Applications of the Operator ${}_r\Phi_s$ in q-identities

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Abstract

In this paper, we set up the general operator ${}_r\Phi_s$, and then we find some of its operator identities that will be used to generalize some well-known q-identities, such as Cauchy identity, Heine's transformation formula and the q-Pfaff-Saalschütz summation formula. By giving special values to the parameters in the obtained identities, some new results are achieved and/or others are recovered.

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1. Introduction

We adopt the following notations and terminology in [8]. We assume that $0 < q < 1$. The q-shifted factorial is given by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

and the multiple q-shifted factorials is given by

$$(a_1,a_2,\dots,a_r;q)_m = (a_1;q)_m(a_2;q)_m\cdots(a_r;q)_m.$$

where $m \in Z \; or \; \infty$.

The basic hypergeometric series ${}_r\phi_s$ is defined as follows [8]:

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, x\right) = \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_r;q)_k}{(q,b_1,\dots,b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} x^k,$$

where $r, s \in \mathbb{N}$; $a_1,\dots,a_r,b_1,\dots,b_s \in \mathbb{C}$; and none of the denominator factors evaluate to zero. The above series is absolutely convergent for all $x \in \mathbb{C}$ if $r < s+1$, for $|x| < 1$ if $r = s+1$ and for $x = 0$ if $r > s+1$.

The q-binomial coefficient is presented as follows [8]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & if \;\; 0 \le k \le n; \\ 0, & otherwise, \end{cases}$$

where n, k are nonnegative integers.

In this paper, we will repeatedly use the following equations [8]:

$$(b;q)_{-k} = \frac{(-1)^k q^{\binom{k}{2}}(q/b)^k}{(q/b;q)_k} \quad . \tag{1.1}$$

$$(b;q)_{n-k} = \frac{(b;q)_n}{(q^{1-n}/b;q)_k}\;(-1)^k q^{\binom{k}{2}-nk}\;\left(\frac{q}{b}\right)^k \quad . \tag{1.2}$$

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}\;(-1)^k\; q^{\binom{k}{2}-nk} \quad . \tag{1.3}$$

$$(bq^{-n};q)_\infty = (-1)^n\; b^n\; q^{-\binom{n+1}{2}}\;(q/b;q)_n\;(b;q)_\infty \quad . \tag{1.4}$$

The Cauchy identity is given by:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\;x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{1.5}$$

The special case of the Cauchy identity (1.5), given by Euler, is [8]

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} x^n = (x;q)_\infty. \tag{1.6}$$

q-Chu-Vandermonde's identities are [8]

$${}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, cq^n/b\right) = \frac{(c/b;q)_n}{(c;q)_n}, \qquad |c/b| < 1. \tag{1.7}$$

$${}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q\right) = \frac{(c/b;q)_n}{(c;q)_n}\, b^n. \tag{1.8}$$

The q-Pfaff-Saalschütz sum is given by [8]

$${}_3\phi_2\left(\begin{matrix} q^{-n}, a, b \\ c, q^{1-n}ab/c \end{matrix}; q, q\right) = \frac{(c/a, c/b;q)_n}{(c, c/ab;q)_n}. \tag{1.9}$$

The q-Gauss summation formula is given by [8]

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, c/ab\right) = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}, \quad \left|\frac{c}{ab}\right| < 1. \tag{1.10}$$

Heine's transformation formula is given by [8]

$${}_2\phi_1\left(\begin{matrix} a, b \\ c, \end{matrix}; q, z\right) = \frac{(c/b, zb;q)_\infty}{(c, z;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} abz/c, b \\ zb \end{matrix}; q, \frac{c}{b}\right), \tag{1.11}$$

where $\max\{|x|, |c/b|\} < 1$.

The transformation formula [8, Appendix III, equation (III.9)] is given by:

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, de/abc\right) = \frac{(e/a, de/bc;q)_\infty}{(e, de/abc;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, \mathrm{e/a}\right). \tag{1.12}$$

Definition 1.1 ([2], [3], [10]). *The D_q operator or the q-derivative is defined as follows*:

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}. \tag{1.13}$$

Theorem 1.2 ([2], [10]). *For $n \geq 0$, we have*

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(a)\} D_q^{n-k}\{g(aq^k)\}. \tag{1.14}$$

Theorem 1.3 ([2], [16]). *Let D_q be defined as in* (1.13), *then*

$$D_q^k\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\} = t^k (v/t;q)_k \frac{(avq^k;q)_\infty}{(at;q)_\infty}, \quad |at| < 1. \tag{1.15}$$

In 2010, Fang [5] defined the finite operator as follows:

Definition 1.4 [5]. *The q-exponential operator* ${}_1\Phi_0\left(\begin{matrix} q^{-M} \\ - \end{matrix}; q, cD_q\right)$ *is defined by:*

$${}_1\Phi_0\left(\begin{matrix} q^{-M} \\ - \end{matrix}; q, cD_q\right) = \sum_{k=0}^{M} \frac{(q^{-M};q)_k}{(q;q)_k}(cD_q)^k. \tag{1.16}$$

