

Applications of the Operator ${}_r\Phi_s$ in q -Integrals

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<u>ARTICLE INFO</u>	ABSTRACT
<p>Keywords q-operator, q-integrals, Andrews-Askey integral, Gasper integral, Askey-Wilson integral.</p>	<p>We establish general q-operator ${}_r\Phi_s$ and subsequently discover some of its operator identities, which we use to generalize various q-integrals such as the Andrews-Askey integral, Gasper integral, and Askey-Wilson integral. In q-integrals, we specify exact values for achieving certain new results or reproofing others.</p>

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1. Introduction

In this paper, we employ the notations and terminology from [10], assuming that $0 < q < 1$. Let a be a complex variable. The q -shifted factorial is defined as follows:

$$(a; q)_0 = 1, \quad (a; q)_m = \prod_{k=0}^{m-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

The multiple q -shifted factorial is given by:

$$(a_1, a_2, \dots, a_r; q)_m = (a_1; q)_m (a_2; q)_m \cdots (a_r; q)_m,$$

where $m \in \mathbb{Z}$ or ∞ , $r \in \mathbb{Z}^+$ and $a_1, \dots, a_r \in \mathbb{C}$.

For non-negative integers r and s , the basic hypergeometric series ${}_r\Phi_s$ is known as [10]

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x \right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} x^k,$$

where $r, s \in \mathbb{N}$; $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{C}$.

In this paper, we will use the following identities [10]:

$$(a; q)_{n-k} = \frac{(a; q)_n}{(q^{1-n}/a; q)_k} (-1)^k q^{\binom{k}{2}-nk} \left(\frac{q}{a}\right)^k. \tag{1.1}$$

$$(q^{-n}; q)_k = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk}. \tag{1.2}$$

$$(aq^k; q)_{n-k} = \frac{(a; q)_n}{(a; q)_k}. \tag{1.3}$$

The q -binomial coefficient is presented as follows [10]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

where $n, k \in \mathbb{N}$.

The Cauchy identity is

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1.$$

Euler provided the following special case of Cauchy identity [10]



$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} x^n = (x; q)_{\infty}.$$

-Chu-Vandermonde’s identity is given as [10]: q

$${}_2\phi_1 \left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q \right) = \frac{(c/b; q)_n}{(c; q)_n} b^n. \tag{1.4}$$

The q -Gauss sum is given as [10]:

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, c/ab \right) = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}}. \tag{1.5}$$

The transformation of ${}_3\phi_2$ is given by [10, Appendix III, equation (III.9)]:

$${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, de/abc \right) = \frac{(e/a, de/bc; q)_{\infty}}{(e, de/abc; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, e/a \right). \tag{1.6}$$

The transformation of ${}_3\phi_2$ is given by [10, Appendix III, equation (III.12)]:

$${}_3\phi_2 \left(\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix}; q, q \right) = \frac{(e/c; q)_n}{(e; q)_n} c^n {}_3\phi_2 \left(\begin{matrix} q^{-n}, c, d/b \\ d, cq^{1-n}/e \end{matrix}; qbq/e \right). \tag{1.7}$$

Definition 1.1 [5]. The q -differential operator, or q -derivative, is defined by

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}. \tag{1.8}$$

Theorem 1.2 [5]. For $n \in \mathbb{N}$, we have

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(a)\} D_q^{n-k}\{g(aq^k)\}. \tag{1.9}$$

Theorem 1.3 [19]. Let D_q be defined as in (1.8). then

$$D_q^k \left\{ \frac{(av; q)_{\infty}}{(at; q)_{\infty}} \right\} = t^k (v/t; q)_k \frac{(avq^k; q)_{\infty}}{(at; q)_{\infty}}, \quad |at| < 1. \tag{1.10}$$

The Andrews-Askey integral is given by [1, 2]:

$$\int_c^d \frac{(qt/c, qt/d; q)_{\infty}}{(at, bt; q)_{\infty}} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_{\infty}}{(ac, ad, bc, bd; q)_{\infty}}, \tag{1.11}$$

where $\max\{|a|, |b|, |c|, |d|\} < 1, cd \neq 0$.

