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A New Approximate Method for Solving Linear and Non-Linear Differential Equation Systems

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ABSTRACT. In this work, a new approximate method is proposed to enhance the accuracy and convergence of solutions for differential equations. This method uses the Taylors' series in its derivation. The proposed method has used for solving different types of linear and nonlinear differential equation systems with the initial conditions. Four examples have presented to check the effectiveness, accuracy, and convergence of the method. The solutions of the proposed method also compared with those obtained by the Adomian decomposition method (ADM), and the Homotopy analysis method (HAM). The results of the proposed method outperformed the other methods in terms of accuracy and convergence.

Keywords: Differential Equations System, Taylors' Series, Approximate Solution, Accuracy, Non-Linear Operator.

INTRODUCTION

It is very rare that a real-life phenomenon can be modeled by a single partial differential equation. Different systems of ODEs and PDEs have applied to describe these phenomena. There are a lot of numerical and approximation methods that use to solve these systems.

Recently, several numerical methods have been developed to solve ODEs and PDEs systems, which can be found in [1], [2], [3], [4], [5], [6], [7], [8]. These techniques consider a necessitate as computational resources to solve some problems that appear in other sciences such as in image processing [9]. Still, these techniques perhaps complicated and require a high computational cost for solving different linear and nonlinear systems of ODEs and PDEs. To avoid these defects, analytic approximate methods have used to solve the linear and nonlinear systems of differential equations.

There are many used approximate methods to find the exact solution, such as nonlinear self-adjointness and conservation method [10], multiplication generating method [11], a generalization of the overdetermined method [12], etc. Also, there are other types of approximate techniques that combine the exact and approximate methods, which can be found in [13], [14], [15], [16], [17], [18], [19], [20].

One of these methods which has received much concern is the Adomian decomposition method (ADM) [13]. The ADM has been employed to solve various linear and nonlinear models. The ADM yields a rapidly convergent series solution with much less computational work [21]. The ADM is unlike the traditional numerical methods, where ADM is used extensively to solve nonlinear differential equations because it works based on calculation Adomian polynomials for non-linear terms [22] and [23].

While, the approximate solutions of differential transform method (DTM) [15], are as a polynomial form which is different from the traditional higher-order Taylor series method because the Taylor series method needs huge computational for large orders. So, the DTM uses a different procedure to obtain an analytic Taylor series solution of the PDE [24].

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On the other hand, many complicated problems in different applied sciences have been successfully solved by the homotopy analysis method (HAM) [19], and [25]. Homotopy perturbation method (HPM) [26] which has been handled successfully to solve many linear and non-linear partial differential equations systems.

Whereas, the variational iteration method (VIM) [27], can be applied to solve all types of linear or nonlinear differential equations systems, constant or variable coefficients with homogeneous or inhomogeneous [28] because it effectively used to solve these nonlinear equations with a good convergent to the exact solutions.

This study aimed to find approximate solutions for the linear and nonlinear systems of Differential equations with initial conditions by using a proposed approximate method, which is considered as extending and developing to that in [29]. This method is based on the Taylor series, which is efficient to solve linear and nonlinear systems of ODEs and PDEs. Several test problems are given, and their results are compared with the solutions obtained by ADM [13], and HAM [19] to confirm the excellent accuracy and small absolute errors of the proposed method.

GENERATING AN APPROXIMATE METHOD

In this section, the basic ideas for constructing a proposed approximate method will be discussed. Let us consider the initial value problems:

$$\begin{aligned} u_t(x,t) &= F(u_t, v_t, u, v, u_x, v_x, u_{xx}, v_{xx}, \dots) \\ v_t(x,t) &= G(u_t, v_t, u, v, u_x, v_x, u_{xx}, v_{xx}, \dots) \end{aligned} \quad (1)$$

with initial condition

$$\begin{aligned} u(x,0) &= f_0(x) \\ v(x,0) &= g_0(x) \end{aligned} \quad (2)$$

By using the integral for the two sides of Eq.(1) from 0 to t ,we obtain

$$u(x,t) - u(x,0) = \int_0^t F[u.v]\,dt$$

$$u(x,t) - f_0(x) = \int_0^t F[u,v]\,dt$$

$$u(x,t) = f_0(x) + \int_0^t F[u,v]\,dt \quad (3)$$

$$v(x,t) - v(x,0) = \int_0^t G[u,v]\,dt$$

$$v(x,t) - g_0(x) = \int_0^t G[u,v]\,dt$$

$$v(x,t) = g_0(x) + \int_0^t G[u,v]\,dt \quad (4)$$

where $F[u,v] = F(u_t, v_t, u, v, u_x, v_x, u_{xx}, v_{xx}, \dots)$
and $G[u,v] = G(u_t, v_t, u, v, u_x, v_x, u_{xx}, v_{xx}, \dots)$
The expand Taylor series for $F[u,v]$ about $t = 0$ is

$$F[u,v] = F[u_0,v_0] + F'[u_0,v_0]t + F''[u_0,v_0]\frac{t^2}{2!} + F'''[u_0,v_0]\frac{t^3}{3!} + \cdots + F^{(n)}[u_0,v_0]\frac{t^n}{n!} + \cdots \quad (5)$$

By substituting Eq.(5) in Eq.(3) we get

$$\begin{aligned} u(x,t) &= f_0(x) + F[u_0,v_0]t + F'[u_0,v_0]\frac{t^2}{2!} + F''[u_0,v_0]\frac{t^3}{3!} + \cdots + F^{(n-1)}[u_0,v_0]\frac{t^n}{n!} + \cdots \\ &= a_0 + a_1 t + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + \cdots + a_n\frac{t^n}{n!} + \cdots \end{aligned} \quad (6)$$

Where $a_0 = f_0(x)$

$a_1 = F[u_0,v_0]$

$a_2 = F'[u_0,v_0]$

$a_3 = F''[u_0,v_0]$

$a_n = F^{(n-1)}[u_0,v_0]$

The formal in Eq.(6) is expand Taylor’s series for u about t=0 which mean

$a_0 = u(x,0)$

$a_1 = \frac{\partial}{\partial t}u(x,0)$

$a_2 = \frac{\partial^2}{\partial t^2} u(x,0)$
$a_3 = \frac{\partial^3}{\partial t^3} u(x,0)$
$a_n = \frac{\partial^n}{\partial t^n} u(x,0)$
In the same way we find the approximate solution of v

TEST PROBLEMS

Problem 1. Solve the following system of nonlinear ordinary differential equations [30]:

$$\frac{d^2x}{dt^2} + y^2\left(\frac{dx}{dt}\right)^2 - x = 1$$
$$\frac{d^2y}{dt^2} + x^2\left(\frac{dy}{dt}\right)^2 - y = 1 \qquad (7)$$

with initial condition
$x(0) = x'(0) = y(0) = 1 \;, y'(0) = -1$
and the exact solution for equations are:
$x(t) = e^t \;, y(t) = e^{-t}$
Solution:
By the following equation (1), we can note after rewrite equation (7):
$F[x,y] = 1 - \left(y^2\left(\frac{dx}{dt}\right)^2 - x\right)$
$G[x,y] = 1 - \left(x^2\left(\frac{dy}{dt}\right)^2 - y\right)$
$a_0 = f_0(t) = 1$
$b_0 = g_0(t) = 1$
$a_1 = f_1(t) = 1$
$b_1 = g_1(t) = -1$
$a_2 = F[x_0, y_0] = 1$
$b_2 = G[x_0, y_0] = 1$
$a_3 = F'[x_0, y_0] = 1$
$b_3 = G'[x_0, y_0] = -1$
From Eq.(6)

$$x(t) = a_0 + a_1 t + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + \cdots + a_n\frac{t^n}{n!} + \cdots$$
$$= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$
$$= e^t$$
$$y(t) = b_0 + b_1 t + b_2\frac{t^2}{2!} + b_3\frac{t^3}{3!} + \cdots + b_n\frac{t^n}{n!} + \cdots$$
$$= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots$$
$$= e^{-t}$$

