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The bivariate Rogers–Szegö polynomials

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Abstract
We present an operator approach to deriving Mehler's formula and the Rogers formula for the bivariate Rogers–Szegö polynomials $h_n(x, y|q)$. The proof of Mehler's formula can be considered as a new approach to the nonsymmetric Poisson kernel formula for the continuous big q-Hermite polynomials $H_n(x; a|q)$ due to Askey, Rahman and Suslov. Mehler's formula for $h_n(x, y|q)$ involves a ${}_3\phi_2$ sum and the Rogers formula involves a ${}_2\phi_1$ sum. The proofs of these results are based on parameter augmentation with respect to the q-exponential operator and the homogeneous q-shift operator in two variables. By extending recent results on the Rogers–Szegö polynomials $h_n(x|q)$ due to Hou, Lascoux and Mu, we obtain another Rogers-type formula for $h_n(x, y|q)$. Finally, we give a change of base formula for $H_n(x; a|q)$ which can be used to evaluate some integrals by using the Askey–Wilson integral.

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1. Introduction

The Rogers–Szegö polynomials $h_n(x|q)$ have been extensively studied since the end of the 19th century. Two classical results for the Rogers–Szegö polynomials are Mehler's formula and the Rogers formula which respectively correspond to the Poisson kernel formula and the linearization formula. In this paper, we extend Mehler's formula and the Rogers formula to the bivariate Rogers–Szegö polynomials $h_n(x, y|q)$ by using the q-exponential operator as studied in [11] and the homogeneous q-shift operator recently introduced by Chen, Fu and Zhang [10]. It should be noted that Mehler's formula for $h_n(x, y|q)$ is equivalent to the nonsymmetric Poisson kernel formula for the continuous big q-Hermite polynomials due to Askey, Rahman and Suslov [5]. So, our proof of Mehler's formula for $h_n(x, y|q)$ may be considered as a new approach to the nonsymmetric Poisson kernel for the continuous big

q-Hermite polynomials. As can be seen, the bivariate version of the Rogers–Szegö polynomials is easier to deal with from the operator point of view.

Let us review some common notation and terminology for the basic hypergeometric series in [13]. Throughout this paper, we assume that $|q| < 1$. The q-shifted factorial is defined by

$$(a;q)_0 = 1, \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad n \in \mathbb{Z}.$$

The following notation stands for the multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$
$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \cdots (a_m;q)_\infty.$$

The q-binomial coefficients, or the Gaussian coefficients, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

The basic hypergeometric series ${}_{r+1}\phi_r$ are defined by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} x^n.$$

We will be mainly concerned with the *bivariate Rogers–Szegö polynomials* as given below:

$$h_n(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, y),$$

where $P_n(x, y) = (x-y)(x-qy)\cdots(x-q^{n-1}y)$ are the Cauchy polynomials with the generating function

$$\sum_{n=0}^{\infty} P_n(x, y)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \qquad |xt| < 1. \tag{1.1}$$

Note that the Cauchy polynomials $P_n(x, y)$ naturally arise in the q-umbral calculus as studied by Andrews [2, 3], Goldman and Rota [14], Goulden and Jackson [15], Ihrig and Ismail [17], Johnson [21] and Roman [27]. The generating function (1.1) is also the homogeneous version of the Cauchy identity or the q-binomial theorem [13]:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z| < 1. \tag{1.2}$$

Putting $a = 0$, (1.2) becomes Euler's identity [13]

$$\sum_{n=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}, \qquad |z| < 1 \tag{1.3}$$

and its inverse relation takes the form

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} z^k}{(q;q)_k} = (z;q)_\infty. \tag{1.4}$$

The continuous big q-Hermite polynomials [23] are defined by

$$H_n(x; a|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a\,\mathrm{e}^{\mathrm{i}\theta}; q)_k\, \mathrm{e}^{\mathrm{i}(n-2k)\theta}, \qquad x = \cos\theta.$$

We first observe that the bivariate Rogers–Szegö polynomials $h_n(x, y|q)$ introduced by Chen, Fu and Zhang [10] are equivalent to the continuous big q-Hermite polynomials owing to the following relation:

$$H_n(x; a|q) = \mathrm{e}^{\mathrm{i}n\theta} h_n(\mathrm{e}^{-2\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta}|q), \qquad x = \cos\theta. \tag{1.5}$$

The polynomials $h_n(x, y|q)$ have the generating function [10]:

$$\sum_{n=0}^{\infty} h_n(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(t, xt; q)_\infty}, \qquad |t| < 1, \quad |xt| < 1, \tag{1.6}$$

which is equivalent to the generating function for the big continuous q-Hermite polynomials, see, for example, Koekoek–Swarttouw [23]. Note that the classical *Rogers–Szegö polynomials*,

