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New application of the Cauchy operator on the homogeneous Rogers–Szegö polynomials

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Abstract
This paper is mainly concerned with using the Cauchy operator $T(a, b; D_q)$ in proving identities that involve the homogeneous Rogers–Szegö polynomials $h_n(x, y|q)$. We introduce some operator identities for the Cauchy operator and represent the homogeneous Rogers–Szegö polynomials by the Cauchy operator. Also we use the Cauchy operator to derive the basic identities for $h_n(x, y|q)$: generating function, Mehler's formula and Rogers formula. Then, we give several extended identities for $h_n(x, y|q)$ such as extended generating function, extended Mehler's formula, extended Rogers formula and other extended identities.

Keywords Rogers–Szegö polynomials · homogenous Rogers–Szegö polynomials · Cauchy operator · Generating function · Mehler's formula, Rogers formula · q-Difference equation

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1 Introduction and notation

Let us review some common notation and terminology for basic hypergeometric series in [14], where $|q| < 1$. The q-shifted factorial is given by

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$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \quad (1.1)$$

Notice that

$$(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty.$$
$$(a;q)_{n+k} = (a;q)_k\,(aq^k;q)_n\,.$$

We shall use the following notations of multiple q-shifted factorials:

$$(a_1,a_2,\cdots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n\,,$$
$$(a_1,a_2,\cdots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty\,.$$

The q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

The generalized basic hypergeometric series is defined by

$$ {}_r\phi_s\left(\begin{matrix} a_1,a_2,\cdots,a_r \\ b_1,b_2,\cdots,b_s \end{matrix};q,x\right) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\cdots(b_s;q)_n} \times\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n,$$

where $q \neq 0$ when $r > s+1$. Notice that

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_{r+1} \\ b_1,b_2,\cdots,b_r \end{matrix};q,x\right) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_{r+1};q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\cdots(b_r;q)_n}x^n.$$

The Cauchy identity is defined as

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x|<1. \quad (1.2)$$

Euler found the following special case of the Cauchy identity (1.2):

$$\sum_{n=0}^{\infty}\frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \quad |x|<1. \quad (1.3)$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} x^k}{(q;q)_k} = (x;q)_\infty. \quad (1.4)$$

The Cauchy polynomial is defined by

$$P_n(x, y) = (y/x; q)_n\, x^n = (x - y)(x - qy) \cdots (x - q^{n-1}y).$$

The Cauchy polynomial is the homogeneous form of the q-shifted factorial (1.1). It has the following generating function (the homogeneous form of the Cauchy identity):

$$\sum_{k=0}^{\infty} P_k(x, y) \frac{t^k}{(q; q)_k} = \frac{(yt; q)_\infty}{(xt; q)_\infty}, \quad |xt| < 1.$$

Setting $y = 0$ and $x = 0$, the Cauchy identity becomes Euler's identities (1.3) and (1.4), respectively.

The classical Rogers–Szegö polynomial is defined by

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{1.5}$$

which has following generating function:

$$\sum_{n=0}^{\infty} h_n(x|q) \frac{t^n}{(q; q)_n} = \frac{1}{(t, xt; q)_\infty}, \qquad \max\{|t|, |xt|\} < 1. \tag{1.6}$$

Mehler's formula for $h_n(x|q)$ is

$$\sum_{n=0}^{\infty} h_n(x|q)\, h_n(y|q) \frac{t^n}{(q; q)_n} = \frac{(xyt^2; q)_\infty}{(t, xt, yt, xyt; q)_\infty}, \tag{1.7}$$

where $\max\{|t|, |xt|, |yt|, |xyt|\} < 1$ and the Rogers formula of $h_n(x|q)$ is

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(xst; q)_\infty}{(t, s, xt, xs; q)_\infty}, \tag{1.8}$$

where $\max\{|s|, |t|, |xs|, |xt|\} < 1$. We will call the identities (1.6), (1.7) and (1.8) as the basic identities for $h_n(x|q)$. The Rogers–Szegö polynomials play an important role in the theory of the orthogonal polynomials, particularly in the study of the Askey–Wilson polynomials [6–8,17,18].