Problem 1 for x

TABLE 1. Comparison of absolute errors among different methods and present method, for problem 1 with $t = 1$,

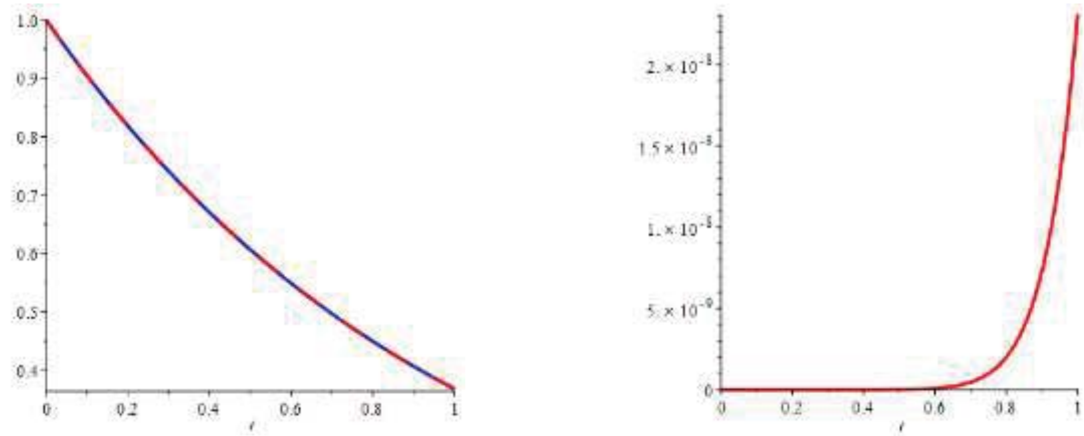
Method	x	n=5	n=6	n=7	n=8	n=9	n=10
Proposed method	0.2	2.53E-09	2.53E-09	6.32E-12	6.32E-12	4.90E-12	4.90E-12
ADE method		2.44E-07	2.87E-07	3.24E-11	4.42E-10	2.71E-09	2.16E-11
HAM method		1.54E-07	3.51E-06	1.92E-10	2.90E-11	3.84E-11	6.47E-11
Proposed method	0.4	3.24E-07	3.24E-07	7.23E-10	7.23E-10	6.33E-13	6.33E-13
ADE method		2.23E-06	6.14E-05	6.11E-08	4.76E-09	7.66E-11	2.16E-11
HAM method		3.54E-05	4.27E-06	1.53E-09	5.33E-07	3.27E-12	3.83E-11
Proposed method	0.6	5.53E-06	5.53E-06	2.77E-08	2.77E-08	8.57E-11	8.57E-11
ADE method		5.11E-05	6.03E-05	6.03E-06	3.88E-07	6.56E-09	4.11E-10
HAM method		3.03E-04	3.62E-04	4.16E-07	7.36E-06	2.45E-10	2.74E-08
Proposed method	0.8	4.12E-05	4.12E-05	3.68E-07	3.68E-07	2.14E-09	2.14E-09
ADE method		6.54E-04	2.82E-04	7.15E-05	4.11E-06	5.78E-07	4.33E-08
HAM method		4.76E-03	5.67E-04	5.44E-06	7.54E-05	2.14E-08	2.54E-07
Proposed method	1	1.96E-04	1.96E-04	2.73E-06	2.73E-06	2.49E-08	2.49E-08
ADE method		6.23E-03	4.65E-03	7.73E-05	5.22E-04	6.25E-07	6.15E-06
HAM method		3.90E-02	3.32E-03	4.51E-04	3.12E-05	5.32E-06	3.41E-07

Problem 1 for y

TABLE 2. Comparison of absolute errors among different methods and present method, for problem 1 with $t = 1$,

Method	x	n=5	n=6	n=7	n=8	n=9	n=10
Proposed method	0.2	8.64E-08	2.48E-09	6.41E-11	6.01E-13	2.01E-12	1.98E-12
ADE method		1.46E-05	2.05E-08	2.52E-09	2.77E-10	2.74E-11	2.48E-10
HAM method		1.32E-06	1.85E-07	2.28E-10	2.50E-11	2.48E-10	2.23E-11
Proposed method	0.4	5.38E-06	3.10E-07	1.56E-08	6.90E-10	3.25E-11	3.64E-12
ADE method		1.20E-03	1.68E-06	2.06E-06	2.27E-09	2.24E-10	2.02E-11
HAM method		1.08E-04	1.52E-05	1.87E-07	2.05E-08	2.03E-10	1.83E-11
Proposed method	0.6	5.96E-05	5.16E-06	3.90E-07	2.62E-08	1.58E-09	9.06E-11
ADE method		9.80E-04	1.37E-04	1.69E-06	1.86E-06	1.84E-07	1.66E-09
HAM method		8.86E-03	1.24E-05	1.53E-05	1.68E-07	1.66E-08	1.50E-10
Proposed method	0.8	3.26E-04	3.78E-05	3.82E-06	3.42E-07	2.76E-08	2.01E-09
ADE method		8.02E-03	1.12E-04	1.38E-05	1.52E-05	1.50E-07	1.36E-08
HAM method		7.26E-03	1.02E-03	1.25E-04	1.37E-06	1.36E-07	1.23E-07
Proposed method	1	1.21E-03	1.76E-04	2.23E-05	2.50E-06	2.52E-07	2.31E-08
ADE method		6.57E-02	9.20E-02	1.13E-04	1.24E-04	1.23E-05	1.11E-07
HAM method		5.94E-02	8.32E-03	1.02E-04	1.13E-05	1.11E-06	1.01E-07

When the value of n increases the closeness between the exact solution and the approximate solution increases and the absolute error decreases.



(a) (b)

FIGURE 1. (a) Exact solution, (b) Approximate solution and Absolute error for problem 1.

Therefore, the graph of exact, analytical solutions, and absolute errors of Problem 1 for $t = 1$ are given in **Error! Reference source not found.** and 2. Tables 1 and 2 show the absolute error using the proposed technique, ADM, and HAM of Problem 1 with $t = 1, k = 0.001$. When the value of n increases, the approximate solution gradually approaches the exact solution and the absolute error will decrease.

<u>Problem 2.</u> Solve the following system of nonlinear ordinary differential equations [30]:

$$\frac{d^4x}{dt^4} + y^3\left(\frac{dx}{dt}\right) - x + x^4 + 2\,(x\,y)^2 = 1$$
$$\frac{d^4y}{dt^4} + x^3\left(\frac{dy}{dt}\right) - y + y^4 + 2\,(x\,y)^2 = 1 \qquad (8)$$

with initial condition

$x(0) = x''(0) = 0, x'(0) = 1, x'''(0) = -1$

$y'(0) = y'''(0) = 0\,, y(0) = 1, y''(0) = -1$

and the exact solution for equations are:

$x(t) = \sin(t)\,, y(t) = cos(t)$

<u>Solution:</u>

By the following equation (1), we can note after rewrite equation (8):

$F[x, y] = 1 - \left(y^3\left(\frac{dx}{dt}\right) - x + x^4 + 2\,(x\,y)^2\right)$

$G[x, y] = 1 - \left(x^3\left(\frac{dy}{dt}\right) - y + y^4 + 2\,(x\,y)^2\right)$