Fang used the q-exponential operator ${}_1\Phi_0\left(\begin{matrix} q^{-M} \\ - \end{matrix}; q, cD_q\right)$ to prove the following result:

Theorem 1.5 [5]. *Let* ${}_1\Phi_0\left(\begin{matrix} q^{-M} \\ - \end{matrix}; q, cD_q\right)$ *be defined as in* (1.16), *then*

$${}_3\phi_2\left(\begin{matrix} q^{-M}, \frac{c_1}{d_2}, xd_1 \\ cd_1q^{-M}, xc_1 \end{matrix}; q, cd_2\right)$$

$$= \frac{(cd_2,q)_M}{(cd_1,q)_M}\left(\frac{d_2}{d_1}\right)^M {}_3\phi_2\left(\begin{matrix} q^{-M}, \frac{c_1}{d_1}, xd_2 \\ cd_2q^{-M}, xc_1 \end{matrix}; q, cd_1\right).$$

In 2010, Zhang and Yang [15] constructed the finite q-Exponential Operator ${}_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, cD_q\right]$ with two parameters as follows:

Definition 1.6 [15]. *The finite q-Exponential Operator* ${}_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, cD_q\right]$ *is defined by*

$${}_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, cD_q\right] = \sum_{n=0}^{N} \frac{(q^{-N},w;q)_n}{(q,v;q)_n}(cD_q)^n. \tag{1.18}$$

Zhang and Yang used the operator ${}_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, cD_q\right]$ to get a generalization of q-Chu-Vandermond formula (1.8) as follows:

Theorem 1.7 [15]. *Let* ${}_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, cD_q\right]$ *be defined as in* (1.18), *then*

$$\sum_{m=0}^{n}\sum_{k=0}^{N} \frac{(q^{-n},a;q)_m}{(q,c;q)_m} \frac{(q^{-N},w;q)_k}{(q,v;q)_k} c^k q^{m+mk}$$

$$= a^n w^N \frac{(c/a;q)_n}{(c;q)_n} \frac{(v/w;q)_N}{(v;q)_N} {}_4\phi_2\left(\begin{matrix} q^{-N}, w, \frac{q^{1-n}}{c}, \frac{aq}{c} \\ \frac{aq^{1-n}}{c}, \frac{wq^{1-N}}{v} \end{matrix}; q, \frac{c}{v}\right). \tag{1.19}$$

Also, by using the operator ${}_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, cD_q\right]$, they obtained the following result:

$$ {}_2\phi_1\left(\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, c\right) = w^N \frac{(v/w;q)_N}{(v;q)_N} \ {}_3\phi_1\left(\begin{matrix} q^{-N}, w, \frac{q}{c} \\ \frac{wq^{1-N}}{v} \end{matrix}; q, \frac{c}{v}\right) \tag{1.20} $$

In 2016, Li-Tan [9] constructed the generalized q-exponential operator $\mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix} | q; cD_q\right]$ with three parameters as follows:

Definition 1.8 [9]. *The generalized q-exponential operator $\mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix} | q; cD_q\right]$ is defined by*

$$ \mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix} | q; cD_q\right] = \sum_{n=0}^{\infty} \frac{(u,v;q)_n}{(q,w;q)_n} (cD_q)^n. \tag{1.21} $$

Li and Tan used the generalized q-exponential operator $\mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix} | q; cD_q\right]$ to get a generalization for q-Chu-Vandermonde sum (1.8), as follows:

Theorem 1.9 [9]. *Let $\mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix} | q; cD_q\right]$ be defined as in* (1.21), *then*

$$ \sum_{k=0}^{n} \frac{(q^{-n},x;q)_k}{(q,c;q)_k} q^k \ {}_2\phi_1\left[\begin{matrix} u, v \\ w \end{matrix}; q, tq^k\right] $$
$$ = x^n \frac{(c/x;q)_n}{(c;q)_n} \sum_{i,k\geq 0} \frac{(u,v;q)_{i+k}}{(q;q)_i (w;q)_{i+k}} \frac{(q^{1-n}/c, qx/c;q)_k}{(q,q^{1-n}x/c;q)_k} t^{i+k} \left({}^{q}/{}_{c}\right)^i. \tag{1.22} $$

The Cauchy polynomials $P_n(x,y)$ is defined by [7]

$$ P_n(x,y) = \begin{cases} (x-y)(x-qy)(x-q^2y)\cdots(x-q^{n-1}y), & if \ \ n > 0; \\ 1, & if \ \ n = 0. \end{cases} \tag{1.23} $$