Hahn polynomials are defined as follows [5,15,16,23]:

$$\phi_n^{(a)}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k\, x^k. \tag{1.9}$$

In 1965, Al-Salam and Carlitz [5] obtained the following results:

Theorem 1.1 [5]
The generating function for $\phi_n^{(a)}(x)$ *is*

$$\sum_{n=0}^{\infty}\phi_n^{(a)}(x)\frac{t^n}{(q;q)_n}=\frac{(axt;q)_\infty}{(t,xt;q)_\infty},\quad \max\{|t|,|xt|\}<1. \tag{1.10}$$

Mehler's formula for $\phi_n^{(a)}(x)$ *is*

$$\sum_{n=0}^{\infty}\phi_n^{(a)}(x)\phi_n^{(b)}(y)\frac{t^n}{(q;q)_n}=\frac{(axt,byt;q)_\infty}{(t,xt,yt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,b,t\\ axt,byt\end{matrix};q,xyt\right), \tag{1.11}$$

where $\max\{|t|,|xt|,|yt|,|xyt|\}<1$.

The q-differential operator D_q is defined by

$$D_q\{f(a)\}=\frac{f(a)-f(aq)}{a}. \tag{1.12}$$

The following identities are easy to verify [9]:

$$D_q^k\left\{\frac{1}{(at;q)_\infty}\right\}=\frac{t^k}{(at;q)_\infty}, \tag{1.13}$$

$$D_q^n\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\}=t^n\,(v/t;q)_n\,\frac{(avq^n;q)_\infty}{(at;q)_\infty}. \tag{1.14}$$

In 1997, Chen and Liu [9] defined the q-exponential operator $T(bD_q)$ as follows:

$$T(bD_q)=\sum_{n=0}^{\infty}\frac{(bD_q)^n}{(q;q)_n}.$$

They used the q-exponential operator $T(D_q)$ to represent the classical Rogers–Szegö polynomials $h_n(x|q)$ to derive Mehler's formula and Rogers formula for $h_n(x|q)$.

In 2003, Chen et al. [11] introduced the homogeneous q-difference operator D_{xy} on functions in two variables as follows:

$$D_{xy}\{f(x,y)\}=\frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y},$$

which turns out to be suitable for dealing with the homogeneous form of the q-binomial theorem. They derived the Leibniz formula for this operator. Based on the homogeneous q-difference operator, they built up the homogeneous q-shift operator $\mathbb{E}(D_{xy})$ as follows:

$$\mathbb{E}(D_{xy})=\sum_{k=0}^{\infty}\frac{D_{xy}^k}{(q;q)_k}.$$

Also they introduced the homogeneous Rogers–Szegö polynomials as

$$h_n(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(x,y), \tag{1.15}$$

with the generating function

$$\sum_{n=0}^{\infty}h_n(x,y|q)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(t,xt;q)_\infty},\quad \max\{|t|,|xt|\}<1. \tag{1.16}$$

The Rogers–Szegö polynomials $h_n(x|q)$ are special case of $h_n(x,y|q)$ when y is set to zero.

In 2007, Chen et. al. [12] used the q-exponential operator $T(bD_q)$ and the homogeneous q-shift operator $\mathbb{E}(D_{xy})$ to recover Mehler's formula and derive the Rogers formula for the homogeneous Rogers–Szegö polynomials $h_n(x,y|q)$.

Theorem 1.2 [12] *Mehler's formula for* $h_n(x,y|q)$ *is as follows:*

$$\sum_{n=0}^{\infty}h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n}=\frac{(yt,xvt;q)_\infty}{(t,xt,xut;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}y,xt,v/u\\yt,xvt\end{matrix};q,ut\right), \tag{1.17}$$

where $\max\{|t|,|xt|,|ut|,|xut|\}<1$.
The Roger's formula for $h_n(x,y|q)$ *is as follows*:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}=\frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}y,xs\\ys\end{matrix};q,t\right), \tag{1.18}$$

where $\max\{|t|,|s|,|xt|,|xs|\}<1$.

In 2008, Chen and Gu [10] introduced the Cauchy operator for basic hypergeometric series as follows:

$$T(a,b;D_q)=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(bD_q)^n. \tag{1.19}$$

The Cauchy operator is the reminiscent of the Cauchy identity (1.2). Compared with the q-exponential operator $T(bD_q)$, the Cauchy operator (1.19) involves two parameters. Clearly, the q-exponential operator $T(bD_q)$ can be considered as a special case of the Cauchy operator (1.19) for $a=0$. They obtained the following operator identities, where its assumed that the Cauchy operator $T(a,b;D_q)$ acts on the parameter c:

Theorem 1.3 [10] *Let D_q and $T(a,b;D_q)$ be defined as in (1.12) and (1.19), respectively. Then*

$$T(a,b;D_q)\{c^n\}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(a;q)_k\,b^k\,c^{n-k}. \tag{1.20}$$

$$T(a,b;D_q)\left\{\frac{1}{(ct;q)_\infty}\right\}=\frac{(abt;q)_\infty}{(bt,ct;q)_\infty},\quad |bt|<1. \tag{1.21}$$

$$T(a,b;D_q)\left\{\frac{1}{(cs,ct;q)_\infty}\right\}=\frac{(abt;q)_\infty}{(bt,cs,ct;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}a,ct\\abt\end{matrix};q,bs\right),$$
$$\max\{|bs|,|bt|\}<1. \tag{1.22}$$

$$T(a,b;D_q)\left\{\frac{(cv;q)_\infty}{(cs,ct;q)_\infty}\right\}=\frac{(abs,cv;q)_\infty}{(bs,cs,ct;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,cs,v/t\\abs,cv\end{matrix};q,bt\right),$$
$$\max\{|bs|,|bt|\}<1. \tag{1.23}$$

They found that the Cauchy operator is suitable for studying the homogeneous Rogers–Szegö polynomials $h_n(x,y|q)$ because of the following fact:

$$\lim_{c\to1}T(y/x,x;D_q)\{c^n\}=h_n(x,y|q). \tag{1.24}$$

They used (1.24) to derive Mehler's formula and Rogers formula for $h_n(x,y|q)$.

In 2010, Saad and Sukhi [20] gave another formula for the homogeneous Rogers–Szegö polynomials $h_n(x,y|q)$ as

$$h_n(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(y;q)_k\,x^{n-k}. \tag{1.25}$$

In 2014, Zhou and Luo [23] obtained some new generating functions for q-Hahn polynomials and gave their proofs based on the homogeneous q-difference operator $\mathbb{E}(D_{xy})$. They obtained the Rogers formula for $\phi_n^{(a)}(x)$ as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi_{m+n}^{(a)}(x)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}=\frac{(axs;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}xa,xs\\axs\end{matrix};q,t\right), \tag{1.26}$$

where $\max\{|s|,|t|,|xs|,|xt|\}<1$.

Notice that

$$\phi_n^{(y/x)}(x)=h_n(x,y|q). \tag{1.27}$$

$$\phi_n^{(y)}(1/x)=x^{-n}h_n(x,y|q). \tag{1.28}$$

The homogeneous Rogers–Szegö polynomials $h_n(x,y|q)$ introduced by Chen et al. [11] are equivalent to the Hahn polynomials $\phi_n^{(a)}(x)$ which were first studied by

Hahn [15] and then by Al-Salam and Carlitz [5] and Zhou and Luo [23]. The results of Al-Salam and Carlitz [5] and Zhou and Luo [23] for the polynomials $\phi_n^{(a)}(x)$ can be used to get results for the polynomials $h_n(x, y|q)$ and vice versa. For example, setting $a = y/x$ in the generating function for $\phi_n^{(a)}(x)$ (1.10), we get the generating function for $h_n(x, y|q)$ (1.16). Also, setting $a = y$, $x \to 1/x$, $t \to xt$, $b = v/u$, $y = u$ in the Mehler's formula for $\phi_n^{(a)}(x)$ (1.11), we get Mehler's formula for $h_n(x, y|q)$ (1.17). Specifying $y = ax$ in the Rogers formula for $h_n(x, y|q)$ (1.18), we get Rogers formula for $\phi_n^{(a)}(x)$ (1.26).

In this paper, we use the Cauchy operator to represent the homogeneous Rogers–Szegö polynomials $h_n(x, y|q)$ in the form (1.25). We use our operator representation to give a short and simple operator proof for the basic identities for $h_n(x, y|q)$. Also we give some extensions for the basic identities. The technique of using operators in representing q-polynomials is more used as in [1–4,12,13,19–22].

2 Operator identities for the Cauchy operator

In this section, we derive some operator identities for the Cauchy operator $T(a, b; D_q)$. These identities are very important for proving the extended identities for the homogeneous Rogers–Szegö polynomials $h_n(x, y|q)$.