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k,$$

are a special case of $h_n(x, y|q)$ when y is set to zero, and in this case (1.6) reduces to

$$\sum_{n=0}^{\infty} h_n(x|q) \frac{t^n}{(q; q)_n} = \frac{1}{(t, xt; q)_\infty}, \qquad |t| < 1. \tag{1.7}$$

The Rogers–Szegö polynomials play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey–Wilson polynomials, see [1, 4, 8, 9, 18, 20, 24, 29]. They are closely related to the q-Hermite polynomials

$$H_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{e}^{\mathrm{i}(n-2k)\theta}, \qquad x = \cos\theta.$$

In fact, the following relations hold:

$$H_n(x|q) = H_n(x; 0|q) = \mathrm{e}^{\mathrm{i}n\theta} h_n(\mathrm{e}^{-2\mathrm{i}\theta}|q), \qquad x = \cos\theta. \tag{1.8}$$

The continuous big q-Hermite polynomials $H_n(x; a|q)$ are connected with the q-Hermite polynomials $H_n(x|q)$ via the following relation [7, 12]:

$$H_n(x; a|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} a^k H_{n-k}(x|q), \tag{1.9}$$

and the inverse expansion of (1.9) becomes

$$H_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^k H_{n-k}(x; a|q). \tag{1.10}$$

This paper is motivated by the natural question of extending Mehler's formula to $h_n(x, y|q)$, where Mehler's formula for the Rogers–Szegö polynomials reads

$$\sum_{n=0}^{\infty} h_n(x|q) h_n(y|q) \frac{t^n}{(q; q)_n} = \frac{(xyt^2; q)_\infty}{(t, xt, yt, xyt; q)_\infty}. \tag{1.11}$$

Formula (1.11) has been extensively studied, see [11, 18, 22, 24, 29, 30]. Based on the recurrence relation for $H_n(x|q)$, Bressoud [9] gave a proof of the equivalent formula, or the Poisson kernel formula, for the q-Hermite polynomials $H_n(x|q)$. Ismail, Stanton and Viennot [20] found a combinatorial proof of the Poisson kernel formula for $H_n(x|q)$ by using the vector space interpretation of the q-binomial coefficients. Askey, Rahman and Suslov [5] derived the nonsymmetric Poisson kernel formula for the continuous big q-Hermite polynomials $H_n(x;a|q)$:

$$\sum_{n=0}^{\infty} H_n(x;a|q)H_n(y;b|q)\frac{t^n}{(q;q)_n} = \frac{(at\,\mathrm{e}^{\mathrm{i}\beta}, b\,\mathrm{e}^{-\mathrm{i}\beta}, t^2;q)_\infty}{(t\,\mathrm{e}^{\mathrm{i}(\theta+\beta)}, t\,\mathrm{e}^{\mathrm{i}(\theta-\beta)}, t\,\mathrm{e}^{-\mathrm{i}(\theta+\beta)}, t\,\mathrm{e}^{-\mathrm{i}(\theta-\beta)};q)_\infty} \times {}_3\phi_2\left(\begin{matrix} t\,\mathrm{e}^{\mathrm{i}(\theta+\beta)}, t\,\mathrm{e}^{-\mathrm{i}(\theta-\beta)}, at/b \\ at\,\mathrm{e}^{\mathrm{i}\beta}, t^2 \end{matrix}; q, b\,\mathrm{e}^{-\mathrm{i}\beta}\right), \tag{1.12}$$

where $x = \cos\theta$, $y = \cos\beta$. The above formula can be viewed as Mehler's formula for $H_n(x;a|q)$. Moreover, it can be restated in terms of $h_n(x,y|q)$ (theorem 2.1). The first result of this paper is an operator approach to Mehler's formula for $h_n(x,y|q)$. We will present a simple proof by using the exponential operators involving the classical q-differential operator and a bivariate q-differential operator introduced by Chen, Fu and Zhang [10].

The second result of this paper is the Rogers formula for $h_n(x,y|q)$. The Rogers formula [11, 24, 25] for the classical Rogers–Szegö polynomials $h_n(x|q)$ reads:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = (xst;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_n(x|q)h_m(x|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}. \tag{1.13}$$

One of the most important applications of the Rogers formula is to deduce the following linearization formula for $h_n(x|q)$ (cf [9, 18, 26]):

$$h_n(x|q)h_m(x|q) = \sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}(q;q)_k x^k h_{n+m-2k}(x|q). \tag{1.14}$$

Based on a recent approach of Hou, Lascoux and Mu to the Rogers–Szegö polynomials, we derive a second Rogers-type formula for $h_n(x,y|q)$ which leads to a simpler linearization formula compared with the first one we have obtained.

We conclude this paper with a change of base formula for $H_n(x;a|q)$. This formula along with other identities can be used to compute some integrals with the aid of the Askey–Wilson integral.

2. Mehler's formula for $h_n(x, y|q)$

In this section, we aim to present an operator approach to Mehler's formula for the bivariate Rogers–Szegö polynomials $h_n(x,y|q)$.