The Gasper integral [9] is given by:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta}; q)_{\infty}}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta}; q)_{\infty}} d\theta$$



$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf; q)_\infty}{(q, ac, ad, bc, bd, cf, df; q)_\infty}, \tag{1.12}$$

where $\max\{|a|, |b|, |c|, |d|\} < 1, cd\rho \neq 0$.

The Askey-Wilson integral is stated as follows [3, 11, 12]:

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)} d\theta = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \tag{1.13}$$

where $\max\{|a|, |b|, |c|, |d|\} < 1$.

In 1997, Chen and Liu [5] constructed the following q -exponential operator:

Definition 1.4 [5]. *The q -exponential operator $T(bD_q)$ is defined by*

$$T(bD_q) = \sum_{k=0}^\infty \frac{(bD_q)^k}{(q; q)_k}. \tag{1.14}$$

In light of parameter augmentation, they employed the q -exponential operator $T(bD_q)$ to obtain an extension to the Andrews-Askey integral (1.11) as follows:

Theorem 1.5 [5]. *Let $T(bD_q)$ be defined as in (1.14), then*

$$\int_c^d \frac{(qt/c, qt/d, abcde; q)_\infty}{(at, bt, et; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd, bcde, acde; q)_\infty}{(ac, ad, bc, bd, ce, de; q)_\infty}. \tag{1.15}$$

In 2005, Zhang and Wang [18] employed the q -exponential operator $T(bD_q)$ to offer the following extension to the Gasper integral (1.12):

Theorem 1.6 [18]. *Let $T(bD_q)$ be defined as in (1.14), then*

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^\pi \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, pce^{-i\theta}, qe^{i\theta}/c\rho, abcdf e^{i\theta}, bcdf g e^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, fe^{i\theta}, ce^{-i\theta}, de^{-i\theta}, ge^{i\theta}; q)_\infty} \\ & \times {}_3\phi_2 \left(\begin{matrix} bcdf, fe^{i\theta}, be^{i\theta} \\ abcdf e^{i\theta}, bcdf g e^{i\theta} \end{matrix}; q, acdg \right) d\theta \\ & = \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf, bcdg, cdfg; q)_\infty}{(q, ac, ad, bc, bd, cf, df, cg, dg; q)_\infty}. \end{aligned} \tag{1.16}$$



In 2008, Chen and Gu [4] introduced the Cauchy operator $T(a, b; D_q)$ as follows:

Definition 1.7 [4]. *The Cauchy operator $T(a, b; D_q)$ is defined by*

$$T(a, b; D_q) = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} (bD_q)^k. \tag{1.17}$$

Chen and Gu [4] used the operator $T(a, b; D_q)$ to give an extension for the Gasper integral (1.12) as follows:

Theorem 1.8 [4]. *Let $T(a, b; D_q)$ be defined as in (1.17), then*

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdf e^{i\theta}, ghe^{i\theta}; q)_{\infty}}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta}, he^{i\theta}; q)_{\infty}} \\ & \times {}_3\phi_2 \left(\begin{matrix} g, ae^{i\theta}, fe^{i\theta} \\ abcdf e^{i\theta}, ghe^{i\theta} \end{matrix}; q, bcdh \right) d\theta \\ & = \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf, cgh; q)_{\infty}}{(q, ac, ad, bc, bd, cf, ch, df; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} g, ac, fc \\ cgh, acdf \end{matrix}; q, dh \right). \end{aligned} \tag{1.18}$$

In 2016, Li and Tan [13] constructed the generalized q -exponential operator $\mathbb{T} \left[\begin{smallmatrix} u, v \\ w \end{smallmatrix} | q; cD_q \right]$ with three parameters as follows:

Definition 1.9 [13]. *The generalized q -exponential operator $\mathbb{T} \left[\begin{smallmatrix} u, v \\ w \end{smallmatrix} | q; cD_q \right]$ is defined by*

$$\mathbb{T} \left[\begin{smallmatrix} u, v \\ w \end{smallmatrix} | q; cD_q \right] = \sum_{n=0}^{\infty} \frac{(u, v; q)_n}{(q, w; q)_n} (cD_q)^n. \tag{1.19}$$

Li and Tan [13] used the operator $\mathbb{T} \left[\begin{smallmatrix} u, v \\ w \end{smallmatrix} | q; cD_q \right]$ to obtain the following generalization for the Askey-Wilson integral (1.13):

Theorem 1.10 [13]. *Let $\mathbb{T} \left[\begin{smallmatrix} u, v \\ w \end{smallmatrix} | q; cD_q \right]$ be defined as in (1.19), then*

$$\begin{aligned} & \int_0^{\pi} \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)} \sum_{n, k \geq 0} \frac{(u, v; q)_{n+k}}{(q; q)_n (w; q)_{n+k}} \frac{(ae^{-i\theta}, be^{-i\theta}; q)_k}{(q, ab; q)_k} f^{n+k} e^{(k-n)i\theta} d\theta \\ & = \frac{2\pi(abcd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}} \sum_{n, k \geq 0} \frac{(u, v; q)_{n+k}}{(q; q)_n (w; q)_{n+k}} \frac{(ad, bd)_k}{(q, abcd; q)_k} f^{n+k} d^n c^k. \end{aligned} \tag{1.20}$$



The following is how our paper is organized. In section 2, we build a generalized q -operator ${}_r\Phi_s$ and then determine its identities. In section 3, we employ the operator ${}_r\Phi_s$ to generalize the Andrews-Askey, Gasper, and Askey-Wilson integrals.

2. The General Operator ${}_r\Phi_s$ and its Identities

In this section, we introduce the general operator ${}_r\Phi_s \left(\begin{smallmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{smallmatrix}; q, cD_q \right)$. Then we give some of its identities.

Definition 2.1 We define the general q -operator ${}_r\Phi_s$ as follows:

$${}_r\Phi_s \left(\begin{smallmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{smallmatrix}; q, cD_q \right) = \sum_{n=0}^{\infty} \frac{W_n}{(q; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} (cD_q)^n, \tag{2.1}$$

where $W_n = \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n}$.

In order to obtain many previously stated operators, several specific values may be assigned to the general q -operator ${}_r\Phi_s$ as follows:

- Setting $r = 1, s = 0, a_1 = 0$ and $c = b$, we get on the exponential operator $T(bD_q)$ defined by Chen and Liu [5] in 1997.
- If $r = 1, s = 0, a_1 = a$ and $c = b$, we get on the Cauchy operator $T(a, b; D_q)$ defined by Chen and Gu [4] in 2008.
- When $r = 2, s = 1, a_1 = u, a_2 = v$ and $b_1 = w$, we get on the generalized exponential operator with three parameters $\mathbb{T} \left[\begin{smallmatrix} u, v \\ w \end{smallmatrix} | q; cD_q \right]$ constructed by Li and Tan [13] in 2016.

Other special values for the generalized q -operator ${}_r\Phi_s$ can be used to describe operators like those in [6,7,8, 14, 15, 16, 17,19].

The following operator identities will be derived using q -Liebniz formula (1.9).

Theorem 2.2 Let ${}_r\Phi_s \left(\begin{smallmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{smallmatrix}; q, cD_q \right)$ be defined as in (2.1), then

$${}_r\Phi_s \left(\begin{smallmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{smallmatrix}; q, cD_q \right) \left\{ \frac{(av, au; q)_{\infty}}{(at, aw; q)_{\infty}} \right\} = \frac{(av, au; q)_{\infty}}{(at, aw; q)_{\infty}} \times \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{W_{n+k}}{(q; q)_n} \frac{(v/t, aw; q)_k}{(q, av; q)_k} \frac{(u/w; q)_n}{(au; q)_{n+k}} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} (cw)^n (ct)^k, \tag{2.2}$$



provided that $\max\{|at|, |aw|\} < 1$.