Theorem 2.1 *For $n \in N$, we have*

$$T(a, b; D_q)\left\{\frac{c^n}{(cs, ct; q)_\infty}\right\} = \frac{(abt; q)_\infty}{(bt, cs, ct; q)_\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, ct; q)_{j+l} (cs; q)_j}{(abt; q)_{j+l}\, (q; q)_l} \times b^{j+l}\, c^{n-j}\, s^l, \tag{2.1}$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$\begin{aligned} &T(a, b; D_q)\left\{\frac{c^n}{(cs, ct; q)_\infty}\right\} \\ &= \sum_{k=0}^{\infty} \frac{(a; q)_k\, b^k}{(q; q)_k} D_q^k \left\{\frac{c^n}{(cs, ct; q)_\infty}\right\} \\ &= \sum_{k=0}^{\infty} \frac{(a; q)_k\, b^k}{(q; q)_k} \sum_{j=0}^{k} q^{j(j-k)} \begin{bmatrix} k \\ j \end{bmatrix} D_q^j\{c^n\}\, D_q^{k-j} \left\{\frac{1}{(csq^j, ctq^j; q)_\infty}\right\} \\ &= \sum_{k=0}^{\infty} \frac{(a; q)_k\, b^k}{(q; q)_k} \sum_{j=0}^{k} q^{j(j-k)} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q; q)_n}{(q; q)_{n-j}} c^{n-j}\, D_q^{k-j} \left\{\frac{1}{(csq^j, ctq^j; q)_\infty}\right\} \\ &= \sum_{j=0}^{n} \sum_{k=0}^{\infty} \frac{(a; q)_{k+j}\, b^{k+j}}{(q; q)_k} q^{-jk} \begin{bmatrix} n \\ j \end{bmatrix} c^{n-j}\, D_q^k \left\{\frac{1}{(csq^j, ctq^j; q)_\infty}\right\} \end{aligned}$$

$$= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (a;q)_j\, b^j\, c^{n-j} \sum_{k=0}^{\infty} \frac{(aq^j;q)_k\,(bq^{-j})^k}{(q;q)_k}\, D_q^k \left\{ \frac{1}{(csq^j, ctq^j;q)_\infty} \right\}$$

$$= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (a;q)_j\, b^j\, c^{n-j}\, T(aq^j, bq^{-j}; D_q) \left\{ \frac{1}{(csq^j, ctq^j;q)_\infty} \right\}$$

$$= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (a;q)_j\, b^j\, c^{n-j}\, \frac{(abtq^j;q)_\infty}{(bt, csq^j, ctq^j;q)_\infty}\, {}_2\phi_1 \left(\begin{matrix} aq^j, ctq^j \\ abtq^j \end{matrix} ; q, bs \right)$$

$$= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (a;q)_j\, b^j\, c^{n-j}\, \frac{(abtq^j;q)_\infty}{(bt, csq^j, ctq^j;q)_\infty} \sum_{l=0}^{\infty} \frac{(aq^j, ctq^j;q)_l}{(abtq^j, q;q)_l}\, (bs)^l$$

$$= \frac{(abt;q)_\infty}{(bt, cs, ct;q)_\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, ct;q)_{j+l}(cs;q)_j}{(abt;q)_{j+l}\,(q;q)_l}\, b^{j+l}\, c^{n-j}\, s^l.$$

□

Setting $n = 0$ in (2.1), we get (1.22).

Setting $s = 0$ in (2.1), we get the following corollary:

Corollary 2.1 *For $n \in N$, we have*

$$T(a, b; D_q) \left\{ \frac{c^n}{(ct;q)_\infty} \right\} = \frac{(abt;q)_\infty}{(bt, ct;q)_\infty} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, ct;q)_j}{(abt;q)_j}\, b^j\, c^{n-j}, \tag{2.2}$$

where $|bt| < 1$.

Setting $n = 0$ in (2.2), we get (1.21).

Theorem 2.2 *For $n \in N$, we have*

$$T(a, b; D_q) \left\{ \frac{c^n\,(cv;q)_\infty}{(cs, ct;q)_\infty} \right\} = \frac{(abs, cv;q)_\infty}{(bs, cs, ct;q)_\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \times \frac{(a, cs;q)_{j+l}(ct;q)_j(v/t;q)_l}{(abs, cv;q)_{j+l}(q;q)_l} \times b^{j+l} c^{n-j} t^l, \tag{2.3}$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$T(a, b; D_q) \left\{ c^n \frac{(cv;q)_\infty}{(cs, ct;q)_\infty} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{(a;q)_k\, b^k}{(q;q)_k}\, D_q^k \left\{ c^n \frac{(cv;q)_\infty}{(cs, ct;q)_\infty} \right\}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{(a;q)_k\, b^k}{(q;q)_k} \sum_{j=0}^{k} q^{j(j-k)} \begin{bmatrix} k \\ j \end{bmatrix} D_q^j\{c^n\}\, D_q^{k-j} \left\{ \frac{(cvq^j;q)_\infty}{(csq^j, ctq^j;q)_\infty} \right\} \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (a;q)_j\, b^j\, c^{n-j} \sum_{k=0}^{\infty} \frac{(aq^j;q)_k\, (bq^{-j})^k}{(q;q)_k} D_q^k \left\{ \frac{(cvq^j;q)_\infty}{(csq^j, ctq^j;q)_\infty} \right\} \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (a;q)_j\, b^j\, c^{n-j}\, T(aq^j, bq^{-j}; D_q) \left\{ \frac{(cvq^j;q)_\infty}{(csq^j, ctq^j;q)_\infty} \right\} \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (a;q)_j\, b^j\, c^{n-j} \frac{(absq^j, cvq^j;q)_\infty}{(bs, csq^j, ctq^j;q)_\infty}\, {}_3\phi_2 \left(\begin{matrix} aq^j, csq^j, v/t \\ absq^j, cvq^j \end{matrix} ; q, bt \right) \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (a;q)_j\, b^j\, c^{n-j} \frac{(absq^j, cvq^j;q)_\infty}{(bs, csq^j, ctq^j;q)_\infty} \sum_{l=0}^{\infty} \frac{(aq^j, csq^j, v/t;q)_l}{(absq^j, cvq^j, q;q)_l} (bt)^l \\
&= \frac{(abs, cv;q)_\infty}{(bs, cs, ct;q)_\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, cs;q)_{j+l} (ct;q)_j (v/t;q)_l}{(abs, cv;q)_{j+l} (q;q)_l} b^{j+l} c^{n-j} t^l.
\end{aligned}
$$