$a_0 = f_0(t) = 0$

$b_0 = g_0(t) = 1$

$a_1 = f_1(t) = 1$

$b_1 = g_1(t) = 0$

$a_2 = f_2(t) = 0$

$b_2 = g_2(t) = -1$

$a_3 = f_3(t) = -1$

$b_3 = g_3(t) = 0$

$a_4 = F[x_0, y_0] = 0$

$b_4 = G[x_0, y_0] = 1$

$a_5 = F'[x_0, y_0] = 1$

$b_5 = G'[x_0, y_0] = 0$

From Eq.(6)

$x(t) = a_0 + a_1 t + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + \cdots + a_n\frac{t^n}{n!} + \cdots$

$$= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$
$$= \sin(t)$$
$$y(t) = b_0 + b_1 t + b_2 \frac{t^2}{2!} + b_3 \frac{t^3}{3!} + \cdots + b_n \frac{t^n}{n!} + \cdots$$
$$= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$
$$= \cos(t)$$

Problem 2 for x

TABLE 3. Comparison of absolute errors among different methods and present method, for problem 2 with $t = 1$,

Method	x	n=5	n=6	n=7	n=8	n=9	n=10
Proposed method	0.2	2.53E-09	2.53E-09	6.32E-12	6.32E-12	4.90E-12	4.90E-12
ADE method		1.37E-07	1.56E-07	4.29E-10	4.04E-10	4.04E-11	4.04E-10
HAM method		8.64E-06	2.48E-07	6.41E-09	6.01E-10	2.01E-10	1.98E-11
Proposed method	0.4	3.24E-07	3.24E-07	7.23E-10	7.23E-10	6.33E-13	6.33E-13
ADE method		9.71E-06	4.18E-07	1.57E-09	5.44E-08	1.34E-11	1.76E-12
HAM method		5.38E-06	3.10E-06	1.56E-08	6.90E-09	3.25E-11	3.64E-11
Proposed method	0.6	5.53E-06	5.53E-06	2.77E-08	2.77E-08	8.57E-11	8.57E-11
ADE method		2.02E-04	1.46E-05	9.18E-06	5.13E-06	2.54E-10	1.47E-09
HAM method		5.96E-05	5.16E-05	3.90E-07	2.62E-07	1.58E-09	9.06E-10
Proposed method	0.8	4.12E-05	4.12E-05	3.68E-07	3.68E-07	2.14E-09	2.14E-09
ADE method		1.48E-04	1.50E-03	1.33E-06	1.04E-06	7.31E-08	4.69E-08
HAM method		3.26E-04	3.78E-04	3.82E-06	3.42E-06	2.76E-08	2.01E-08
Proposed method	1	1.96E-04	1.96E-04	2.73E-06	2.73E-06	2.49E-08	2.49E-08
ADE method		6.53E-03	8.52E-03	9.70E-04	9.79E-05	8.88E-06	7.31E-06
HAM method		1.21E-03	1.76E-03	2.23E-05	2.50E-05	2.52E-07	2.31E-07

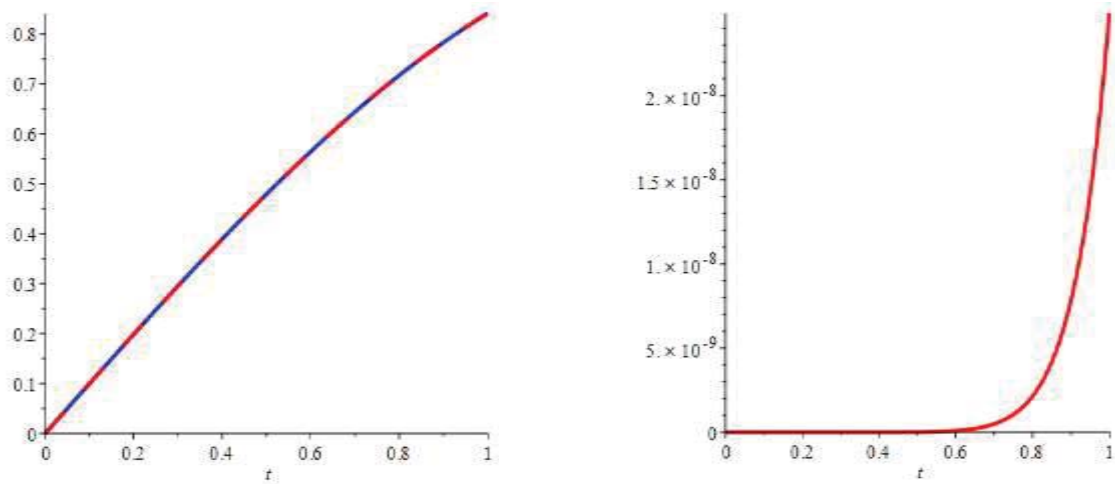
Problem 2 for y

TABLE 4. Comparison of absolute errors among different methods and present method, for problem 2 with $t = 1$,

Method	x	n=5	n=6	n=7	n=8	n=9	n=10
Proposed method	0.2	8.88E-08	6.22E-11	6.22E-11	1.27E-12	1.27E-12	1.24E-12
ADE method		1.43E-06	4.44E-08	2.46E-09	2.44E-11	2.43E-10	2.43E-11
HAM method		9.15E-07	2.64E-09	1.05E-10	4.13E-10	3.99E-11	3.99E-11
Proposed method	0.4	5.67E-06	1.62E-08	1.62E-08	3.18E-11	3.18E-11	2.88E-12
ADE method		1.06E-05	4.51E-06	1.71E-07	7.99E-10	2.57E-10	2.40E-11
HAM method		6.03E-04	3.42E-07	1.70E-06	7.11E-10	1.17E-09	4.05E-10
Proposed method	0.6	6.44E-05	4.15E-07	4.15E-07	1.66E-09	1.66E-09	4.86E-12
ADE method		2.34E-04	1.65E-05	1.03E-05	5.66E-07	2.82E-07	1.29E-11
HAM method		7.08E-03	6.00E-06	4.46E-05	2.95E-08	1.77E-07	1.05E-10
Proposed method	0.8	3.60E-04	4.13E-06	4.13E-06	2.94E-08	2.94E-08	1.45E-10

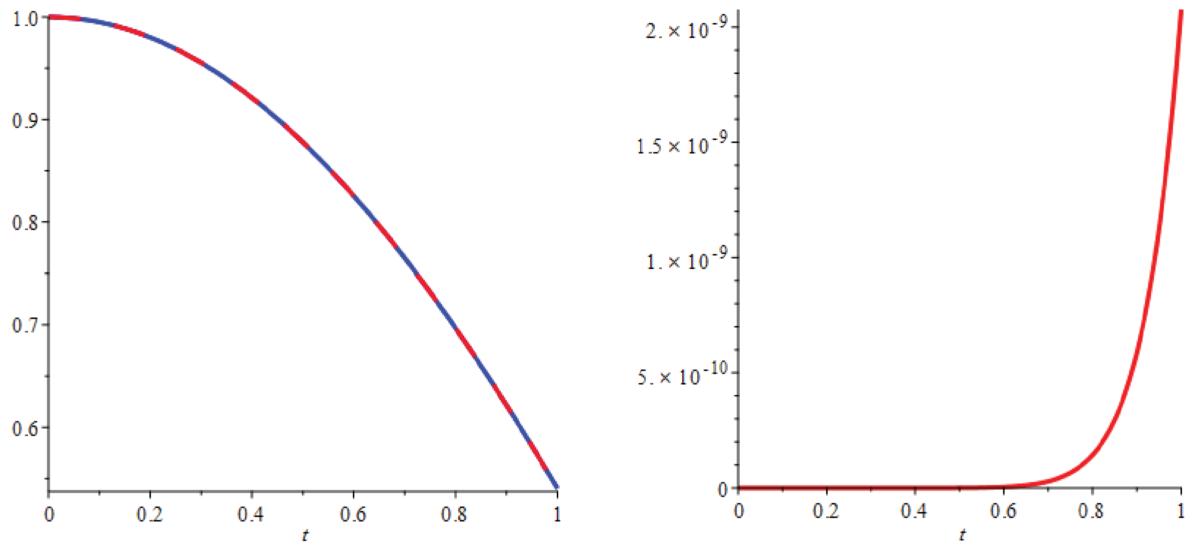
Method	x	n=5	n=6	n=7	n=8	n=9	n=10
ADE method	0.8	1.81E-02	1.79E-05	1.55E-05	1.20E-07	8.34E-07	5.55E-08
HAM method		4.10E-03	4.62E-04	4.56E-05	4.02E-07	3.19E-06	2.31E-09
Proposed method	1	1.36E-03	2.45E-05	2.45E-05	2.73E-07	2.73E-07	2.08E-09
ADE method		8.45E-02	1.07E-04	1.18E-03	1.17E-06	1.05E-06	8.54E-07
HAM method		1.62E-02	2.26E-03	2.79E-04	3.06E-05	3.03E-05	2.73E-06