Theorem 2.1 (Mehler's formula for $h_n(x,y|q)$)**.** *We have*

$$\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n} = \frac{(yt, vxt;q)_\infty}{(t, xt, uxt;q)_\infty}{}_3\phi_2\left(\begin{matrix} y, xt, v/u \\ yt, vxt \end{matrix}; q, ut\right), \tag{2.1}$$

provided that $|t|, |xt|, |ut|, |uxt| < 1$.

Obviously, Mehler's formula (1.11) for $h_n(x|q)$ can be deduced from the above theorem by setting $y = 0$, $v = 0$ and $u = y$. We note that it is not difficult to reformulate (2.1) as the nonsymmetric Poisson kernel formula (1.12) for $H_n(x;a|q)$. To this end, we first make the variable substitutions $x \to \mathrm{e}^{-2\mathrm{i}\theta}$, $y \to a\,\mathrm{e}^{-\mathrm{i}\theta}$, $u \to e^{-2i\beta}$, $v \to be^{-i\beta}$, so that we may use relation (1.5) to transform $h_n(x,y|q)$ and $h_n(u,v|q)$ into $H_n(x;a|q)$ and $H_n(y;b|q)$. Then the formula (1.12) follows from the ${}_3\phi_2$ transformation [13, appendix III, equation (III.9)]:

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right) = \frac{(e/a, de/bc;q)_\infty}{(e, de/abc;q)_\infty}{}_3\phi_2\left(\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, \frac{e}{a}\right). \tag{2.2}$$

Our operator approach to theorem 2.1 involves two identities (lemmas 2.2 and 2.3) in connection with the q-exponential operator and the homogeneous q-shift operator. The q-differential operator, or the q-derivative, acting on the variable a, is defined by

$$D_q f(a) = \frac{f(a) - f(aq)}{a},$$

and the q-exponential operator is given by

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}.$$

Evidently,

$$T(D_q)\{x^n\} = h_n(x|q). \tag{2.3}$$

Lemma 2.2. *We have*

$$T(bD_q)\left\{\frac{(av;q)_\infty}{(as,at;q)_\infty}\right\} = \frac{(bv;q)_\infty}{(as,bs,bt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} v/t, bs \\ bv \end{matrix}; q, at\right), \tag{2.4}$$

provided that $|bs|, |bt| < 1$.

From the Leibniz rule for D_q (see [28])

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k\{f(a)\} D_q^{n-k}\{g(q^k a)\},$$

(2.4) can be verified by straightforward computation. Here, we also note that a more general relation has been established by Zhang and Wang [31]:

$$\begin{aligned} T(bD_q)\left\{\frac{(av;q)_\infty}{(as,at,aw;q)_\infty}\right\} &= (av,bv;q)_\infty \frac{(abstw/v;q)_\infty}{(as,at,aw,bs,bt,bw;q)_\infty} \\ &\quad \times {}_3\phi_2\left(\begin{matrix} v/s, v/t, v/w \\ av, bv \end{matrix}; q, abstw/v\right), \end{aligned} \tag{2.5}$$

where $|bs|, |bt|, |bw|, |abstw/v| < 1$. Setting $w = 0$ in (2.5), by virtue of Jackson's transformation [13, appendix III, equation (III.4)] and Heine's transformation [13, appendix III, equation (III.1)], (2.4) becomes a consequence of (2.5).

In [10], Chen, Fu and Zhang introduced the homogeneous q-difference operator

$$D_{xy} f(x,y) = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}$$

and the homogeneous q-shift operator

$$\mathbb{E}(D_{xy}) = \sum_{k=0}^{\infty} \frac{D_{xy}^k}{(q;q)_k}.$$

The following basic facts have been observed in [10]:

$$\begin{aligned} &D_{xy}\{P_n(x,y)\} = (1-q^n)P_{n-1}(x,y), \\ &\mathbb{E}(D_{xy})\{P_n(x,y)\} = h_n(x,y|q). \end{aligned} \tag{2.6}$$

Lemma 2.3. *We have*

$$\mathbb{E}(D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_n(x,y)}{(yt;q)_n}\right\} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(y,xt;q)_k}{(yt;q)_k}x^{n-k},$$

provided that $|t|, |xt| < 1$.