In 1983, Goulden and Jackson [7] gave the following identity:

$$ P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k x^{n-k}. $$

The generating function for Cauchy polynomials $P_n(x,y)$ [1] is

$$ \sum_{k=0}^{\infty} P_n(x,y) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt| < 1. \tag{1.24} $$

In 2003, Chen et al [1] introduced the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ as:

$$ h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,y), $$

where $P_k(x,y)$ is defined as in (1.23). In 2010, Saad and Sukhi [11] gave another formula for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ as:

$$ h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (y;q)_k \, x^{n-k}. $$

The generating function for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ is [1]

$$\sum_{k=0}^{\infty} h_n(x,y|q)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}, \quad \max\{|t|,|xt|\}<1. \tag{1.25}$$

The generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(a,b)}(x,y)$ was introduced in 2020 by Srivastava and Arjika [14] as

$$\phi_n^{(a,b)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(a_1,a_2,\dots,a_{s+1};q)_k}{(b_1,b_2,\dots,b_s;q)_k}\, x^k\ y^{n-k},$$

which has the following generating function:

$$\sum_{n=0}^{\infty}\phi_n^{(a,b)}(x,y)\ \frac{t^n}{(q;q)_n}=\ \frac{1}{(yt;q)_\infty}\ {}_{s+1}\phi_s\left(\begin{matrix} a_1,a_2,\dots,a_{s+1} \\ b_1,b_2,\dots,b_s \end{matrix};q,xt\right), \tag{1.26}$$

where $\max\{|xt|,|yt|\}<1$.

The paper is organized as follows. In section 2, we built the general operator ${}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_q\right)$. We also provide some operator identities, which will be used in section 3. In section 3, we generalize some well-known q-identities, such as Cauchy identity, Heine's transformation formula and the q-Pfaff-Saalschütz summation formula. Then, in these generalizations, we may assign the parameters unique values, we get several results.

2. The General Operator ${}_r\Phi_s$ and its Identities

In this section, we establish the general operator ${}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_q\right)$. We also give some identities to this operator, which will be used in the next section.

Definition 2.1 *We define the generalized q-operator ${}_r\Phi_s$ as follows*:

$${}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_q\right)=\sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\ \left(cD_q\right)^n, \tag{2.1}$$

where $W_n=\frac{(a_1,\cdots,a_r;q)_n}{(b_1,\dots,b_s;q)_n}$.

Some special values may be given to the general q-operator ${}_r\Phi_s$ to obtain several previously specified operators, as follows:

- Setting $r=1,\ s=0,\ a_1=0$ and $c=b$, we get on the exponential operator $T(bD_q)$ defined by Chen and Liu [2] in 1997.
- If $r=1, s=0$ and $a_1=b$, we get on the Cauchy operator ${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix};q,cD_q\right)$ which was defined by Fang[4] in 2008.

• If $r = 1, s = 0$ and $a_1 = q^{-M}$, we get on the finite operator ${}_1\Phi_0\left(\begin{matrix} q^{-M} \\ - \end{matrix}; q, cD_q\right)$ described by Fang[5] in 2010.

• If $r = 2, s = 1$,$a_1 = q^-N, a_2 = w$ and $b_1 = v$, we get on the finite exponential operator ${}_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, cD_q\right]$ with two parameters specified by Zhang and Yang[15] in 2010.

• If $r = s = 0$, we get on the q-exponential operator $R(bDq)$ which is defined by Saad and Sukhi [12] in 2013.

• Setting $r = s + 1$, we get the generalized q-operator $F(a_0, \dots, a_s; b_1, \dots, b_s; cD_q)$ described by Fang [6] in 2014 and the homogeneous q-difference operator $\mathbb{T}(a, b, cD_q)$ specified by Srivastava and Arjika [14] in 2020.

• If $r = 2, s = 1$,$a_1 = u, a_2 = v$ and $b_1 = w$, we get on the generalized exponential operator $\mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix}|q; cD_q\right]$ with three parameters constructed by Li and Tan [9] in 2016.

• Setting $r = 3,\ s = 2,\ a_1 = a,\ a_2 = b,\ a_3 = c,\ b_1 = d,\ b_2 = e$ and $c = f$, we get the operator $\phi\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, fD_q\right)$ with five parameters defined by Saad and Jaber [13] in 2020.

The following operator identities will be derived using q-Leibniz formula (1.14):

Theorem 2.2 *Let* ${}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q\right)$ *be defined as in* (2.1), *then*

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_q\right)\left\{\frac{(av, au; q)_\infty}{(at, aw; q)_\infty}\right\} = \frac{(av, au; q)_\infty}{(at, aw; q)_\infty} $$
$$ \times \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{W_{n+k}}{(q;q)_n}\frac{(v/t, aw; q)_k}{(q, av; q)_k}\frac{(u/w; q)_n}{(au; q)_{n+k}}\left[(-1)^{n+k} q^{\binom{n+k}{2}}\right]^{1+s-r} (cw)^n\,(ct)^k, \qquad (2.2) $$

provided that $\max\{|at|, |aw|\} < 1$.