When the value of n increases the closeness between the exact solution and the approximate solution increases and the absolute error decreases.



(a) (b)

FIGURE 2. (a) Exact solution, (b) Approximate solution and Absolute error for Problem 2.



(a) (b)

FIGURE 3. (a) Exact solution, (b) Approximate solution and Absolute error for Problem 3.

Therefore, the graph of exact, analytical solutions, and absolute errors of Problem 2 for $t = 1$ are given in Figures 3 and 4. Tables 3 and 4 show the absolute error using the proposed technique, ADM, and HAM of Problem 2 with $t = 1, k = 0.001$. When the value of n increases, the approximate solution gradually approaches the exact solution and the absolute error will decrease.

Problem 3. Solve the following system of linear partial differential equations [31]:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + u = 1$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} - v = 1 \qquad (9)$$

with initial condition

$u(x,0) = e^{x}, v(x,0) = e^{-x}$

and the exact solution for equations are:

$u(x,t) = e^{x-t}, v(x,t) = e^{t-x}$

Solution:

By the following equation (1), we can note after rewrite equation (9):

$F[u,v] = 1 - \left(v\frac{\partial u}{\partial x} + u\right)$

$G[u,v] = 1 - \left(u\frac{\partial v}{\partial x} - v\right)$

$a_0 = f_0(t) = e^{x}$

$b_0 = g_0(t) = e^{-x}$

$a_1 = F[u_0, v_0] = -e^{x}$

$b_1 = G[u_0, v_0] = e^{-x}$

$a_2 = F'[u_0, v_0] = e^{x}$

$b_2 = G'[u_0, v_0] = e^{-x}$

$a_3 = F''[u_0, v_0] = -e^{x}$

$b_3 = G''[u_0, v_0] = e^{-x}$

From Eq.(6)

$$u(x,t) = a_0 + a_1 t + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + \cdots + a_n\frac{t^n}{n!} + \cdots$$
$$= e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)$$
$$= e^{x-t}$$

$$v(x,t) = b_0 + b_1 t + b_2\frac{t^2}{2!} + b_3\frac{t^3}{3!} + \cdots + b_n\frac{t^n}{n!} + \cdots$$

$$= e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$
$$= e^{t-x}$$

Problem 3 for u when $t = 1$

TABLE 5. Comparison of absolute errors among different methods and present method, for problem 3 with $t = 1$,

Method	x	n=5	n=6	n=7	n=8	n=9	n=10
Proposed method	0.2	1.48E-03	2.15E-04	2.72E-05	3.06E-06	3.08E-07	2.82E-08
ADE method		9.38E-02	9.38E-03	1.06E-03	1.06E-03	7.82E-05	7.82E-07
HAM method		1.96E-02	1.96E-02	2.73E-04	2.73E-04	2.49E-06	2.49E-06
Proposed method	0.4	1.81E-03	2.63E-04	3.33E-05	3.73E-06	3.77E-07	3.45E-08
ADE method		1.62E-02	1.62E-03	1.11E-04	1.11E-05	4.91E-05	4.91E-07
HAM method		4.12E-02	4.12E-02	3.68E-04	3.68E-04	2.14E-06	2.14E-06
Proposed method	0.6	2.21E-03	3.21E-04	4.06E-05	4.56E-06	4.60E-07	4.22E-08
ADE method		1.54E-02	1.54E-03	5.37E-03	5.37E-04	1.68E-05	1.68E-06
HAM method		5.53E-02	5.53E-02	2.77E-04	2.77E-05	8.57E-06	8.57E-07
Proposed method	0.8	2.70E-03	3.92E-04	4.96E-05	5.57E-06	5.62E-07	5.14E-08
ADE method		4.33E-02	4.33E-03	5.29E-04	5.29E-05	1.38E-06	1.38E-07
HAM method		3.24E-02	3.24E-03	7.23E-04	7.23E-04	6.33E-05	6.33E-06
Proposed method	1	3.30E-03	4.79E-04	6.06E-05	6.80E-06	6.86E-07	6.28E-08
ADE method		1.97E-02	1.97E-02	1.78E-03	1.78E-04	1.75E-06	1.75E-06
HAM method		2.53E-02	2.53E-03	6.32E-04	6.32E-05	4.90E-06	4.90E-07

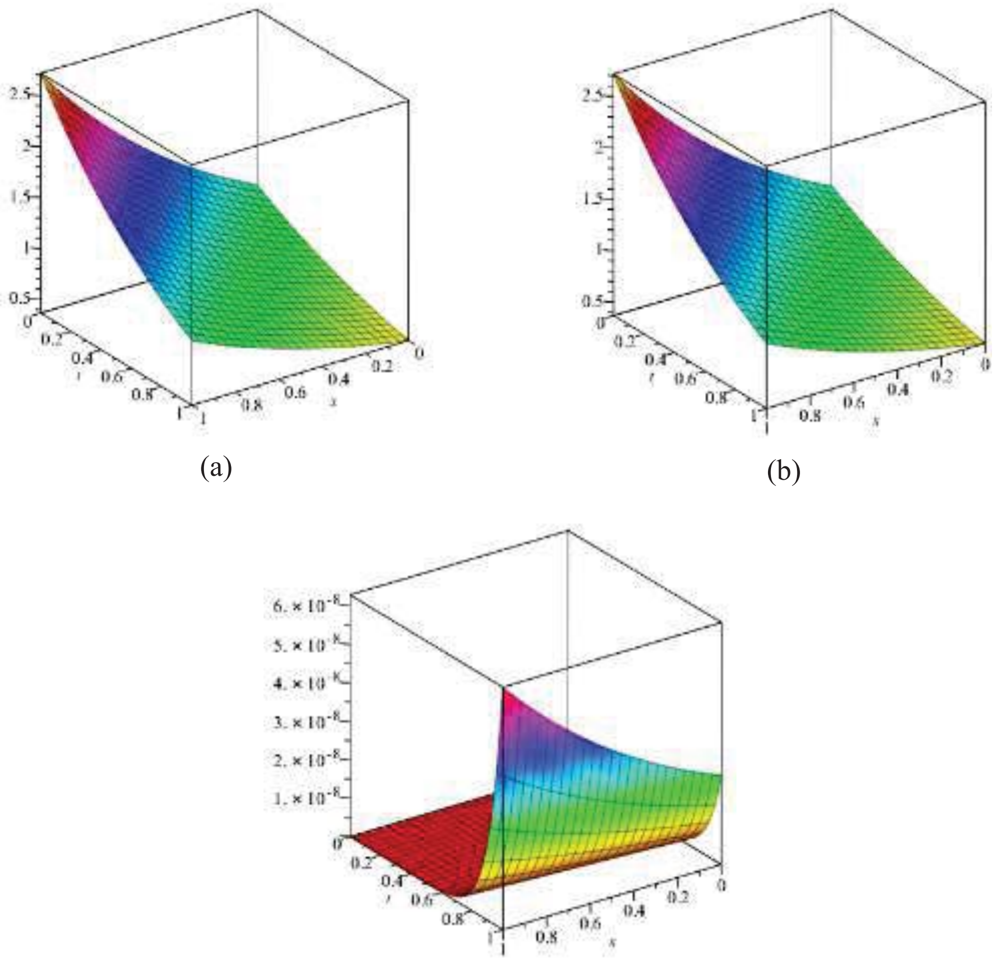
Problem 3 for v when $t = 1$

TABLE 6. Comparison of absolute errors among different methods and present method, for problem 3 with $t = 1$,