Proof. Let us compute the following sum in two ways:

$$\sum_{n=0}^{\infty} h_n(x, y|q) h_n(z|q) \frac{t^n}{(q;q)_n}. \tag{2.7}$$

We may either express $h_n(z|q)$ as $T(D_q)\{z^n\}$ by (2.3) or express $h_n(x, y|q)$ as $\mathbb{E}(D_{xy})\{P_n(x, y)\}$ by (2.6). Invoking $h_n(z|q) = T(D_q)\{z^n\}$, the sum (2.7) equals

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(x, y|q) T(D_q)\{z^n\} \frac{t^n}{(q;q)_n} \\
&\quad = T(D_q)\left\{\sum_{n=0}^{\infty} h_n(x, y|q) \frac{(zt)^n}{(q;q)_n}\right\} \qquad (|zt| < 1, |xzt| < 1) \\
&\quad = T(D_q)\left\{\frac{(yzt;q)_\infty}{(xzt, zt;q)_\infty}\right\} \qquad (|t| < 1, |xt| < 1).
\end{aligned}$$

According to lemma 2.2, (2.7) can be expressed in the following form

$$\frac{(yt;q)_\infty}{(xzt, xt, t;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} y, xt \\ yt \end{matrix}; q, zt\right). \tag{2.8}$$

On the other hand, (2.7) also equals

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathbb{E}(D_{xy})\{P_n(x, y)\} h_n(z|q) \frac{t^n}{(q;q)_n} \\
&\quad = \mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty} P_n(x, y) h_n(z|q) \frac{t^n}{(q;q)_n}\right\} \\
&\quad = \mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty} P_n(x, y) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} z^k \frac{t^n}{(q;q)_n}\right\} \\
&\quad = \mathbb{E}(D_{xy})\left\{\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} P_n(x, q^k y) \frac{t^n}{(q;q)_n}\right) P_k(x, y) \frac{(zt)^k}{(q;q)_k}\right\} \\
&\quad = \sum_{k=0}^{\infty} \frac{(zt)^k}{(q;q)_k} \mathbb{E}(D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{P_k(x, y)}{(yt;q)_k}\right\},
\end{aligned}$$

where $|xt| < 1$. Now, we see that

$$\sum_{k=0}^{\infty} \frac{(zt)^k}{(q;q)_k} \mathbb{E}(D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{P_k(x, y)}{(yt;q)_k}\right\} = \frac{(yt;q)_\infty}{(xzt, xt, t;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} y, xt \\ yt \end{matrix}; q, zt\right).$$

Employing Euler's identity (1.3) for $1/(xzt;q)_\infty$ and expanding the ${}_2\phi_1$ summation on the right-hand side of the above identity, we obtain

$$\sum_{k=0}^{\infty} \frac{(zt)^k}{(q;q)_k} \mathbb{E}(D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{P_k(x, y)}{(yt;q)_k}\right\} = \frac{(yt;q)_\infty}{(t, xt;q)_\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(y, xt;q)_n}{(q, yt;q)_n} \frac{z^{n+k} t^{n+k} x^k}{(q;q)_k}.$$

Equating the coefficients of z^n, the desired identity follows. □

We are now ready to present the proof of theorem 2.1.

Proof. From (2.6) it follows that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n}\\
&\qquad= \mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty} P_n(x,y)h_n(u,v|q)\frac{t^n}{(q;q)_n}\right\}\\
&\qquad= \mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}P_k(u,v)\right\}\\
&\qquad= \mathbb{E}(D_{xy})\left\{\sum_{k=0}^{\infty} P_k(u,v)P_k(x,y)\frac{t^k}{(q;q)_k}\left(\sum_{n=0}^{\infty} P_n(x,q^k y)\frac{t^n}{(q;q)_n}\right)\right\} \qquad (|xt|<1)\\
&\qquad= \mathbb{E}(D_{xy})\left\{\sum_{k=0}^{\infty} P_k(u,v)P_k(x,y)\frac{t^k}{(q;q)_k}\frac{(q^k yt;q)_\infty}{(xt;q)_\infty}\right\}\\
&\qquad= \sum_{k=0}^{\infty} P_k(u,v)\frac{t^k}{(q;q)_k}\mathbb{E}(D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_k(x,y)}{(yt;q)_k}\right\} \qquad (|t|,|xt|<1).
\end{aligned}
$$

In view of lemma 2.3, the above summation equals

$$
\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{\infty} P_k(u,v)\frac{t^k}{(q;q)_k}\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}\frac{(y,xt;q)_j}{(yt;q)_j}x^{k-j}.
$$

Exchanging the order of summations, we get

$$
\begin{aligned}
&\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{j=0}^{\infty} P_j(u,v)\frac{(y,xt;q)_j}{(q,yt;q)_j}t^j\sum_{k=0}^{\infty}\frac{(xt)^k P_k(u,q^j v)}{(q;q)_k} \qquad (|uxt|<1)\\
&\qquad= \frac{(yt,vxt;q)_\infty}{(t,xt,uxt;q)_\infty}\sum_{j=0}^{\infty} P_j(u,v)\frac{(y,xt;q)_j}{(q,yt,vxt;q)_j}t^j\\
&\qquad= \frac{(yt,vxt;q)_\infty}{(t,xt,uxt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} y,xt,v/u\\ yt,vxt\end{matrix};q,ut\right) \qquad (|ut|<1).
\end{aligned}
$$

This completes the proof. □

3. The Rogers formula for $h_n(x, y|q)$

In this section, we obtain the Rogers formula for the bivariate Rogers–Szegö polynomials $h_n(x, y|q)$ using the operator $\mathbb{E}(D_{xy})$ and the technique of parameter augmentation [10, 11]. This Rogers formula implies a linearization formula for $h_n(x, y|q)$. We also get another Rogers-type formula for $h_n(x, y|q)$ which leads to a simpler linearization formula.