Proof.

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_q\right)\left\{\frac{(av, au; q)_\infty}{(at, aw; q)_\infty}\right\} $$
$$ = \sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} c^n D_q^n\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\frac{(au;q)_\infty}{(aw;q)_\infty}\right\} \qquad (by\ \ using\ \ (2.1)) $$
$$ = \sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} c^n $$

$$\times \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk} D_q^k \left\{ \frac{(av;q)_\infty}{(at;q)_\infty} \right\} D_q^{n-k} \left\{ \frac{(au;q)_\infty}{(aw;q)_\infty} \right\} \qquad (by\ using\ (1.14))$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{W_n}{(q;q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} c^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk}$$

$$\times t^k \frac{(v/t;q)_k (avq^k;q)_\infty}{(at;q)_\infty} (wq^k)^{n-k} \frac{(u/w;q)_{n-k} (auq^n;q)_\infty}{(awq^k;q)_\infty} \qquad (by\ using\ (1.15)) \quad (2.3)$$

$$= \frac{(av,au;q)_\infty}{(at,aw;q)_\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{W_{n+k}}{(q;q)_n} \frac{(v/t,aw;q)_k}{(q,av;q)_k} \frac{(u/w;q)_n}{(au;q)_{n+k}} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} (cw)^n (ct)^k.$$

Setting $u = 0$ in equation (2.2), we get the following corollary:

Corollary 2.2.1 *Let* ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_q \right)$ *be defined as in* (2.1), *then*

$${}_r\Phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, cD_q \right) \left\{ \frac{(av;q)_\infty}{(at,aw;q)_\infty} \right\} = \frac{(av;q)_\infty}{(at,aw;q)_\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{W_{n+k}}{(q;q)_n} \frac{(v/t,aw;q)_k}{(q,av;q)_k}$$

$$\times \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} (cw)^n (ct)^k, \qquad (2.4)$$

where $\max\{|at|, |aw|\} < 1$.

In view of symmetry of t and w on the left hand side of equation (2.4), we get the following formula:

Theorem 2.3

$$\sum_{n,k \geq 0} \frac{W_{n+k}}{(q;q)_n} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} \frac{(v/t,aw;q)_k}{(q,av;q)_k} (cw)^n (ct)^k$$

$$= \sum_{n,k \geq 0} \frac{W_{n+k}}{(q;q)_n} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} \frac{(v/w,at;q)_k}{(q,av;q)_k} (ct)^n (cw)^k \ . \qquad (2.5)$$

- If $r = 1, s = 0$ in equation (2.5) and then using (1.5), we get Hall's transformation (1.12).

- If $r = 1, s = 0$ and $a_1 = q^{-N}$ in equation (2.5), then using equations (1.4) and (1.5), we get Theorem 3.5. obtained by Fang [5] (equation (1.17)).

Theorem 2.4 *Let* ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_q \right)$ *be defined as in* (2.1), *then*

$${}_r\Phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, cD_q \right) \left\{ a^n \frac{(ax;q)_\infty}{(ay;q)_\infty} \right\} = a^n \frac{(ax;q)_\infty}{(ay;q)_\infty}$$

$$\times \sum_{i,j=0}^{\infty} \frac{W_{i+j}}{(q;q)_i} \frac{(x/y;q)_i}{(axq^j;q)_i} \frac{(ay;q)_j}{(ax;q)_j} \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} \begin{bmatrix} n \\ j \end{bmatrix} (cy)^i (\frac{c}{a})^j, \quad |ay| < 1. \quad (2.6)$$

Proof.

$$
\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_q\right)\left\{a^n \frac{(ax;q)_\infty}{(ay;q)_\infty}\right\} \\
&= \sum_{i=0}^{\infty} \frac{W_i}{(q;q)_i} \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} c^i \; D_q^i \left\{a^n \frac{(ax;q)_\infty}{(ay;q)_\infty}\right\} \qquad (by\ using\ (2.1)) \\
&= \sum_{i=0}^{\infty} \frac{W_i}{(q;q)_i} \times \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} c^i \\
&\quad \times \sum_{j=0}^{i} q^{j^2-ij} \begin{bmatrix} i \\ j \end{bmatrix} D_q^j \; a^n D_q^{i-j} \left\{\frac{(axq^j;q)_\infty}{(ayq^j;q)_\infty}\right\} \qquad (by\ using\ (1.14)) \\
&= \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} \frac{W_i \; c^i}{(q;q)_{i-j}} \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{j^2-ij} \begin{bmatrix} n \\ j \end{bmatrix} \; a^{n-j} D_q^{i-j} \left\{\frac{(axq^j;q)_\infty}{(ayq^j;q)_\infty}\right\} \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{W_{i+j} \; c^{i+j}}{(q;q)_i} \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} q^{-ij} \begin{bmatrix} n \\ j \end{bmatrix} a^{n-j} \\
&\quad \times D_q^i \left\{\frac{(axq^j;q)_\infty}{(ayq^j;q)_\infty}\right\} \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{W_{i+j} \; c^{i+j}}{(q;q)_i} \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} \begin{bmatrix} n \\ j \end{bmatrix} a^{n-j} \; y^i \left\{\frac{(x/y;q)_i \; (axq^{i+j};q)_\infty}{(ayq^j;q)_\infty}\right\} \\
&= a^n \frac{(ax;q)_\infty}{(ay;q)_\infty} \sum_{i,j=0}^{\infty} \frac{W_{i+j}}{(q;q)_i} \frac{(x/y;q)_i}{(axq^j;q)_i} \frac{(ay;q)_j}{(ax;q)_j} \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} \begin{bmatrix} n \\ j \end{bmatrix} (cy)^i (\frac{c}{a})^j
\end{aligned}
$$