Method	x	n=5	n=6	n=7	n=8	n=9	n=10
Proposed method	0.2	1.32E-03	1.85E-04	2.28E-05	2.50E-06	2.48E-07	2.23E-08
ADE method		7.28E-03	1.06E-03	1.06E-04	9.55E-04	9.55E-05	5.86E-07
HAM method		1.36E-02	2.45E-04	2.45E-03	2.73E-04	2.73E-06	2.08E-06
Proposed method	0.4	1.08E-03	1.52E-04	1.87E-05	2.05E-06	2.03E-07	1.83E-08
ADE method		1.62E-02	1.42E-03	1.42E-03	7.76E-05	7.76E-06	2.40E-06
HAM method		3.60E-02	4.13E-03	4.13E-04	2.94E-04	2.94E-06	1.45E-05
Proposed method	0.6	8.86E-04	1.24E-04	1.53E-05	1.68E-06	1.66E-07	1.50E-08
ADE method		2.16E-03	9.66E-02	9.66E-03	2.69E-04	2.69E-05	1.03E-07
HAM method		6.44E-03	4.15E-03	4.15E-04	1.66E-05	1.66E-06	4.86E-06
Proposed method	0.8	7.26E-04	1.02E-04	1.25E-05	1.37E-06	1.36E-07	1.23E-08
ADE method		1.01E-03	1.63E-03	1.63E-04	2.77E-05	2.77E-06	4.40E-06
HAM method		5.67E-03	1.62E-03	1.62E-03	3.18E-04	3.18E-05	2.88E-07

Method	x	n=5	n=6	n=7	n=8	n=9	n=10
Proposed method	1	5.94E-04	8.32E-05	1.02E-05	1.13E-06	1.11E-07	1.01E-08
ADE method		1.39E-03	2.22E-03	2.22E-04	1.97E-04	1.97E-06	1.97E-04
HAM method		8.88E-02	6.22E-04	6.22E-04	1.27E-05	1.27E-05	1.24E-06

When the value of n increases the closeness between the exact solution and the approximate solution increases and the absolute error decreases.



(a) (b) (c)

FIGURE 4. (a) Exact solution of u, (b) Approximate solution of u (c) Absolute errors of u for Problem 3.

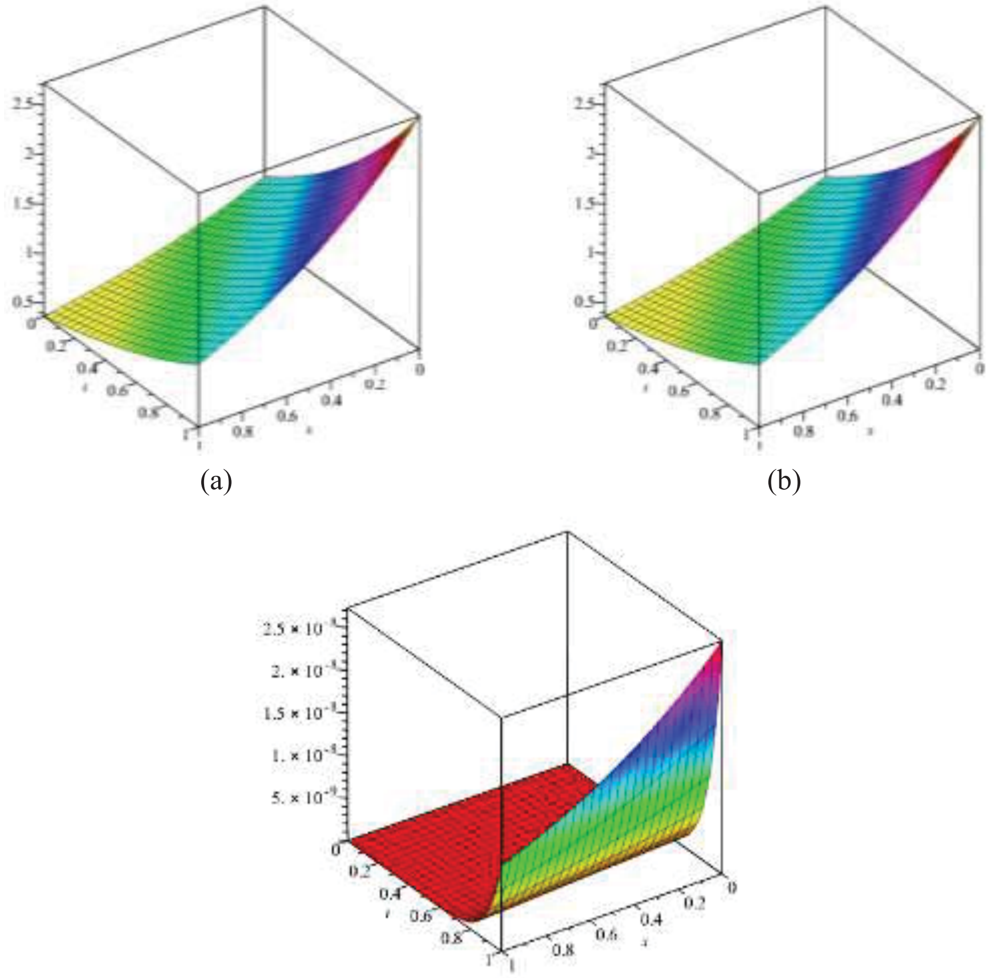


FIGURE 5. (a) Exact solution of v, (b) Approximate solution of v (c) Absolute errors of v for Problem 3.

Therefore, the graph of exact, analytical solutions, and absolute errors of Problem 3 for $t = 1$ are given in Figures 5 and 6. Tables 5 and 6 show the absolute error using the proposed technique, ADM, and HAM of Problem 3 with $t = 1, k = 0.001$. When the value of n increases, the approximate solution gradually approaches the exact solution and the absolute error will decrease.

$$\frac{\partial v}{\partial t} = v\frac{\partial v}{\partial x} + u\frac{\partial v}{\partial y} \tag{10}$$

with initial condition
$u(x, y, 0) = v(x, y, 0) = x + y$
and the exact solution for equations are:
$u(x, y, t) = v(x, y, t) = \frac{x+y}{1-2t}$

<u>Solution:</u>

By the following equation (1), we can note after rewrite equation (10):
$F[u, v] = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$
$G[u, v] = v\frac{\partial v}{\partial x} + u\frac{\partial v}{\partial y}$
$a_0 = f_0(t) = x + y$
$b_0 = g_0(t) = x + y$