Theorem 3.1 (the Rogers formula for $h_n(x,y|q)$)**.** *We have*

$$
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} y,xs\\ ys\end{matrix};q,t\right), \tag{3.1}
$$

provided that $|t|, |s|, |xt|, |xs| < 1$.

Proof. By (2.6), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&\qquad=\mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\right\}\\
&\qquad=\mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\left(\sum_{m=0}^{\infty} P_m(x,q^n y)\frac{s^m}{(q;q)_m}\right)\right\} \qquad (|xs|<1)\\
&\qquad=\mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\frac{(q^n ys;q)_\infty}{(xs;q)_\infty}\right\}\\
&\qquad=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\mathbb{E}(D_{xy})\left\{\frac{(ys;q)_\infty P_n(x,y)}{(xs;q)_\infty (ys;q)_n}\right\} \qquad (|s|<1,\ |xs|<1).
\end{aligned}$$

Employing lemma 2.3, we find

$$\begin{aligned}
&\frac{(ys;q)_\infty}{(s,xs;q)_\infty}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(y,xs;q)_k}{(ys;q)_k}x^{n-k}\\
&\qquad=\frac{(ys;q)_\infty}{(s,xs;q)_\infty}\sum_{k=0}^{\infty}\frac{(y,xs;q)_k}{(q,ys;q)_k}t^k\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n} \qquad (|xt|<1)\\
&\qquad=\frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(y,xs;q)_k}{(q,ys;q)_k}t^k\\
&\qquad=\frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}y,xs\\ ys\end{matrix};q,t\right) \qquad (|t|<1),
\end{aligned}$$

as desired. □

Clearly, the Rogers formula (1.13) for $h_n(x|q)$ is a special case of (3.1) when $y=0$. From the above theorem and (1.5), we get the equivalent formula for $H_n(x;a|q)$:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_{n+m}(x;a|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&\qquad=\frac{(as;q)_\infty}{(s\,e^{i\theta},s\,e^{-i\theta},t\,e^{-i\theta};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}a\,e^{-i\theta},s\,e^{-i\theta}\\ as\end{matrix};q,t\,e^{i\theta}\right),
\end{aligned}$$

where $x=\cos\theta$ and $|t\,e^{i\theta}|, |s\,e^{i\theta}|, |t\,e^{-i\theta}|, |s\,e^{-i\theta}|<1$.

As in the classical case, the Rogers formula can be used to derive linearization formula. For the bivariate case, we obtain the linearization formula for $h_n(x,y|q)$ as a double summation identity.

Corollary 3.1.1. *We have*

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{l=0}^{m}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\l\end{bmatrix}(y;q)_k(y/x;q)_l x^l h_{n+m-k-l}(x,y|q)\\
&\qquad=\sum_{k=0}^{n}\sum_{l=0}^{m}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\l\end{bmatrix}(y;q)_k(y/x;q)_l(xq^k)^l h_{n-k}(x,y|q)h_{m-l}(x,y|q).
\end{aligned}$$

Proof. We rewrite theorem 3.1 in the following form:

$$\frac{(ys;q)_\infty(yt;q)_\infty}{(xs;q)_\infty(t;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\sum_{k=0}^{\infty}\frac{(y;q)_k(ysq^k;q)_\infty}{(q;q)_k(xsq^k;q)_\infty}t^k\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_n(x,y|q)h_m(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}.$$

Expanding $(ys;q)_\infty/(xs;q)_\infty$, $(yt;q)_\infty/(t;q)_\infty$, $(ysq^k;q)_\infty/(xsq^k;q)_\infty$ by the Cauchy identity (1.2), and equating the coefficients of t^ns^m, the required formula is justified. □

We note that Hou, Lascoux and Mu [16] represented the Rogers–Szegö polynomials $h_n(x|q)$ as a special case of the complete symmetric functions. By computing the Hankel forms, they obtained the Askey–Ismail formula (see [4, 11])

$$h_{m+n}(x|q)=\sum_{k=0}^{\min\{m,n\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_kq^{\binom{k}{2}}(-x)^kh_{n-k}(x|q)h_{m-k}(x|q),\tag{3.2}$$

which can be regarded as the inverse relation of the linearization formula (1.14) for the Rogers–Szegö polynomials. Applying the technique of Hou, Lascoux and Mu to the bivariate Rogers–Szegö polynomials $h_n(x,y|q)$, the following relation can be verified:

$$\sum_{k=0}^{\min\{m,n\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(-1)^k(q;q)_kq^{\binom{k}{2}}(x^kh_{n-k}(x,y|q)h_{m-k}(x,y|q)-y^kh_{n+m-k}(x,y|q))=0.\tag{3.3}$$

The details of the proof are omitted. Multiplying the above equation by

$$\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$

and summing over n and m, we get another Rogers-type formula.