□

Setting $v = 0$ in (2.3), we get (2.1). Setting $v = s = 0$ in (2.3), we get (2.2) and setting $n = 0$ in (2.3) we get (1.23).

Theorem 2.3 *We have*

$$
T(a, b; D_q) \left\{ \frac{1}{(cs, ct, cv;q)_\infty} \right\} = \frac{(abt;q)_\infty}{(bt, cs, ct, cv;q)_\infty} \sum_{k,j=0}^{\infty} \frac{(a, ct;q)_{k+j} (cs;q)_k}{(abt;q)_{k+j}} \times \frac{(bs)^j}{(q;q)_j} \frac{(bv)^k}{(q;q)_k}, \tag{2.4}
$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$
\begin{aligned}
&T(a, b; D_q) \left\{ \frac{1}{(cs, ct, cv;q)_\infty} \right\} \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n\, b^n}{(q;q)_n} \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \left\{ \frac{1}{(cv;q)_\infty} \right\} D_q^{n-k} \left\{ \frac{1}{(csq^k, ctq^k;q)_\infty} \right\} \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n\, b^n}{(q;q)_n} \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{v^k}{(cv;q)_\infty} D_q^{n-k} \left\{ \frac{1}{(csq^k, ctq^k;q)_\infty} \right\} \\
&= \frac{1}{(cv;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\, (bv)^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(aq^k;q)_n\, (bq^{-k})^n}{(q;q)_n} D_q^n \left\{ \frac{1}{(csq^k, ctq^k;q)_\infty} \right\}
\end{aligned}
$$

$$= \frac{1}{(cv;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(bv)^k}{(q;q)_k} T(aq^k, bq^{-k}; D_q) \left\{ \frac{1}{(csq^k, ctq^k; q)_\infty} \right\}$$
$$= \frac{1}{(cv;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(bv)^k}{(q;q)_k} \frac{(abtq^k;q)_\infty}{(bt, csq^k, ctq^k; q)_\infty} {}_2\phi_1\left(\begin{matrix} aq^k, ctq^k \\ abtq^k \end{matrix} ; q, bs \right)$$
$$= \frac{1}{(cv;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(bv)^k}{(q;q)_k} \frac{(abtq^k;q)_\infty}{(bt, csq^k, ctq^k; q)_\infty} \sum_{j=0}^{\infty} \frac{(aq^k, ctq^k; q)_j}{(abtq^k, q; q)_j} (bs)^j$$
$$= \frac{(abt;q)_\infty}{(bt, cs, ct, cv; q)_\infty} \sum_{k,j=0}^{\infty} \frac{(a, ct; q)_{k+j}\,(cs;q)_k}{(abt;q)_{k+j}} \frac{(bs)^j}{(q;q)_j} \frac{(bv)^k}{(q;q)_k}.$$

□

Setting $v = 0$ in (2.4), we get (1.22).