$a_1 = F[u_0, v_0] = 2x + 2y$
$b_1 = G[u_0, v_0] = 2x + 2y$
$a_2 = F'[u_0, v_0] = 8x + 8y$
$b_2 = G'[u_0, v_0] = 8x + 8y$
$a_3 = F''[u_0, v_0] = 48x + 48y$
$b_3 = G''[u_0, v_0] = 48x + 48y$
From Eq.(5)

$$u(x,t) = a_0 + a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + \cdots + a_n \frac{t^n}{n!} + \cdots$$
$$= (x+y) + (2x+2y)t + (8x+8y)\frac{t^2}{2!} + (48x+48y)\frac{t^3}{3!} + \cdots$$
$$= \frac{x+y}{1-2t}$$
$$v(x,t) = b_0 + b_1 t + b_2 \frac{t^2}{2!} + b_3 \frac{t^3}{3!} + \cdots + b_n \frac{t^n}{n!} + \cdots$$
$$= (x+y) + (2x+2y)t + (8x+8y)\frac{t^2}{2!} + (48x+48y)\frac{t^3}{3!} + \cdots$$
$$= \frac{x+y}{1-2t}$$

Problem 4 for u and v when t =0.1

TABLE 7. Comparison of absolute errors among different methods and present method, for problem 4 with $t = 1$,

Method	x	y	n=5	n=6	n=7	n=8	n=9	n=10
Proposed method	0.1	0.5	1.48E-03	2.15E-04	2.72E-05	3.06E-06	3.08E-07	2.82E-08
ADE method			1.46E-03	2.05E-03	2.52E-04	2.77E-04	2.74E-05	2.48E-08
HAM method			1.08E-02	1.52E-02	1.87E-04	2.05E-04	2.03E-04	1.83E-08
Proposed method	0.5	0.1	1.81E-03	2.63E-04	3.33E-05	3.73E-06	3.77E-07	3.45E-08
ADE method			1.32E-02	1.85E-03	2.28E-04	2.50E-05	2.48E-05	2.23E-06
HAM method			1.08E-02	1.52E-03	1.87E-03	2.05E-04	2.03E-04	1.83E-06
Proposed method	0.5	0.9	2.21E-03	3.21E-04	4.06E-05	4.56E-06	4.60E-07	4.22E-08
ADE method			7.26E-02	1.02E-03	1.25E-03	1.37E-04	1.36E-05	1.23E-06
HAM method			8.86E-02	1.24E-02	1.53E-04	1.68E-05	1.66E-04	1.50E-05
Proposed method	0.9	0.5	2.70E-03	3.92E-04	4.96E-05	5.57E-06	5.62E-07	5.14E-08
ADE method			6.57E-01	9.20E-02	1.13E-04	1.24E-05	1.23E-06	1.11E-06
HAM method			7.26E-02	1.02E-03	1.25E-03	1.37E-04	1.36E-06	1.23E-06
Proposed method	0.9	0.9	3.30E-03	4.79E-04	6.06E-05	6.80E-06	6.86E-07	6.28E-08
ADE method			6.57E-02	9.20E-02	1.13E-03	1.24E-04	1.23E-05	1.11E-06
HAM method			5.94E-02	8.32E-03	1.02E-04	1.13E-03	1.11E-05	1.01E-05

When the value of n increases the closeness between the exact solution and the approximate solution increases and the absolute error decreases.

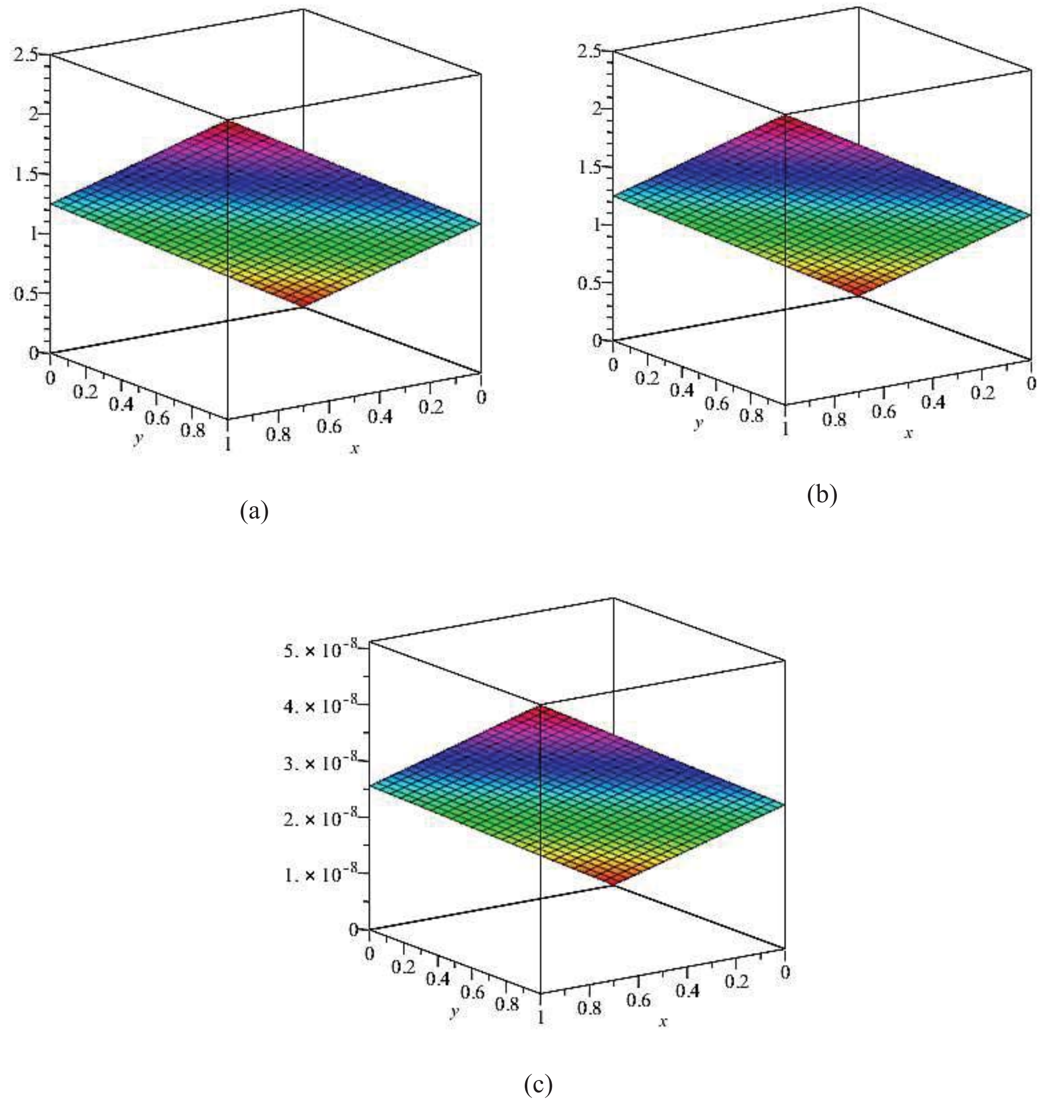


FIGURE 6. (a) Exact solution of u and v, (b) Approximate solution of u and v (c) Absolute errors of u and v for Problem 4.

Therefore, the graph of exact, analytical solutions, and absolute errors of Problem 4 for $t = 1$ are given in Figure 7. Table 7 show the absolute error using the proposed technique, ADM, and HAM of Problem 4 with$t = 1, k = 0.001$. When the value of n increases, the approximate solution gradually approaches the exact solution and the absolute error will decrease.

