xi_n = -\sqrt{\eta}\{\zeta\}_n$ then $\{[\zeta']\} = -\eta\lambda\{\xi\}$ and since $\kappa = \sqrt{\lambda}$ hence, we get an individual boundary condition for the component $\{\zeta\}$;

$$\{[\zeta']\} = -\eta\kappa^2\{\xi\}$$

which is exactly the second part of the equation (9). □

5. Applications of weighted operator

There are a lot of weighted operator applications in different sciences such as in image processing. Many problems appeared in image processing like image inpainting, deblurring, segmentation, and denoising which can solve them by modeling weighted operator [24]. The benefit of this study (i.e. the convergence of weighted operator) is used in analysis the results of weighted operator model in image processing. This analysis will help to modify this model and to enhance its results in the image processing. Two types of image processing models which are statistical, and partial differential equations. The weighted operator is used in statistical model for different problems in image processing [27]. While the differential operators are used in PDE models in image processing [28, 29, 30, 31]. The convergence of

6. Conclusions

We have introduced the spectum of the weighted operator, which it has a different situations in physics such as the photonic crystal problem and a hydrogen atom. We have used the concept of a transformation matrix instead Bloch solutions. We have provided here the spectrum of a weighted operator $\mathcal{W}_\omega$ is absolutely continuous and consists of intervals $(\mathrm{I}_n, \mathrm{I}_{n+1})$ separated by gaps $\mathcal{G}_n = (\pi n_L, \pi n_R)$; with the lengths $\mid \mathcal{G}_n \mid$. Moreover, the resolvent convergence of the periodic weighted operator sequence is described. We have proved that there exist a corresponding resolvent between the weighted operator $\mathcal{W}_\omega$ and differential operator with the individual boundary conditions. In the future work, the weighted operator will use for solving different image processing problems, the convergence of other types of operators will study and we will apply them in image processing.

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and W. Kirsch, *On point interactions in one dimension*, Journal of Operator Theory, 1984, 101-126.

[2] S. Albeverio, L. Dabrowski, P. Kurasov, *Symmetries of Schrödinger operator with point interactions*, Letters in Mathematical Physics, 20 (1998), 33-47.

[3] S. Albeverio, P. Kurasov, *Singular perturbations of differential operators: solvable Schrödinger-type operators*, Cambridge University Press, 2000.

[4] L-E. Andersson, *Inverse eigenvalue problems for a Sturm-Liouville equation in impedance form*, Inverse Problems, 4 (1988), 929.

[5] F. V. Atkinson, A. B. Mingarelli, *Asymptotics of the number of zeros and of the eigenvalues of general weighted Sturm-Liouville problems*, Journal fur die Reine und Angewandte Mathematik, 375 (1987), 380-393.

[6] P. Exner, G. Harald, *Some properties of the one-dimensional generalized point interactions (a torso)*, arXiv preprint math-ph/9910029, 1999.

[7] N. E. Firsova, *Riemann surface of quasimomentum and scattering theory for the perturbed Hill operator*, Journal of Soviet Mathematics, 11.3 (1979), 487-497.

[8] P. Kargaev, E. Korotyaev, *Effective masses and conformal mappings*, Communications in Mathematical Physics, 169.3 (1995), 597-625.

[9] P. Kargaev, E. Korotyaev, *The inverse problem for the Hill operator, a direct approach*, Inventiones Mathematicae 129.3 (1997), 567-594.

[10] M. Klein, E. Korotyaev, *Parametrization of periodic weighted operators in terms of gap lengths*, Inverse Problems 16.6 (2000), 1839.

[11] E. Korotyaev, D. Chelkak, *The inverse sturm Liouville problem with mixed boundary conditions*, Petersburg Mathematical Journal, 21.5 (2010), 761-778.

[12] E. Korotyaev, *Metric properties of conformal mappings on the complex plane with parallel slits*, International Mathematics Research Notices, 1996.10 (1996), 493-503.

[13] E. Korotyaev, *The propagation of the waves in periodic media at large time*, Asymptotic Analysis, 15.1 (1997), 1-24.

[14] E. Korotyaev, *Estimates of periodic potentials in terms of gap lengths*, Communications in mathematical physics, 197.3 (1998), 521-526.

[15] E. Korotyaev, *Estimates for the Hill operator, I*, Journal of Differential Equations, 162.1 (2000), 1-26.

[16] E. Korotyaev, *Inverse problem for periodic weighted operators*, Journal of Functional Analysis, 170.1 (2000), 188-218.

[17] E. Korotyaev, *Marchenko-Ostrovki mapping for periodic Zakharov Shabat systems*, Journal of Differential Equations, 175.2 (2001), 244-274.

[18] E. Korotyaev, *Periodic weighted operators*, Journal of Differential Equations, 189.2 (2003), 461-486.

[19] E. Korotyaev, *Inverse problem and estimates for periodic Zakharov-Shabat systems*, Journal fur die reine und angewandte Mathematik, 2005.583 (2005), 87-115.

[20] M. Krein, *On the characteristic function $A(\lambda)$ of a linear canonical system of differential equations of second order with periodic coefficients*, Prikl. Mat. Meh., 21 (1957), 320-329.

[21] A. Lyapunov, *Sur une équation transcendante et les équations différentielles linéaires du second ordrea coefficients périodiques*, CR Acad. Sci. Paris, 128 (1899), 1085-1088.

[22] W. Magnus, S. Winkler, *Hill's equation*, Dover, New York, 1979.

[23] V. A. Marchenko, I. V. Ostrovskii, *A characterization of the spectrum of Hill's operator*, Matematicheskii Sbornik, 139 (1975), 540-606.

[24] L. François, A. Elmoataz, O. Lézoray, *Partial difference operators on weighted graphs for image processing on surfaces and point clouds*, IEEE Transactions on Image Processing, 23 (2014), 3896-3909.

[25] J. Liang, R. Lai, T. Wai Wong, H. Zhao, *Geometric understanding of point clouds using laplace-beltrami operator*, in CVPR, 23 (2012), 214-221.

[26] H.H. Bauschke, P.L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, Springer, 23 (2011).

[27] Tamalika Chaira, *Medical image processing: advanced fuzzy set theoretic techniques*, CRC Press, 2015, 47-60.

[28] S. Osher, R. Fedkiw, *Level set methods and dynamic implicit surfaces*, Springer, 153 (2004).

[29] M. Bertalmio, L. Cheng, S. Osher, and G. Sapiro, *Variational problems and partial differential equations on implicit surfaces*, J. Comput Phys., 174 (2001), 759-780.

[30] S. Ruuth, B. Merriman, *A simple embedding method for solving partial differential equations on surfaces*, J. Comput Phys., 227 (2008), 1943-1961.

[31] R. Lai, T. F. Chan, *A framework for intrinsic image processing on surfaces*, Comput. Vis., 115 (2011), 1647-1661.

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