Setting $x = 0$ in equation (2.6), we get the following corollary:

Corollary 2 *Let* ${}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q\right)$ *be defined as in* (2.1), *then*

$$
\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_q\right)\left\{\frac{a^n}{(ay;q)_\infty}\right\} \\
&= \frac{a^n}{(ay;q)_\infty} \sum_{i,j\geq 0}^{\infty} \frac{W_{i+j}}{(q;q)_i} \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} (cy)^i \; (ay;q)_j \begin{bmatrix} n \\ j \end{bmatrix} (\frac{c}{a})^j, \quad |ay| < 1. \quad (2.7)
\end{aligned}
$$

3. Applications in q-Identities

In this section, we aim to generalize some well-known q-identities such as Cauchy identity, Heine's transformation of $_2\phi_1$ series and q-Pfaff-Saalschütz sum by using the general operator $_r\Phi_s$. Then, some special results are obtained from these generalizations, some new ones and others are known.

3.1 Generalization of Cauchy Identity

Theorem 3.1 (Generalization of Cauchy identity). *Let Cauchy identity be defined as in* (1.5), *then*

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\;x^k\sum_{i,j\geq 0}\frac{W_{i+j}}{(q;q)_i}\frac{(b/c;q)_i}{(xb;q)_{i+j}}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}(xc;q)_j\begin{bmatrix}k\\j\end{bmatrix}(dc)^i(\frac{d}{x})^j$$
$$=\frac{(xa;q)_\infty}{(x;q)_\infty}\sum_{i,j\geq 0}\frac{W_{i+j}}{(q;q)_i}\frac{(b/c;q)_i}{(xb;q)_{i+j}}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}\frac{(a,xc;q)_j}{(q,xa;q)_j}(dc)^i d^j\;. \qquad (3.1)$$

Proof. Multiply Cauchy identity .خطأ! لم يتم العثور على مصدر المرجع by $\frac{(xb;q)_\infty}{(xc;q)_\infty}$,

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\;x^k\frac{(xb;q)_\infty}{(xc;q)_\infty}=\frac{(ax,xb;q)_\infty}{(x,xc;q)_\infty}. \qquad (3.2)$$

Applying the operator $_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,dD_q\right)$ on both sides of (3.2), we get

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\;{}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,dD_q\right)\left\{x^k\frac{(xb;q)_\infty}{(xc;q)_\infty}\right\}$$
$$=\;{}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,dD_q\right)\left\{\frac{(ax,xb;q)_\infty}{(x,xc;q)_\infty}\right\}\;. \qquad (3.3)$$

By using (2.4), we get

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\;{}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,dD_q\right)\left\{x^k\frac{(xb;q)_\infty}{(xc;q)_\infty}\right\}$$
$$=x^k\;\frac{(xb;q)_\infty}{(xc;q)_\infty}\sum_{i,j\geq 0}\frac{W_{i+j}}{(q;q)_i}\frac{(b/c;q)_i}{(xb;q)_{i+j}}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}(xc;q)_j\begin{bmatrix}k\\j\end{bmatrix}(dc)^i(\frac{d}{x})^j \qquad (3.4)$$

and using (2.2), we get

$${}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,dD_q\right)\left\{\frac{(ax,xb;q)_\infty}{(x,xc;q)_\infty}\right\}$$
$$=\frac{(xa;q)_\infty}{(x;q)_\infty}\sum_{i,j\geq 0}\frac{W_{i+j}}{(q;q)_i}\frac{(b/c;q)_i}{(xb;q)_{i+j}}\left[(-1)^{i+j}q^{\binom{i+j}{2}}\right]^{1+s-r}\frac{(a,xc;q)_j}{(q,xa;q)_j}(dc)^i d^j\;. \qquad (3.5)$$

Substituting (3.4) and (3.5) into (3.3) the proof completed.