CONVERGENCE ANALYSIS

Consider the system of equation in the following form:

$$u = F(u,v),$$
$$v = G(u,v), \quad (11)$$

where F, G are a non-linear operator. The solution by the present approach is equivalent to the following sequence:

$$S_n = \sum_{i=0}^{n} u_i = \sum_{i=0}^{n} a_i \frac{t^i}{(i)!}$$
$$K_n = \sum_{i=0}^{n} v_i = \sum_{i=0}^{n} b_i \frac{t^i}{(i)!}$$

Theorem (Convergence of System Equations)

Let F, G be an operator from a Hilbert space H into H and u, v be the exact solution of equation (11). The approximate solutions

$$\sum_{i=0}^{\infty} u_i = \sum_{i=0}^{\infty} a_i \frac{t^i}{(i)!}$$

and

$$\sum_{i=0}^{\infty} v_i = \sum_{i=0}^{\infty} b_i \frac{t^i}{(i)!}$$

are convergence to exact solutions u, v respectively when $\exists\ 0 \le \alpha < 1$,
$\|u_{i+1}\| \le \alpha \|u_i\|, \forall\, i \in \mathbb{N} \cup \{0\}and\exists\ 0 \le \gamma < 1, \|v_{i+1}\| \le \gamma\|v_i\|, \forall\, i \in \mathbb{N} \cup \{0\}$.

Proof : We want to show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence,
$\|S_{n+1} - S_n\| = \|u_{n+1}\| \le \alpha\|u_n\| \le \alpha^2\|u_{n-1}\| \le \cdots \le \alpha^n\|u_1\| \le \alpha^{n+1}\|u_0\|$.
Now for $n, m \in \mathbb{N}, n \ge m$

$$\|S_n - S_m\| = \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_{m+1} - S_m)\|$$
$$\le \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \cdots + \|S_{m+1} - S_m\|$$
$$\le \alpha^n\|u_0\| + \alpha^{n-1}\|u_0\| + \cdots + \alpha^{m+1}\|u_0\|$$
$$\le (\alpha^{m+1} + \alpha^{m+2} + \cdots + \alpha^n)\|u_0\| =$$
$$\alpha^{m+1}\frac{1-\alpha^{n-m}}{1-\alpha}\|u_0\|$$

Hence, $\lim_{n,m\to\infty}\|S_n - S_m\| = 0$ that is mean $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space H then there exist $S \in H$ such that $\lim_{n\to\infty} S_n = S$, where $S = u$.

In the same way we prove that $\sum_{i=0}^{\infty} b_i \frac{t^i}{(i)!}$ is convergence to exact solution v just replace v by u and α by γ.

Definition 4.2: For every $n \in \mathbb{N} \cup \{0\}$, we define

$$\alpha_n = \begin{cases} \frac{\|u_{n+1}\|}{\|u_n\|}, & \|u_n\| \ne 0 \\ 0, & otherwise \end{cases}$$

Corollary 4.3: From the theorem above $\sum_{i=0}^{\infty} u_i = \sum_{i=0}^{\infty} a_i \frac{t^i}{(i)!}$ convergence to exact solution u when $0 \le \alpha_i < 1,\ i = 0,1,2,\ldots$

Now, To illustrate the convergence of analytical approximate solutions for the four examples we applied Corollary as follows;

In **first** problem, where $t \in (0,1)$ we get for x

$\alpha_0 = \frac{\|x_1\|}{\|x_0\|} = 0.57735026919 < 1$

$\alpha_1 = \frac{\|x_2\|}{\|x_1\|} = 0.38729833463 < 1,$

$\alpha_2 = \frac{\|x_3\|}{\|x_2\|} = 0.28171808490 < 1,$

$\alpha_3 = \frac{\|x_4\|}{\|x_3\|} = 0.22047927593 < 1,$

and for y

$\gamma_0 = \frac{\|y_1\|}{\|y_0\|} = 0.57735026919 < 1,$
$\gamma_1 = \frac{\|y_2\|}{\|y_1\|} = 0.38729833463 < 1,$
$\gamma_2 = \frac{\|y_3\|}{\|y_2\|} = 0.28171808490 < 1,$
$\gamma_3 = \frac{\|y_4\|}{\|y_3\|} = 0.22047927593 < 1,$
In **second** problem, where $\mathrm{t} \in (0,1)$ we get for x
$\alpha_0 = \frac{\|x_1\|}{\|x_0\|} = 0.10910894512 < 1$
$\alpha_1 = \frac{\|x_2\|}{\|x_1\|} = 0.039886201759 < 1,$
$\alpha_2 = \frac{\|x_3\|}{\|x_2\|} = 0.020389258062 < 1,$
$\alpha_3 = \frac{\|x_4\|}{\|x_3\|} = 0.012340601620 < 1,$
$\alpha_4 = \frac{\|x_5\|}{\|x_4\|} = 0.0082626659921 < 1,$
and for y
$\gamma_0 = \frac{\|y_1\|}{\|y_0\|} = 0.22360679775 < 1,$
$\gamma_1 = \frac{\|y_2\|}{\|y_1\|} = 0.062112999376 < 1,$
$\gamma_2 = \frac{\|y_3\|}{\|y_2\|} = 0.027735009811 < 1,$
$\gamma_3 = \frac{\|y_4\|}{\|y_3\|} = 0.015615618431 < 1,$
$\gamma_4 = \frac{\|y_5\|}{\|y_4\|} = 0.0099970601201 < 1,$
In **third** problem, where $(x, \mathrm{t}) \in (0,1)^2$ we get for u
$\alpha_0 = \frac{\|u_1\|}{\|u_0\|} = 0.57735026917 < 1$
$\alpha_1 = \frac{\|u_2\|}{\|u_1\|} = 0.38729833464 < 1,$
$\alpha_2 = \frac{\|u_3\|}{\|u_2\|} = 0.28171808492 < 1,$
$\alpha_3 = \frac{\|u_4\|}{\|u_3\|} = 0.22047927592 < 1,$
and for v
$\gamma_0 = \frac{\|v_1\|}{\|v_0\|} = 0.57735026921 < 1,$
$\gamma_1 = \frac{\|v_2\|}{\|v_1\|} = 0.38729833462 < 1,$
$\gamma_2 = \frac{\|v_3\|}{\|v_2\|} = 0.28171808490 < 1,$
$\gamma_3 = \frac{\|v_4\|}{\|v_3\|} = 0.22047927593 < 1,$
In **four** problem, where $(x, \mathrm{y}) \in (0,1)^2$ and t=0.1we get for u
$\alpha_0 = \frac{\|u_1\|}{\|u_0\|} = 0.2 < 1$
$\alpha_1 = \frac{\|u_2\|}{\|u_1\|} = 0.2 < 1,$
$\alpha_2 = \frac{\|u_3\|}{\|u_2\|} = 0.2 < 1,$
$\alpha_3 = \frac{\|u_4\|}{\|u_3\|} = 0.2 < 1,$
and for v
$\gamma_0 = \frac{\|v_1\|}{\|v_0\|} = 0.2 < 1,$
$\gamma_1 = \frac{\|v_2\|}{\|v_1\|} = 0.2 < 1,$
$\gamma_2 = \frac{\|v_3\|}{\|v_2\|} = 0.2 < 1,$
$\gamma_3 = \frac{\|v_4\|}{\|v_3\|} = 0.2 < 1,$

DISCUSSION AND CONCLUSIONS

The proposed technique is an efficient methodology with good accuracy and convergence and a powerful tool to find approximate analytic solutions for the linear and nonlinear differential equation systems. Four test problems are introduced for confirming the validity of the novel proposed technique, which used to find the solutions of linear and nonlinear systems of ODEs and PDEs. Figures (1-7) showed that the exact solution, approximate solution, and absolute errors at $t = 1, k = 0.001$. Also, the approximate solutions obtained by a proposed method have been compared with the solutions obtained by ADM [13], and HAM [19] in these four test problems, which are given in Tables (1-7). The approximate solutions obtained by a proposed method are more identical to the exact solutions than those obtained using other methods. Therefore, the absolute errors of the proposed method are smaller than ADM and HAM, as introduced in Tables (1-7). These test problems confirm the validity of a proposed approximate method to handle current linear and nonlinear systems of ODEs and PDEs. In the future, this research can be extended to the investigation by applying this method for more complicated problems such as systems of high-order nonlinear PDEs. More precisely, the measurement of absolute errors for these examples guarantees the ability of the proposed method and its accuracy in finding the approximate solutions of linear and nonlinear systems of ODEs and PDEs.