Theorem 3.2. *We have*

$$\sum_{k=0}^{\infty}\frac{(-1)^ky^kq^{\binom{k}{2}}}{(q;q)_k}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}h_{n+m-k}(x,y|q)\frac{t^n}{(q;q)_{n-k}}\frac{s^m}{(q;q)_{m-k}}$$
$$=(xst;q)_\infty\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_n(x,y|q)h_m(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}.\tag{3.4}$$

Clearly, the classical Rogers formula (1.13) is a special case when $y=0$. By equating the coefficients of t^ns^m in the above theorem, we can derive a simpler linearization formula for $h_n(x,y|q)$.

Corollary 3.2.2. *For $n,m\geqslant 0$, we have*

$$h_n(x,y|q)h_m(x,y|q)$$
$$=\sum_{l=0}^{\min\{m,n\}}\sum_{k=0}^{\min\{m,n\}}\begin{bmatrix}m\\l\end{bmatrix}\begin{bmatrix}n\\l\end{bmatrix}\begin{bmatrix}m-l\\k\end{bmatrix}\begin{bmatrix}n-l\\k\end{bmatrix}$$
$$\times(q;q)_k(q;q)_l(-1)^kx^ly^kq^{\binom{k}{2}}h_{n+m-2l-k}(x,y|q).\tag{3.5}$$

The following special case of theorem 3.1 for $y=0$ will be useful to verify the relation between $h_n(x|q)$ and $h_n(x,y|q)$.

Corollary 3.2.3. *For $n, m \geqslant 0$, we have*

$$\sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (q;q)_k x^k h_{n+m-2k}(x|q)$$
$$= \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k h_{n-k}(x,y|q) \right) \left(\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} y^j h_{m-j}(x,y|q) \right). \tag{3.6}$$

Proof. Setting $y = 0$ in theorem 3.1, from the Cauchy identity (1.2) and (1.6) it follows that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$
$$= \frac{1}{(s,xs,xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(xs;q)_k}{(q;q)_k} t^k \qquad (|t|<1)$$
$$= \frac{(xst;q)_\infty}{(ys,yt;q)_\infty} \frac{(yt;q)_\infty}{(t,xt;q)_\infty} \frac{(ys;q)_\infty}{(s,xs;q)_\infty} \qquad (|t|, |s|, |xt|, |xs| < 1)$$
$$= \frac{(xst;q)_\infty}{(ys,yt;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_n(x,y|q) h_m(x,y|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}, \tag{3.7}$$

which can be rewritten as

$$\frac{1}{(xst;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$
$$= \frac{1}{(yt,ys;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_n(x,y|q) h_m(x,y|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m},$$

where $|t|, |s|, |xt|, |xs| < 1$.

Assuming that $|xst|, |yt|, |ys| < 1$, we can expand $1/(xst;q)_\infty, 1/(yt;q)_\infty$ and $1/(ys;q)_\infty$ by Euler's identity (1.3). Equating coefficients of $t^n s^m$ gives (3.6). Since $|t|, |s|, |xt|, |xs| < 1$ and $|xst|, |yt|, |ys| < 1$, we see that $|x|$ and $|y|$ must be finite. This completes the proof. □

When $y = 0$, both (3.5) and (3.6) reduce to the well-known linearization formula (1.14). Setting $m = 0$ in (3.6), we are led to the following relation between $h_n(x|q)$ and $h_n(x,y|q)$:

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k h_{n-k}(x,y|q), \tag{3.8}$$

which is a special case of a relation of Askey–Wilson [6, equation (6.4)]. The inverse relation of (3.8) is as follows:

$$h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k h_{n-k}(x|q). \tag{3.9}$$

Note that (3.8) and (3.9) are equivalent to relations (1.10) and (1.9) between $H_n(x|q)$ and $H_n(x;a|q)$.

In fact, we can go one step further from (3.7). Reformulating (3.7) by multiplying $(ys,yt;q)_\infty$ on both sides and expanding $(ys;q)_\infty, (yt;q)_\infty$ and $(xst;q)_\infty$ using Euler's formula (1.4), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{\binom{j}{2}+\binom{k}{2}}(-y)^{j+k}}{(q;q)_j(q;q)_k}h_{n+m}(x|q)\frac{t^{n+j}}{(q,q)_n}\frac{s^{m+k}}{(q,q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}(-x)^k}{(q;q)_k}h_n(x,y|q)h_m(x,y|q)\frac{t^{n+k}}{(q,q)_n}\frac{s^{m+k}}{(q,q)_m}.$$

Comparing the coefficients of $t^n s^m$, we reach the following identity:

$$\sum_{j=0}^{n}\sum_{k=0}^{m}\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}q^{\binom{j}{2}+\binom{k}{2}}(-y)^{j+k}h_{n+m-(j+k)}(x|q)$$
$$=\sum_{k=0}^{\min\{n,m\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k q^{\binom{k}{2}}(-x)^k h_{n-k}(x,y|q)h_{m-k}(x,y|q). \tag{3.10}$$

Setting $y=0$ in the above identity, we are led to the Askey–Ismail formula (3.2).