Proof.

$$\begin{aligned}
 & {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q \right) \left\{ \frac{(av, au; q)_\infty}{(at, aw; q)_\infty} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{W_n}{(q; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} c^n D_q^n \left\{ \frac{(av; q)_\infty (au; q)_\infty}{(at; q)_\infty (aw; q)_\infty} \right\} \quad (\text{by using (2.1)}) \\
 &= \sum_{n=0}^{\infty} \frac{W_n}{(q; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} c^n \\
 &\quad \times \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk} D_q^k \left\{ \frac{(av; q)_\infty}{(at; q)_\infty} \right\} D_q^{n-k} \left\{ \frac{(auq^k; q)_\infty}{(awq^k; q)_\infty} \right\} \quad (\text{by using (1.9)}) \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{W_n}{(q; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} c^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk} \\
 &\quad \times t^k \frac{(v/t; q)_k (avq^k; q)_\infty}{(at; q)_\infty} (wq^k)^{n-k} \frac{(u/w; q)_{n-k} (auq^n; q)_\infty}{(awq^k; q)_\infty} \quad (\text{by using (1.10)}) \quad (2.3) \\
 &= \frac{(av, au; q)_\infty}{(at, aw; q)_\infty} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{W_{n+k}}{(q; q)_n} \frac{(v/t, aw; q)_k}{(q, av; q)_k} \frac{(u/w; q)_n}{(au; q)_{n+k}} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} (cw)^n (ct)^k.
 \end{aligned}$$

■

Setting $u = 0$ in equation (2.2), we get the following corollary:

Corollary 2.1 Let ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q \right)$ be defined as in (2.1), then

$$\begin{aligned}
 & {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q \right) \left\{ \frac{(av; q)_\infty}{(at, aw; q)_\infty} \right\} = \frac{(av; q)_\infty}{(at, aw; q)_\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{W_{n+k}}{(q; q)_n} \frac{(v/t, aw; q)_k}{(q, av; q)_k} \\
 &\quad \times \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} (cw)^n (ct)^k, \quad \max\{|at|, |aw|\} < 1. \quad (2.4)
 \end{aligned}$$

Setting $u = 0$ and then $w = 0$ in equation (2.2), we get the following corollary:

Corollary 2.2 Let ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q \right)$ be defined as in (2.1), then

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q \right) \left\{ \frac{(av; q)_\infty}{(at; q)_\infty} \right\} = \frac{(av; q)_\infty}{(at; q)_\infty} \sum_{k=0}^{\infty} \frac{W_k}{(q; q)_k} \frac{(v/t; q)_k}{(av; q)_k}$$

$$\times \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} (ct)^k, \quad |at| < 1. \quad (2.5)$$

3. Applications in q -Integrals

In this section, we will use the general q -operator ${}_r\Phi_s$ to generalize various q -integrals, such as the Andrews-Askey integral, the Gasper integral, and the Askey-Wilson integral. We can use special values in q -integrals to achieve new findings or recover others.

3.1 Generalization of the Andrews-Askey Integral

Theorem 3.1. (Generalization of the Andrews-Askey integral). Let ${}_r\Phi_s \left(\begin{smallmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{smallmatrix}; q, cD_q \right)$ be defined as in (2.1), then

$$\begin{aligned} & \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} \sum_{k=0}^\infty \frac{W_k}{(q; q)_k} \frac{(f/t; q)_k}{(af; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} (et)^k d_q t \\ &= \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty} \sum_{n, k \geq 0} \frac{W_{n+k}}{(q; q)_n} \frac{(f/d; q)_n}{(af; q)_{n+k}} (ed)^n \\ & \quad \times \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} \frac{(ad, bd; q)_k}{(q, abcd; q)_k} (ec)^k. \end{aligned} \quad (3.1)$$

Proof. Multiplying equation (1.11) by $(af; q)_\infty$, we have

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty (af; q)_\infty}{(bt; q)_\infty (at; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d; q)_\infty (af, abcd; q)_\infty}{(bc, bd; q)_\infty (ac, ad; q)_\infty}. \quad (3.2)$$