REFERENCES

1. M. A. El Salam, "Solving high-order partial differential equations in unbounded domains by means of double exponential second kind Chebyshev approximation," *Computational Methods for Differential Equations*, 2015.
2. S. C. S. Rao and V. Srivastava, "Parameter-Robust Numerical Method for Time-Dependent Weakly Coupled Linear System of Singularly Perturbed Convection-Diffusion Equations," *Differ. Equations Dyn. Syst.*, vol. 25, no. 2, pp. 301–325, Apr. 2017.
3. R. Shekari Beidokhti and A. Malek, "Solving initial-boundary value problems for systems of partial differential equations using neural networks and optimization techniques," *J. Franklin Inst.*, vol. 346, no. 9, pp. 898–913, Nov. 2009.
4. R. C. M. and R. Bhatia, "A Collocation Method for Numerical Solution of Hyperbolic Telegraph Equation with Neumann Boundary Conditions," *International Journal of Computational Mathematics*, 2014. .
5. A. S. Nielsen, G. Brunner, and J. S. Hesthaven, "Communication-aware adaptive Parareal with application to a nonlinear hyperbolic system of partial differential equations," *J. Comput. Phys.*, vol. 371, pp. 483–505, Oct. 2018.
6. F. Gao and C. Chi, "Unconditionally stable difference schemes for a one-space-dimensional linear hyperbolic equation," *Appl. Math. Comput.*, vol. 187, no. 2, pp. 1272–1276, Apr. 2007.
7. H. Y. Xu, S. Y. Liu, and Q. P. Li, "Entire solutions for several systems of nonlinear difference and partial differential-difference equations of Fermat-type," *J. Math. Anal. Appl.*, vol. 483, no. 2, p. 123641, Mar. 2020.
8. Y. Sheng and T. Zhang, "The finite volume method for two-dimensional Burgers' equation," *Pers. Ubiquitous Comput.*, vol. 22, no. 5–6, pp. 1133–1139, Oct. 2018.
9. A. K. Al-Jaberi and E. M. Hameed, "A Review of PDE Based Local Inpainting Methods," *J. Phys. Conf. Ser.*, vol. 1818, no. 1, p. 012149, Mar. 2021.
10. K. Singla and R. K. Gupta, "Conservation laws for certain time fractional nonlinear systems of partial differential equations," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 53, pp. 10–21, Dec. 2017.
11. J. Jonasson, "Multiplication of solutions for linear overdetermined systems of partial differential equations," *J. Geom. Phys.*, vol. 58, no. 8, pp. 1015–1029, Aug. 2008.
12. N. Taghizadeh and A. Neirameh, "Generalization of some overdetermined systems of complex partial differential equations," *J. King Saud Univ. - Sci.*, vol. 23, no. 2, pp. 175–177, Apr. 2011.
13. W. Li and Y. Pang, "Application of Adomian decomposition method to nonlinear systems," *Adv. Differ. Equations*, vol. 2020, no. 1, pp. 1–17, Dec. 2020.
14 H. Al-Badrani, S. Saleh, H. O. Bakodah, M. Al-Mazmumy, A.-M. Al-Munawarah, and S. Arabia, "Numerical Solution for Nonlinear Telegraph Equation by Modified Adomian Decomposition Method," *Nonlinear Anal. Differ. Equations*, vol. 4, no. 5, pp. 243–257, 2016.
15. J. K. " Zhou, "Differential transformation and its applications for electrical circuits."," pp. 1279-1289., 1986.
16. J. H. He, "Variational iteration method-Some recent results and new interpretations," *J. Comput. Appl. Math.*, vol. 207, no. 1, pp. 3–17, Oct. 2007.
17. H. Jafari, H. K. Jassim, S. P. Moshokoa, V. M. Ariyan, and F. Tchier, "Reduced differential transform method for partial differential equations within local fractional derivative operators," *Adv. Mech. Eng.*, vol. 8, no. 4, p. 168781401663301, Apr. 2016.

18. H. Eltayeb and A. Kiliçman, "Application of sumudu decomposition method to solve nonlinear system of partial differential equations," *Abstr. Appl. Anal.*, vol. 2012, 2012.
19. A. Sami Bataineh, M. S. M. Noorani, and I. Hashim, "Approximate analytical solutions of systems of PDEs by homotopy analysis method," *Comput. Math. with Appl.*, vol. 55, no. 12, pp. 2913–2923, Jun. 2008.
20 S. Rathore, D. Kumar, J. Singh, and S. Gupta, "Homotopy Analysis Sumudu Transform Method for Nonlinear Equations," 2012.
21. K. Abbaoui and Y. Cherruault, "New ideas for proving convergence of decomposition methods," *Comput. Math. with Appl.*, vol. 29, no. 7, pp. 103–108, Apr. 1995.
22. S. M. El-Sayed and D. Kaya, "On the numerical solution of the system of two-dimensional Burgers' equations by the decomposition method," *Appl. Math. Comput.*, vol. 158, no. 1, pp. 101–109, Oct. 2004.
23. M. Inc, "Exact solutions with solitary patterns for the Zakharov-Kuznetsov equations with fully nonlinear dispersion," *Chaos, Solitons and Fractals*, vol. 33, no. 5, pp. 1783–1790, Aug. 2007.
24. R. Abazari and A. Borhanifar, "Numerical study of the solution of the Burgers and coupled Burgers equations by a differential transformation method," *Comput. Math. with Appl.*, vol. 59, no. 8, pp. 2711–2722, Apr. 2010.
25. A. A. M. Arafa, S. Z. Rida, and H. Mohamed, "Homotopy Analysis Method for Solving Biological Population Model," *Commun. Theor. Phys.*, vol. 56, no. 5, p. 797, 2011.
26. P. Roul, "Application of Homotopy Perturbation Method to Biological Population Model," *Appl. Appl. Math. An Int. J.*, vol. 05, no. 2, pp. 1369–1378, 2010.
27. J. He, "A new approach to nonlinear partial differential equations," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 2, no. 4, pp. 230–235, Dec. 1997.
28. R. Yulita Molliq, M. S. M. Noorani, I. Hashim, and R. R. Ahmad, "Approximate solutions of fractional Zakharov-Kuznetsov equations by VIM," *J. Comput. Appl. Math.*, vol. 233, no. 2, pp. 103–108, Nov. 2009.
29. M. Ahmed K. Al-JABERI.1 , Ehsan M. HAMEED.2, "A NOVEL ANALYTIC METHOD FOR SOLVING LINEAR AND NON-LINEAR TELEGRAPH EQUATION," *PERIÓDICO TCHÊ QUÍMICA (2179-0302) 17 (35), 536-548*, 2020.
30. M. A. Muheidat, "Analytical Solutions for Nonlinear Dynamic System by using homotopy analysis method," Zarqa University, 2017.
31. A. A. Manan, M. Noah, M. Hassan, and M. Khabir, "Homotopy Perturbation Method for Solving Advection and Integral Equations A Thesis submitted in partial fulfillment for the Degree of M.Sc. in Mathematics," Sudan University of Science and Technology, Apr. 2016.
32. A. H. Ali and A. S. J. Al-Saif, "Adomian decomposition method for solving some models of nonlinear partial differential equations," 2008.