4. A change of base formula for $H_n(x;a|q)$

In this section, we give an extension of the q-Hermite change of base formula to the continuous big q-Hermite polynomials. The corresponding statement for $h_n(x,y|q)$ is omitted because we find that it is more convenient to work with $H_n(x;a|q)$ for this purpose. This formula can be used to evaluate certain integrals.

In [19, p 7], Ismail and Stanton gave the following q-Hermite change of base formula for $H_n(x|p)$:

$$H_n(x|p)=\sum_{j=0}^{n/2}c_{n,n-2j}(p,q)H_{n-2j}(x|q), \tag{4.1}$$

where $x=\cos\theta$ and

$$c_{n,n-2k}(p,q)=\sum_{j=0}^{k}(-1)^j p^{k-j}q^{\binom{j+1}{2}}\begin{bmatrix}n-2k+j\\j\end{bmatrix}_q\left(\begin{bmatrix}n\\k-j\end{bmatrix}_p-p^{n-2k+2j+1}\begin{bmatrix}n\\k-j-1\end{bmatrix}_p\right).$$

From (1.9), (1.10) along with the above relation, we obtain *a change of base formula for* $H_n(x;a|q)$:

$$H_n(x;a|p)=\sum_{j=0}^{n}d_{n,j,l,m}(p,q)H_{n-j-2l-m}(x;a|q), \tag{4.2}$$

where $x=\cos\theta$ and

$$d_{n,j,l,m}(p,q)=\begin{bmatrix}n\\j\end{bmatrix}_p(-1)^j p^{\binom{j}{2}}a^j\sum_{l=0}^{(n-j)/2}\sum_{m=0}^{n-j-2l}\begin{bmatrix}n-j-2l\\m\end{bmatrix}_q c_{n-j,n-j-2l}(p,q)a^m.$$

Based on the orthogonality relation of q-Hermite polynomials $H_n(x|q)$

$$\frac{(q;q)_\infty}{2\pi}\int_0^\pi H_m(x|q)H_n(x|q)(\mathrm{e}^{2\mathrm{i}\theta},\mathrm{e}^{-2\mathrm{i}\theta};q)_\infty\,\mathrm{d}\theta=(q;q)_n\delta_{mn},$$

Ismail and Stanton [19] found two generating functions for the q-Hermite polynomials:

$$\sum_{n=0}^{\infty}\frac{H_{2n}(x|q)}{(q^2;q^2)_n}t^n=\frac{(-t;q)_\infty}{(t\,\mathrm{e}^{2\mathrm{i}\theta},t\,\mathrm{e}^{-2\mathrm{i}\theta};q^2)_\infty}, \tag{4.3}$$

$$\sum_{n=0}^{\infty} \frac{H_n(x|q^2)}{(q;q)_n} t^n = \frac{(qt^2;q^2)_\infty}{(t\,\mathrm{e}^{\mathrm{i}\theta}, t\,\mathrm{e}^{-\mathrm{i}\theta};q)_\infty}, \tag{4.4}$$

where $x = \cos\theta$.

Similarly, from the orthogonality relation [23] of $H_n(x;a|q)$

$$\frac{(q;q)_\infty}{2\pi} \int_0^\pi H_n(x;a|q) H_m(x;a|q) \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta};q)_\infty}{(a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta};q)_\infty} \,\mathrm{d}\theta = (q;q)_n \delta_{mn}, \tag{4.5}$$

it follows the identities

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n/2} \frac{q^{\binom{n-2k}{2}} a^{n-2k} t^{n-k}}{(q^2;q^2)_k (q;q)_{n-2k}} \right) H_n(x;a|q) = \frac{(a^2t;q^2)_\infty (-t;q)_\infty}{(t\,\mathrm{e}^{2\mathrm{i}\theta}, t\,\mathrm{e}^{-2\mathrm{i}\theta};q^2)_\infty}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k q^{k^2} a^k t^{n+k}}{(q^2;q^2)_k (q;q)_{n-k}} \right) H_n(x;a|q^2) = \frac{(at;q)_\infty (qt^2;q^2)_\infty}{(t\,\mathrm{e}^{\mathrm{i}\theta}, t\,\mathrm{e}^{-\mathrm{i}\theta};q)_\infty}, \tag{4.7}$$

where $x = \cos\theta$.