Applying the operator ${}_r\Phi_s$ on both sides of (2.7), we get

$$\begin{aligned} & \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(bt; q)_\infty} {}_r\Phi_s \left(\begin{smallmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{smallmatrix}; q, eD_q \right) \left\{ \frac{(af; q)_\infty}{(at; q)_\infty} \right\} d_q t \\ &= \frac{d(1-q)(q, dq/c, c/d; q)_\infty}{(bc, bd; q)_\infty} {}_r\Phi_s \left(\begin{smallmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{smallmatrix}; q, eD_q \right) \left\{ \frac{(af, abcd; q)_\infty}{(ac, ad; q)_\infty} \right\}. \end{aligned} \quad (3.3)$$

Using equation (2.5), we get

$$\begin{aligned} & {}_r\Phi_s \left(\begin{smallmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{smallmatrix}; q, eD_q \right) \left\{ \frac{(af; q)_\infty}{(at; q)_\infty} \right\} d_q t \\ &= \frac{(af; q)_\infty}{(at; q)_\infty} \sum_{k=0}^\infty \frac{W_k}{(q; q)_k} \frac{(f/t; q)_k}{(af; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} (et)^k. \end{aligned} \quad (3.4)$$



and using equation (2.2), we get

$$\begin{aligned}
 {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, eD_q \right) \left\{ \frac{(af, abcd; q)_\infty}{(ac, ad; q)_\infty} \right\} &= \frac{(af, abcd; q)_\infty}{(ac, ad; q)_\infty} \\
 \times \sum_{n,k \geq 0} \frac{W_{n+k} (f/d; q)_n}{(q; q)_n (af; q)_{n+k}} (ed)^n \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} &\frac{(ad, bd; q)_k}{(q, abcd; q)_k} (ec)^k. \quad (3.5)
 \end{aligned}$$

Substituting (3.4) and 3.5) into equation (3.3), we obtain the desired result. ■

- If $r = s = 0$ and $f = 0$ in equation (3.1), we obtain the following result:

Corollary 3.1.1 *We have*

$$\begin{aligned}
 \int_c^d \frac{(qt/c, qt/d, et; q)_\infty}{(at, bt; q)_\infty} d_q t \\
 = \frac{d(1-q)(q, dq/c, c/d, abcd, ed; q)_\infty}{(ac, ad, bc, bd; q)_\infty} {}_2\Phi_2 \left(\begin{matrix} ad, bd \\ ed, abcd; q, ec \end{matrix} \right).
 \end{aligned}$$

- When $r = 1, s = 0, f = 0$ and $a_1 = abcd$ in equation (3.1), we recover Theorem 6.2. obtained by Chen and Liu [5] (equation (1.15)).
- If $r = 2, s = 1, a_1 = q^{-N}, f = 0$ and $e \rightarrow q/d$ in equation (3.1), by using q -Chu-Vandermond identity (1.4), and then equations (1.1), (1.3), we obtain the following result:

Corollary 3.1.2 *We have*

$$\begin{aligned}
 \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} {}_2\Phi_1 \left(\begin{matrix} q^{-N}, a_2 \\ b_1 \end{matrix}; q, \frac{qt}{d} \right) d_q t \\
 = \frac{a_2^N d(1-q)(q, dq/c, c/d, abcd, b_1 q^N, b_1/a_2; q)_\infty}{(ac, ad, bc, bd, b_1, b_1 q^N/a_2; q)_\infty} \\
 \times {}_4\Phi_2 \left(\begin{matrix} q^{-N}, a_2, ad, bd \\ abcd, a_2 q^{N-1}/b_1 \end{matrix}; q, qc/b_1 d \right).
 \end{aligned}$$

3.2 Generalization of Gasper Integral

Theorem 3.2 (Generalization of Gasper integral). *Let ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q \right)$ be defined as in (2.1), then*