Theorem 2.4 *We have*

$$T(a, b; D_q) \left\{ \frac{(cv;q)_\infty}{(cs, ct, cu; q)_\infty} \right\} = \frac{(abs, cv; q)_\infty}{(bs, cs, ct, cu; q)_\infty} \times \sum_{k,j=0}^{\infty} \frac{(a, cs; q)_{k+j} (v/t; q)_j (ct; q)_k}{(abs, cv; q)_{k+j}} \frac{(bt)^j}{(q;q)_j} \times \frac{(bu)^k}{(q;q)_k}, \tag{2.5}$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$T(a, b; D_q) \left\{ \frac{(cv;q)_\infty}{(cs, ct, cu; q)_\infty} \right\}$$
$$= \sum_{n=0}^{\infty} \frac{(a;q)_n\, b^n}{(q;q)_n} \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \left\{ \frac{1}{(cu;q)_\infty} \right\} D_q^{n-k} \left\{ \frac{(cvq^k;q)_\infty}{(csq^k, ctq^k; q)_\infty} \right\}$$
$$= \sum_{n=0}^{\infty} \frac{(a;q)_n\, b^n}{(q;q)_n} \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{u^k}{(cu;q)_\infty} D_q^{n-k} \left\{ \frac{(cvq^k;q)_\infty}{(csq^k, ctq^k; q)_\infty} \right\}$$
$$= \frac{1}{(cu;q)_\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a;q)_{n+k}\, b^{n+k}}{(q;q)_k\,(q;q)_n} q^{-nk} u^k D_q^n \left\{ \frac{(cvq^k;q)_\infty}{(csq^k, ctq^k; q)_\infty} \right\}$$
$$= \frac{1}{(cu;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(bu)^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(aq^k;q)_n (bq^{-k})^n}{(q;q)_n} D_q^n \left\{ \frac{(cvq^k;q)_\infty}{(csq^k, ctq^k; q)_\infty} \right\}$$

$$
\begin{aligned}
&= \frac{1}{(cu;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(bu)^k}{(q;q)_k}\, T(aq^k, bq^{-k}; D_q) \left\{ \frac{(cvq^k;q)_\infty}{(csq^k, ctq^k;q)_\infty} \right\} \\
&= \frac{1}{(cu;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(bu)^k}{(q;q)_k} \frac{(absq^k, cvq^k;q)_\infty}{(bs, csq^k, ctq^k;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} aq^k, csq^k, v/t \\ absq^k, cvq^k \end{matrix} ; q, bt \right) \\
&= \frac{1}{(bs, cu;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(bu)^k}{(q;q)_k} \frac{(absq^k, cvq^k;q)_\infty}{(csq^k, ctq^k;q)_\infty} \\
&\quad \times \sum_{j=0}^{\infty} \frac{(aq^k, csq^k, v/t;q)_j}{(absq^k, cvq^k, q;q)_j} (bt)^j \\
&= \frac{(abs, cv;q)_\infty}{(bs, cs, ct, cu;q)_\infty} \sum_{k,j=0}^{\infty} \frac{(a, cs;q)_{k+j}\,(v/t;q)_j\,(ct;q)_k}{(abs, cv;q)_{k+j}} \frac{(bt)^j}{(q;q)_j} \frac{(bu)^k}{(q;q)_k}.
\end{aligned}
$$

□

Setting $v = 0$ and replace s with t in (2.5), we get (2.4). Setting $v = u = 0$ (2.5), we get identity (1.22).