Clearly, (4.3) and (4.4) are the special cases when $a = 0$. Recall that the generating function [23] of $H_n(x;a|q)$ is

$$\sum_{n=0}^{m} H_n(x;a|q) \frac{t^n}{(q;q)_n} = \frac{(at;q)_\infty}{(t\,\mathrm{e}^{\mathrm{i}\theta}, t\,\mathrm{e}^{-\mathrm{i}\theta};q)_\infty}, \qquad x = \cos\theta. \tag{4.8}$$

Setting $q \to p$ and multiplying by the weight function $(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta};q)_\infty/(a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta};q)_\infty$, the integrals of the generating functions (4.6–4.8) on base p can be stated as follows:

$$J_{p,q}(a,t) = \frac{(q;q)_\infty (a^2t;p^2)_\infty (-t;p)_\infty}{2\pi} \int_0^\pi \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta};q)_\infty}{(a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta};q)_\infty (t\,\mathrm{e}^{2\mathrm{i}\theta}, t\,\mathrm{e}^{-2\mathrm{i}\theta};p^2)_\infty} \,\mathrm{d}\theta,$$

$$H_{p,q}(a,t) = \frac{(q^2;q^2)_\infty (at;p)_\infty (pt^2;p^2)_\infty}{2\pi} \int_0^\pi \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta};q^2)_\infty}{(a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta};q^2)_\infty (t\,\mathrm{e}^{\mathrm{i}\theta}, t\,\mathrm{e}^{-\mathrm{i}\theta};p)_\infty} \,\mathrm{d}\theta,$$

$$I_{p,q}(a,t) = \frac{(q;q)_\infty (at;p)_\infty}{2\pi} \int_0^\pi \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta};q)_\infty}{(a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta};q)_\infty (t\,\mathrm{e}^{\mathrm{i}\theta}, t\,\mathrm{e}^{-\mathrm{i}\theta};p)_\infty} \,\mathrm{d}\theta.$$

When $a = 0$, they reduce to the integrals $J_{p,q}(t)$, $H_{p,q}(t)$ and $I_{p,q}(t)$ introduced by Ismail and Stanton [19].

We conclude with the observation that for some special values of p and q the above integrals can be computed by the Askey–Wilson integral [13, p 154]:

$$\begin{aligned} &\frac{(q;q)_\infty}{2\pi} \int_0^\pi \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta};q)_\infty}{(a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta}, b\,\mathrm{e}^{\mathrm{i}\theta}, b\,\mathrm{e}^{-\mathrm{i}\theta}, c\,\mathrm{e}^{\mathrm{i}\theta}, c\,\mathrm{e}^{-\mathrm{i}\theta}, d\,\mathrm{e}^{\mathrm{i}\theta}, d\,\mathrm{e}^{-\mathrm{i}\theta};q)_\infty} \,\mathrm{d}\theta \\ &\qquad = \frac{(abcd;q)_\infty}{(ab,ac,ad,bc,bd,cd;q)_\infty}. \end{aligned} \tag{4.9}$$

Because of the change of base formula (4.2) and the orthogonality relation (4.5), we may transform the above integrals into summations:

$$J_{p,q}(a,t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n/2} \frac{p^{\binom{n-2k}{2}} a^{n-2k} t^{n-k}}{(p^2;p^2)_k (p;p)_{n-2k}} \sum_{j=0}^{n} d_{n,j,l,n-j-2l}(p,q),$$

$$H_{p,q}(a,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k p^{k^2} a^k t^{n+k}}{(p^2;p^2)_k (p;p)_{n-k}} \sum_{j=0}^{n} d_{n,j,l,n-j-2l}(p^2,q),$$

$$I_{p,q}(a,t) = \sum_{n=0}^{\infty} \frac{t^n}{(p;p)_n} \sum_{j=0}^{n} d_{n,j,l,n-j-2l}(p,q).$$

Using special cases of the Askey–Wilson integral (4.9), we give some examples of $H_{p,q}(a,t)$ which have closed product formulae:

$$H_{q,q}(a,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k q^{k^2} a^k t^{n+k}}{(q^2;q^2)_k (q;q)_{n-k}} \sum_{j=0}^{n} d_{n,j,l,n-j-2l}(q^2,q) = 1,$$

$$H_{-q,q}(a,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k (-q)^{k^2} a^k t^{n+k}}{(q^2;q^2)_k (-q;-q)_{n-k}} \sum_{j=0}^{n} d_{n,j,l,n-j-2l}(q^2,q) = 1,$$

$$H_{q^2,q}(a,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k q^{2k^2} a^k t^{n+k}}{(q^4;q^4)_k (q^2;q^2)_{n-k}} \sum_{j=0}^{n} d_{n,j,l,n-j-2l}(q^4,q) = (q^2t^2;q^4)_\infty,$$

$$H_{q^2,q^3}(a,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k q^{2k^2} a^k t^{n+k}}{(q^4;q^4)_k (q^2;q^2)_{n-k}} \sum_{j=0}^{n} d_{n,j,l,n-j-2l}(q^4,q^3) = \frac{(at^3q^6;q^6)_\infty}{(t^2q^4;q^4)_\infty}.$$

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