- If $d=0$ in equation (3.1), we obtain Cauchy identity.

• If $b=0$ and then $c=0$ in equation (3.1), we obtain the following formula:

Corollary 3.1.3

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\sum_{j=0}^{\infty}W_j\begin{bmatrix}k\\j\end{bmatrix}\left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r} d^j x^{k-j}$$
$$=\frac{(xa;q)_\infty}{(x;q)_\infty}\sum_{j=0}^{\infty}\frac{W_j}{(q;q)_j}\frac{(a;q)_j}{(xa;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r} d^j \ . \qquad (3.6)$$

• If $r=s=0,\ a=0,\ x\to xt$ and $d\to yt$ in equation (3.6), we get the generating function for Cauchy polynomials $P_k(x,y)$ (1.26).

• If $r=1,\ s=0$ and $a=0$ then replacing $x,\ a_1,\ d$ by $xt,\ y,\ t$ respectively, in equation (3.6), we get on the generating function for bivariate Rogers-Szegö polynomials $h_k(x,y|q)$ (1.25).

• If $r=s+1,\ a=0,\ x\to yt$ and then $d\to xt$ in equation (3.6), we get the generating function for the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(a,b)}(x,y)$ (1.26).

3.2 Generalization of Heine's Transformation of ${}_2\phi_1$ Series

Theorem 3.2 (Generalization of Heine's transformation of ${}_2\phi_1$ series). *Let Heine's identity be defined as in* (1.11), *then*

$$\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}z^k\sum_{n,i\ge 0}\frac{W_{n+i}}{(q;q)_n}\begin{bmatrix}k\\i\end{bmatrix}(zbq^k;q)_i\left[(-1)^{n+i}q^{\binom{n+i}{2}}\right]^{1+s-r}(dbq^k)^n(d/z)^i$$
$$=\frac{(c/b,zb;q)_\infty}{(c,z;q)_\infty}\sum_{k=0}^{\infty}\frac{(abz/c,b;q)_k}{(q,zb;q)_k}(c/b)^k\sum_{n,i\ge 0}\frac{W_{n+i}}{(q;q)_n}\frac{(q^{-k},z;q)_i}{(q,abz/c;q)_i}$$
$$\times\left[(-1)^{n+i}q^{\binom{n+i}{2}}\right]^{1+s-r} d^n\ (dabq^k/c)^i. \qquad (3.7)$$

***Proof*.** Rewrite Heine's formula as follows.

$$\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}\frac{z^k}{(zbq^k;q)_\infty}=\frac{(c/b;q)_\infty}{(c;q)_\infty}\sum_{k=0}^{\infty}\frac{(b;q)_k}{(q;q)_k}(c/b)^k\frac{(abz/c;q)_\infty}{(zbq^k,z;q)_\infty}\ . \qquad (3.8)$$

Applying the general operator ${}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,dD_q\right)$ to both sides of the equation (3.8) gives:

$$\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}\ {}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,dD_q\right)\left\{\frac{z^k}{(zbq^k;q)_\infty}\right\}$$

$$= \frac{(c/b;q)_\infty}{(c;q)_\infty} \sum_{k=0}^{\infty} \frac{(b;q)_k}{(q;q)_k} \ (c/b)^k \ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, dD_q \right) \left\{ \frac{(abz/c;q)_k}{(zbq^k, z;q)_k} \right\} . \tag{3.9}$$

Using (2.7), we get

$${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, dD_q \right) \left\{ \frac{z^k}{(zbq^k;q)_\infty} \right\}$$
$$= \frac{z^k}{(zbq^k;q)_\infty} \sum_{n,i\geq 0} \frac{W_{n+i}}{(q;q)_n} \begin{bmatrix} k \\ i \end{bmatrix} (zbq^k;q)_i \left[(-1)^{n+i} q^{\binom{n+i}{2}} \right]^{1+s-r} (dbq^k)^n (\frac{d}{z})^i . \tag{3.10}$$

and using (2.4), we get

$${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, dD_q \right) \left\{ \frac{(abz/c;q)_k}{(zbq^k, z;q)_k} \right\}$$
$$= \frac{(abz/c;q)_k}{(zbq^k, z;q)_k} \sum_{n,i\geq 0} \frac{W_{n+i}}{(q;q)_n} \ \frac{(q^{-k}, z;q)_i}{(q, abz/c;q)_i} \left[(-1)^{n+i} q^{\binom{n+i}{2}} \right]^{1+s-r} d^n (\frac{dab}{c} q^k)^i . \tag{3.11}$$

Substituting (3.10) and (3.11) in equation (3.9) the proof is completed .

- If $r = s + 1,\ a = 0,\ z \to yt,\ d \to xt,\ c \to cb$ and then $b = 0$ in equation (3.7), we get the generating function for the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(a,b)}(x, y)$ (1.26).