Theorem 2.5 *We have*

$$
\begin{aligned}
&T(a, b; D_q) \left\{ \frac{(cv, cw;q)_\infty}{(cs, ct, cu, cz;q)_\infty} \right\} \\
&= \frac{(abs, cv, cw;q)_\infty}{(bs, cs, ct, cu, cz;q)_\infty} \sum_{j,k,l=0}^{\infty} \frac{(a, cs;q)_{j+k+l}(ct;q)_{k+l}(cu, w/z;q)_k (v/t;q)_j}{(abs, cv;q)_{j+k+l}(cw;q)_k} \\
&\quad \times \frac{(bt)^j}{(q;q)_j} \frac{(bz)^k}{(q;q)_k} \frac{(bu)^l}{(q;q)_l},
\end{aligned} \tag{2.6}
$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$
\begin{aligned}
&T(a, b; D_q) \left\{ \frac{(cv, cw;q)_\infty}{(cs, ct, cu, cz;q)_\infty} \right\} \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n\, b^n}{(q;q)_n} \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \\
&\quad \times \left\{ \frac{(cw;q)_\infty}{(cz;q)_\infty} \right\} D_q^{n-k} \left\{ \frac{(cvq^k;q)_\infty}{(csq^k, ctq^k, cuq^k;q)_\infty} \right\} \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n\, b^n}{(q;q)_n} \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} z^k\, (w/z;q)_k \frac{(cwq^k;q)_\infty}{(cz;q)_\infty} D_q^{n-k} \\
&\quad \times \left\{ \frac{(cvq^k;q)_\infty}{(csq^k, ctq^k, cuq^k;q)_\infty} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(cw;q)_\infty}{(cz;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(w/z;q)_k (bz)^k}{(cw;q)_k (q;q)_k} \sum_{n=0}^{\infty} \frac{(aq^k;q)_n (bq^{-k})^n}{(q;q)_n} \\
&\quad \times D_q^n \left\{ \frac{(cvq^k;q)_\infty}{(csq^k, ctq^k, cuq^k;q)_\infty} \right\} \\
&= \frac{(cw;q)_\infty}{(cz;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(w/z;q)_k (bz)^k}{(cw;q)_k (q;q)_k} T(aq^k, bq^{-k}; D_q) \\
&\quad \times \left\{ \frac{(cvq^k;q)_\infty}{(csq^k, ctq^k, cuq^k;q)_\infty} \right\} \\
&= \frac{(cw;q)_\infty}{(cz;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\,(w/z;q)_k (bz)^k}{(cw;q)_k (q;q)_k} \frac{(absq^k, cvq^k;q)_\infty}{(bs, csq^k, ctq^k, cuq^k;q)_\infty} \\
&\quad \times \sum_{l,j=0}^{\infty} \frac{(aq^k, csq^k;q)_{j+l}\,(v/t;q)_j\,(ctq^k;q)_l}{(absq^k, cvq^k;q)_{j+l}} \frac{(bt)^j}{(q;q)_j} \frac{(bu)^l}{(q;q)_l} \\
&= \frac{(abs, cv, cw;q)_\infty}{(bs, cs, ct, cu, cz;q)_\infty} \\
&\quad \times \sum_{j,k,l=0}^{\infty} \frac{(a, cs;q)_{j+k+l}\,(ct;q)_{k+l}\,(cu, w/z;q)_k\,(v/t;q)_j}{(abs, cv;q)_{j+k+l}\,(cw;q)_k} \\
&\quad \times \frac{(bt)^j}{(q;q)_j} \frac{(bz)^k}{(q;q)_k} \frac{(bu)^l}{(q;q)_l}.
\end{aligned}
$$

□

Setting $w = z = 0$ in (2.6), we get (2.5). Setting $v = w = z = 0$ and replace s with t in (2.6), we get (2.4) and setting $v = w = u = z = 0$ in (2.6), we get (1.22).

3 The basic identities for $h_n(x, y|q)$ and their extensions

In this section, we easily represent the homogeneous Rogers–Szegö polynomials $h_n(x, y|q)$ in the form (1.25) by the Cauchy operator as follows:

$$T(y, 1; D_q)\{x^n\} = h_n(x, y|q). \tag{3.1}$$

Our operator representation for $h_n(x, y|q)$ seems simpler than the one given in (1.24). So we use our operator representation to give a short and simple operator proof for the basic identities for $h_n(x, y|q)$. Also we give extensions for the basic identities.

Theorem 3.1 (The generating function for $h_n(x, y|q)$). *We have*

$$\sum_{n=0}^{\infty} h_n(x, y|q) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t, xt;q)_\infty}, \quad \max\{|t|, |xt|\} < 1. \tag{3.2}$$

Proof By using (3.1) and (1.21), we get

$$\sum_{n=0}^{\infty} h_n(x,y|q)\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty} T(y,1;D_q)\{x^n\}\frac{t^n}{(q;q)_n}$$
$$= T(y,1;D_q)\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\right\}$$
$$= T(y,1;D_q)\left\{\frac{1}{(xt;q)_\infty}\right\}$$
$$= \frac{(yt;q)_\infty}{(t,xt;q)_\infty}.$$

□

Theorem 3.2 (Extended generating function for $h_n(x,y|q)$) *We have*

$$\sum_{n=0}^{\infty} h_{n+k}(x,y|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(y,xt;q)_j}{(yt;q)_j}x^{k-j}, \tag{3.3}$$

where $\max\{|t|,|xt|\}<1$.

Proof

$$\sum_{n=0}^{\infty} h_{n+k}(x,y|q)\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty} T(y,1;D_q)\{x^{n+k}\}\frac{t^n}{(q;q)_n}$$
$$= T(y,1;D_q)\left\{x^k\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\right\}$$
$$= T(y,1;D_q)\left\{\frac{x^k}{(xt;q)_\infty}\right\}$$
$$= \frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(y,xt;q)_j}{(yt;q)_j}x^{k-j}.$$

□

Setting $k=0$ in (3.3), we get the generating function (3.2) for the homogeneous Rogers–Szegö polynomials.

Theorem 3.3 (Mehler's formula for $h_n(x,y|q)$) *We have*

$$\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n} = \frac{(yt,xvt;q)_\infty}{(t,xt,xut;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}y,xt,v/u\\ yt,xvt\end{matrix};q,ut\right), \tag{3.4}$$

where $\max\{|t|,|xt|,|ut|,|xut|\}<1$.