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdf e^{i\theta}; q)_{\infty}}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta}; q)_{\infty}} \\ & \times \sum_{k,n \geq 0} \frac{W_{n+k} (abcd, bcdf; q)_k}{(q; q)_n (q, abcdf e^{i\theta}; q)_k} (ge^{i\theta})^k (gbcd)^n \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} d\theta \\ & = \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf; q)_{\infty}}{(q, ac, ad, bc, bd, cf, df; q)_{\infty}} \\ & \times \sum_{k,n \geq 0} \frac{W_{n+k} (ad, fd; q)_k}{(q; q)_n (q, acdf; q)_k} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} (gc)^k (gd)^n. \end{aligned} \tag{3.6}$$

Proof. Rewrite (1.12) as:

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, fe^{i\theta}; q)_{\infty} (abcdf e^{i\theta}; q)_{\infty}}{(be^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_{\infty} (ae^{i\theta}, abcd; q)_{\infty}} d\theta \\ & = \frac{(\rho c/d, dq/\rho c, \rho, q/\rho, bcdf; q)_{\infty} (acdf; q)_{\infty}}{(q, bc, bd, cf, df; q)_{\infty} (ac, ad; q)_{\infty}}. \end{aligned}$$

Applying the operator ${}_r\Phi_s$ on both sides of (3.7), we get

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, fe^{i\theta}; q)_{\infty}}{(be^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_{\infty}} \\ & \times {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, gD_q \right) \left\{ \frac{(abcdf e^{i\theta}; q)_{\infty}}{(ae^{i\theta}, abcd; q)_{\infty}} \right\} d\theta \\ & = \frac{(\rho c/d, dq/\rho c, \rho, q/\rho, bcdf; q)_{\infty}}{(q, bc, bd, cf, df; q)_{\infty}} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, gD_q \right) \left\{ \frac{(acdf; q)_{\infty}}{(ac, ad; q)_{\infty}} \right\}. \end{aligned} \tag{3.8}$$

Using equation (2.4) on both sides of (3.8), we get

$$\begin{aligned} & {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, gD_q \right) \left\{ \frac{(abcdf e^{i\theta}; q)_{\infty}}{(ae^{i\theta}, abcd; q)_{\infty}} \right\} = \frac{(abcdf e^{i\theta}; q)_{\infty}}{(ae^{i\theta}, abcd; q)_{\infty}} \\ & \times \sum_{k,n \geq 0} \frac{W_{n+k} (abcd, bcdf; q)_k}{(q; q)_n (q, abcdf e^{i\theta}; q)_k} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} (ge^{i\theta})^k (gbcd)^n. \end{aligned} \tag{3.9}$$

and

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, gD_q \right) \left\{ \frac{(acdf; q)_{\infty}}{(ac, ad; q)_{\infty}} \right\}$$



$$= \frac{(acdf; q)_\infty}{(ac, ad; q)_\infty} \sum_{k,n \geq 0} \frac{W_{n+k} (ad, fd; q)_k}{(q; q)_n (q, acdf; q)_k} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} (gc)^k (gd)^n. \quad (3.10)$$

Substituting (3.9) and (3.10) into equation (3.8), we get the desired result. ■

- If $r = s = 0$ in equation (3.6), we get the following result:

Corollary 3.2.3 We have

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdf e^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta}; q)_\infty} \\ & \times {}_2\phi_2 \left(\begin{matrix} abcd, bcdf \\ gbcdf, abcdf e^{i\theta} \end{matrix}; q, ge^{i\theta} \right) d\theta \\ & = \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf, gd; q)_\infty}{(q, ac, ad, bc, bd, cf, df, gbcdf; q)_\infty} {}_2\phi_2 \left(\begin{matrix} ad, fd \\ acdf, gd \end{matrix}; q, gc \right). \end{aligned}$$