- If $r = 1, s = 0$ in equation (3.7), we get the following identity:

Corollary 3.2.4

$$\sum_{k=0}^{\infty} \frac{(a, b, db;q)_k}{(q, c, a_1 db;q)_k} \ z^k \ {}_3\phi_1\left(\begin{matrix} q^{-k}, a_1, zbq^k \\ a_1 dbq^k \end{matrix}; q, dq^k/z \right)$$
$$= \frac{(a_1 d, db, c/b, zb;q)_\infty}{(d, a_1 db, c, z;q)_\infty} \sum_{k=0}^{\infty} \frac{(abz/c, b;q)_k}{(q, zb;q)_k} \ (c/b)^k \ {}_3\phi_2\left(\begin{matrix} q^{-k}, a_1, z \\ abz/c, a_1 d \end{matrix}; q, dabq^k/c \right).$$

3.3 Generalization of q-Pfaff-Saalschütz Sum

Theorem 3.3 (Generalization of q-Pfaff-Saalschütz sum). *Let q-Pfaff-Saalschütz sum be defined as in* (1.9), *then*

$$\sum_{k=0}^{n} \frac{(q^{-n}, a, b;q)_k}{(q, c, abq^{1-n}/c;q)_k} \ q^k \sum_{i,j\geq 0}^{\infty} \frac{W_{i+j}}{(q;q)_i} \ \frac{(q^{-n+k};q)_i}{(abq^{1-n+k}/c;q)_{i+j}} \ \frac{(yq^{-k}, abq/c;q)_j}{(q, ay;q)_j}$$
$$\times \left[(-1)^{i+j} q^{{i+j}_2} \right]^{1+s-r} \ (dbq/c)^i (dq^k)^j$$

$$= \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n} \sum_{i=0}^{\infty} \frac{W_{i+j}}{(q;q)_i} \frac{(yc/q;q)_i}{(ay;q)_{i+j}} \frac{(q^{1-n}/c, aq/c; q)_j}{(aq^{1-n}/c;q)_j}$$
$$\times \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} (dq/c)^i d^j \ . \tag{3.12}$$

Proof. Multiplaying q-Saalschütz identity (1.9) by $(ay;q)_\infty$, we have

$$\sum_{k=0}^{\infty} \frac{(q^{-n}, b; q)_k}{(q, c; q)_k} q^k \frac{(ay, abq^{1-n+k}/c; q)_\infty}{(aq^k, abq/c; q)_\infty} = \frac{b^n (c/b; q)_n}{(c, ; q)_n} \frac{(aq^{1-n}/c, ay; q)_\infty}{(a, aq/c; q)_\infty}. \tag{3.13}$$

Applying the general operator ${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, dD_q\right)$ to both sides of equation (3.13) gives:

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q,c;q)_k} \ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, dD_q\right) \left\{\frac{(ay, abq^{1-n+k}/c; q)_\infty}{(aq^k, abq/c; q)_\infty}\right\}$$
$$= \frac{(-c)^n q^{\binom{n}{2}}}{(c;q)_n} \ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, dD_q\right) \left\{\frac{(aq^{1-n}/c, ay; q)_\infty}{(a, aq/c; q)_\infty}\right\} \tag{3.14}$$

Using (2.2), we get

$${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, dD_q\right) \left\{\frac{(ay, abq^{1-n+k}/c; q)_\infty}{(aq^k, abq/c; q)_\infty}\right\}$$
$$= \frac{(ay, abq^{1-n+k}/c; q)_\infty}{(aq^k, abq/c; q)_\infty} \sum_{i,j\geq 0}^{\infty} \frac{W_{i+j}}{(q;q)_i} \frac{(q^{-n+k};q)_i}{(abq^{1-n+k}/c;q)_{i+j}} \frac{(yq^{-k}, abq/c; q)_j}{(q, ay; q)_j}$$
$$\times \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} (dbq/c)^i (dq^k)^j. \tag{3.15}$$

$${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, dD_q\right) \left\{\frac{(aq^{1-n}/c, ay; q)_\infty}{(a, aq/c; q)_\infty}\right\}$$
$$= \frac{(aq^{1-n}/c, ay; q)_\infty}{(a, aq/c; q)_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{W_{i+j}}{(q;q)_i} \frac{(yc/q;q)_i}{(ay;q)_{i+j}} \frac{(q^{-n};q)_j}{(aq^{1-n}/c;q)_j}$$
$$\times \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} (dq/c)^i (d)^j. \tag{3.16}$$

Substituting (3.15) and (3.16) in equation (3.14), the proof is completed.