- If $r = 1$ and $s = 0$ in (3.6), we get the following equation

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdf e^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta}; q)_\infty} \\ & \times \frac{(a_1 gbcdf; q)_\infty}{(gbcdf; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a_1, bcdf, abcd \\ abcdf e^{i\theta}, a_1 gbcdf \end{matrix}; q, ge^{i\theta} \right) d\theta \\ & = \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf, a_1 gd; q)_\infty}{(q, ac, ad, bc, bd, cf, df, gd; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a_1, fd, ad \\ acdf, a_1 gd \end{matrix}; q, gc \right). \quad (3.11) \end{aligned}$$

For ${}_3\phi_2$ in the left hand side of (3.11), replacing a, b, c, d, e by $a_1, bcdf, abcd, abcdf e^{i\theta}, a_1 gbcdf$, respectively, in transformation formula (1.6), and for ${}_3\phi_2$ in the right hand side of (3.11), replacing a, b, c, d, e by $a_1, df, ad, acdf, a_1 gd$, respectively, in transformations formula (1.6), then substituting the result in equation (3.11), we get the following corollary:

Corollary 3.2.4 We have

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdf e^{i\theta}, a_1 ge^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta}, ge^{i\theta}; q)_\infty} \\ & \times {}_3\phi_2 \left(\begin{matrix} a_1, ae^{i\theta}, fe^{i\theta} \\ abcdf e^{i\theta}, a_1 ge^{i\theta} \end{matrix}; q, gbcdf \right) d\theta \end{aligned}$$

$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf, a_1cg; q)_\infty}{(q, ac, ad, bc, bd, cf, df, gc; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a_1, ac, fc \\ acdf, a_1cg; q, gd \end{matrix} \right). \tag{3.12}$$

* If $g = h$ and then $a_1 = g$ in equation (3.12), we reestablish Theorem 4.1. obtained by Chen and Gu [4] (equation (1.18)).

* Exchanging a and b and then replacing a_1 by $bcdf$ in equation (3.12), then using the q -Gauss sum (1.5), we recover Theorem 4.2. obtained by Zhang and Wang [18] (equation 1.16)).

• If $r = 2, s = 1$ in equation (3.6), we have the following result:

Corollary 3.2.5 *We have*

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdf e^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta}; q)_\infty} \\ & \times \sum_{k=0}^{\infty} \frac{(a_1, a_2, bcdf, abcd; q)_k}{(q, 0, b_1, abcdf e^{i\theta}; q)_k} (ge^{i\theta})^k \sum_{n=0}^{\infty} \frac{(a_1q^k, a_2q^k; q)_n}{(q, b_1q^k; q)_n} (gbcd)^n d\theta \\ & = \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf; q)_\infty}{(q, ac, ad, bc, bd, cf, df; q)_\infty} \sum_{k=0}^{\infty} \frac{(a_1, a_2, ad, fd; q)_k}{(q, 0, b_1, acdf; q)_k} (gc)^k \\ & \times \sum_{n=0}^{\infty} \frac{(a_1q^k, a_2q^k; q)_n}{(q, b_1q^k; q)_n} (gd)^n. \end{aligned}$$

3.3 Generalization of the Askey-Wilson Integral

Theorem 3.3 (Generalization of the Askey-Wilson integral). *Let ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q \right)$ be defined as in (2.1), then*

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)} \sum_{n,k \geq 0} \frac{W_{n+k}}{(q; q)_n} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} f^{n+k} e^{(k-n)i\theta} \frac{(ae^{-i\theta}, be^{-i\theta}; q)_k}{(q, ab; q)_k} d\theta \\ & = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} \end{aligned}$$

$$\times \sum_{n,k \geq 0} \frac{W_{n+k}}{(q; q)_n} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} f^{n+k} d^n c^k \frac{(ad, bd)_k}{(q, abcd; q)_k}. \tag{3.13}$$

Proof. Rewrite equation (1.13) as:

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos 2\theta; b, c, d)} \frac{(ab; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty} d\theta = \frac{2\pi}{(q, bc, bd, cd; q)_\infty} \frac{(abcd; q)_\infty}{(ac, ad; q)_\infty}. \tag{3.14}$$

Applying the operator $r\Phi_s$ on both sides of (3.14), we get:

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos 2\theta; b, c, d)} r\Phi_s \left(a_1, \dots, a_r; q, fD_q \right) \left\{ \frac{(ab; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty} \right\} d\theta \\ &= \frac{2\pi}{(q, bc, bd, cd; q)_\infty} r\Phi_s \left(a_1, \dots, a_r; q, fD_q \right) \left\{ \frac{(abcd; q)_\infty}{(ac, ad; q)_\infty} \right\}. \end{aligned} \tag{3.15}$$

Using equation (2.4), on both sides of (3.15), we have:

$$\begin{aligned} & r\Phi_s \left(a_1, \dots, a_r; q, fD_q \right) \left\{ \frac{(ab; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty} \right\} d\theta = \frac{(ab; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty} \\ & \times \sum_{n,k \geq 0} \frac{W_{n+k}}{(q; q)_n} \frac{(ae^{-i\theta}, be^{-i\theta}; q)_k}{(q, ab; q)_k} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} f^{n+k} e^{(k-n)i\theta}. \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} & r\Phi_s \left(a_1, \dots, a_r; q, fD_q \right) \left\{ \frac{(abcd; q)_\infty}{(ac, ad; q)_\infty} \right\} \\ &= \frac{(abcd; q)_\infty}{(ac, ad; q)_\infty} \sum_{n,k \geq 0} \frac{W_{n+k}}{(q; q)_n} \frac{(ad, bd)_k}{(q, abcd; q)_k} \left[(-1)^{n+k} q^{\binom{n+k}{2}} \right]^{1+s-r} f^{n+k} d^n c^k. \end{aligned} \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), the proof is completed. ■

- If $r = s = 0$ in equation (3.13), we obtain the following result:

Corollary 3.3.6 *We have*

$$\int_0^\pi \frac{h(\cos 2\theta; 1)(fe^{-i\theta}, q)_\infty}{h(\cos \theta; a, b, c, d)} {}_2\phi_2 \left(\begin{matrix} ae^{-i\theta}, be^{-i\theta} \\ ab, fe^{-i\theta} \end{matrix}; q, fe^{i\theta} \right) d\theta$$



$$= \frac{2\pi(abcd, fd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}} {}_2\Phi_2 \left(\begin{matrix} ad, bd \\ abcd, fd \end{matrix}; q, fc \right).$$

- If $r = 1, s = 0$ in equation (3.13), we get the following result:

Corollary 3.3.7 We have

$$\int_0^{\pi} \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)} \frac{(a_1 f e^{-i\theta}; q)_{\infty}}{(f e^{-i\theta}; q)_{\infty}} {}_3\Phi_2 \left(\begin{matrix} a_1, a e^{-i\theta}, b e^{-i\theta} \\ ab, a_1 f e^{-i\theta} \end{matrix}; q, f e^{i\theta} \right) d\theta$$

$$= \frac{2\pi(abcd, a_1 f d; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd, f d; q)_{\infty}} {}_3\Phi_2 \left(\begin{matrix} a_1, ad, bd \\ abcd, a_1 f d \end{matrix}; q, fc \right).$$

- If $r = 2, s = 1$ in equation (3.13), we regain Theorem 22 obtained by Li and Tan [13] (equation (1.20)).

4. Conclusions

Some well-known q -integrals, including the Andrews-Askey, Gasper, and Askey-Wilson integrals, can be successfully generalized using the generalized operator ${}_r\Phi_s$.

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تطبيقات المؤثر ${}_r\Phi_s$ في التكاملات- q

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المستخلص

أنشأنا مؤثر- q العام ${}_r\Phi_s$ بعد ذلك أوجدنا بعض معادلاته ، والتي نستخدمها لتعميم تكاملات- q المختلفة مثل تكامل Andrews-Askey ، تكامل Gasper ، وتكامل Askey-Wilson. في تكاملات- q ، نحدد قيمة دقيقة لأيجاد نتائج جديدة معينة أو إعادة برهان نتائج أخرى.