- If $n = \infty$ in equation (3.12), we get a generalization for q-Gauss sum (1.10) as follows:

Corollary 3.3.5 (Generalization of q-Gauss sum). *Let q-Gauss sum be defined as in* (1.10), *then*

$$\sum_{k=0}^{\infty} \frac{(a,b;q)_k}{(q,c;q)_k} (c/ab)^k \ {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, d/a\right)$$

$$= \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty} \sum_{i,j\geq 0} \frac{W_{i+j}}{(q;q)_i} \frac{(yc/q;q)_i}{(ay;q)_{i+j}} \frac{(aq/c;q)_j}{(q;q)_j} \times \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} (dq/c)^i (d/a)^j .$$

• If $b = \infty$ in equation (3.12), we get a generalization for q-Chu-Vandermonde sum (1.7) as follows:

Corollary 3.3.6 (Generalization to q-Chu-Vandermonde sum(1.7)). *Let q-Chu-Vandermonde sum be defined as in* (1.7), *then*

$$\sum_{k=0}^{n} \frac{(q^{-n}, a; q)_k}{(q, c; q)_k} (cq^n/\mathrm{a})^k \sum_{i,j\geq 0} W_{i+j} \frac{(q^{-n+k};q)_i}{(q;q)_i} \frac{(yq^{-k};q)_j}{(q,ay;q)_j} \times \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} (-1)^i q^{\binom{i}{2}} (\tfrac{dq^n}{aq^{k+j}})^i (dq^n)^j$$
$$= \frac{(c/a;q)_n}{(c;q)_n} \sum_{i=0}^{\infty} \sum_{i,j\geq 0}^{\infty} \frac{W_{i+j}}{(q;q)_i} \frac{(yc/q;q)_i}{(ay;q)_{i+j}} \frac{(q^{1-n}/c, aq/c;q)_j}{(q, aq^{1-n}/c;q)_j} \times \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} (dq/c)^i (d)^j .$$

• If $b = 0$ in equation (3.12), we get a generalization for q-Chu-Vandermonde sum (1.8) as follows:

Corollary 3.3.7 (Generalization to q-Chu-Vandermonde sum (1.8)). *Let q-Chu-Vandermonde sum be defined as in* (1.8), *then*

$$\sum_{k=0}^{n} \frac{(q^{-n}, a; q)_k}{(q, c; q)_k} q^k \; {}_{r+1}\phi_{s+1}\left(\begin{matrix} a_1, \cdots, a_r, yq^{-k} \\ b_1, \cdots, b_s, ay \end{matrix}; q, dq^k\right)$$
$$= \frac{(c/a;q)_n}{(c;q)_n} a^n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{W_{i+j}}{(q;q)_i} \frac{(yc/q;q)_i}{(ay;q)_{i+j}} \frac{(q^{1-n}/c, aq/c;q)_j}{(q, aq^{1-n}/c;q)_j} \times \left[(-1)^{i+j} q^{\binom{i+j}{2}}\right]^{1+s-r} (dq/c)^i d^j . \tag{3.17}$$

• If $r = s = 0$ and $y = 0$ in equation (3.17) then using (1.6), we get the following identity:

Corollary 3.3.8

$${}_3\phi_2\left(\begin{matrix} q^{-n}, a, 0 \\ c, d \end{matrix}; q, q\right) = \frac{(dq/c;q)_\infty}{(d;q)_\infty} \frac{(c/a;q)_n}{(c;q)_n} a^n \; {}_2\phi_2\left(\begin{matrix} q^{1-n}/c, aq/c \\ aq^{1-n}/c, dq/c \end{matrix}; q, d\right)$$

• If $r = 2, s = 1$ and $y = 0$ in equation (3.17), we get Theorem 17 obtained by Li and Tan [9] (equation (1.22)).

• If $r = 2, s = 1,\ y = 0$ and setting $a_1 = q^{-N}$ in equation ??, then using equations (1.1) and (1.7), we get Theorem 3.1 obtained by Zhang and Yang [15] (equation (1.19)).

• If $r = 2,\ s = 1,\ y = 0,\ a_1 = q^{-N}$ and $a = 1$ in equation (3.17), we get Corollary 3.2 obtained by Fang [5] (equation (1.20)).

Conclusions

1. Many operators can be obtained by assigning some special values to the generalized q-operator ${}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix}; q, cD_q\right)$
2. We generalized some well-known q-identities, such as Cauchy identity, Heine's transformation formula and the q-Pfaff-Saalschütz summation formula.

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تطبيقات المؤثر ${}_r\Phi_s$ في المتطابقات-q

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المستخلص:

في هذا البحث، أنشأنا المؤثر العام ${}_r\Phi_s$ ، ثم وجدنا بعض متطابقاته التي سيتم استخدامها لتعميم بعض متطابقات-q المعروفة ، مثل متطابقة كوشي ، وصيغة تحويل هاين ، وصيغة جمع بفاف- سلشوتس. من خلال إعطاء قيم خاصة للمعلمات في المتطابقات التي حصلنا عليها ، تم الحصول على بعض النتائج الجديدة و/اوتم اعادة برهان البعض الآخر.