Proof

$$\begin{aligned}\sum_{n=0}^{\infty} h_n(x, y|q)h_n(u, v|q)\frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} T(y, 1; D_q)\{x^n\}\, h_n(u, v|q)\,\frac{t^n}{(q;q)_n}\\ &= T(y, 1; D_q)\left\{\sum_{n=0}^{\infty} h_n(u, v|q)\,\frac{(xt)^n}{(q;q)_n}\right\}\\ &= T(y, 1; D_q)\left\{\frac{(xvt;q)_\infty}{(xt, xut;q)_\infty}\right\}\\ &= \frac{(yt, xvt;q)_\infty}{(t, xt, xut;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} y, xt, v/u \\ yt, xvt\end{matrix}; q, ut\right).\end{aligned}$$

□

Setting $u = v$ in (3.4), we get the generating function (3.2) for the homogeneous Rogers–Szegö polynomials $h_n(x, y|q)$ and setting $y = v = 0$ in (3.4), we get Mehler's formula (1.7) for the classical Rogers–Szegö polynomials.

Theorem 3.4 (Extended Mehler's formula for $h_n(x, y|q)$) *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty} h_n(x, y|q)\, h_{n+k}(u, v|q)\frac{t^n}{(q;q)_n}\\ &= \frac{(uyt, vt;q)_\infty}{(t, ut, xut;q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\frac{(v, ut;q)_{j+l}\,(xut;q)_j\,(y/x;q)_l}{(uyt, vt;q)_{j+l}\,(q;q)_l}\, u^{k-j}\,(xt)^l,\end{aligned} \tag{3.5}$$

where $\max\{|t|, |xt|, |ut|, |xut|\} < 1$.

Proof

$$\begin{aligned}&\sum_{n=0}^{\infty} h_n(x, y|q)\, h_{n+k}(u, v|q)\frac{t^n}{(q;q)_n}\\ &= \sum_{n=0}^{\infty} h_n(x, y|q)\, T_u(v, 1; D_q)\,\{u^{n+k}\}\,\frac{t^n}{(q;q)_n}\\ &= T_u(v, 1; D_q)\left\{u^k\sum_{n=0}^{\infty} h_n(x, y|q)\,\frac{(ut)^n}{(q;q)_n}\right\}\\ &= T_u(v, 1; D_q)\left\{\frac{u^k\,(uyt;q)_\infty}{(ut, xut;q)_\infty}\right\}\end{aligned}$$

$$= \frac{(uyt, vt; q)_\infty}{(t, ut, xut; q)_\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(v, ut; q)_{j+l}\,(xut; q)_j\,(y/x; q)_l}{(uyt, vt; q)_{j+l}\,(q; q)_l}\, u^{k-j}\,(xt)^l.$$

□

Setting $k = 0$ in (3.5), we get Mehler's formula (3.4) for the homogeneous Rogers–Szegö polynomials.

Now we derive the Rogers formula for the homogeneous Rogers–Szegö polynomials.

Theorem 3.5 (The Rogers formula for $h_n(x, y|q)$) *We have*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(ys; q)_\infty}{(s, xs, xt; q)_\infty}\, {}_2\phi_1\left(\begin{matrix} y, xs \\ ys \end{matrix}; q, t \right), \quad (3.6)$$

where $\max\{|t|, |s|, |xt|, |xs|\} < 1$.

Proof

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\
&\quad = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T(y, 1; D_q) \left\{x^{n+m}\right\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\
&\quad = T(y, 1; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(xt)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(xs)^m}{(q; q)_m} \right\} \\
&\quad = T(y, 1; D_q) \left\{ \frac{1}{(xt, xs; q)_\infty} \right\} \\
&\quad = \frac{(ys; q)_\infty}{(s, xs, xt; q)_\infty}\, {}_2\phi_1\left(\begin{matrix} y, xs \\ ys \end{matrix}; q, t \right).
\end{aligned}$$

□

Setting $y = 0$ in (3.6), we get the Rogers formula (1.8) for the classical Rogers–Szegö polynomials.

Theorem 3.6 (Extended Rogers formula for $h_n(x, y|q)$) *We have*

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m+k}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} &= \frac{(yt; q)_\infty}{(t, xt, xs; q)_\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \\
&\quad \times \frac{(y, xt; q)_{j+l} (xs; q)_j}{(yt; q)_{j+l} (q; q)_l} x^{k-j} s^l, \quad (3.7)
\end{aligned}$$

where $\max\{|t|, |s|, |xt|, |xs|\} < 1$.

Proof

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m+k}(x,y|q)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}\\
&\quad=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(y,1;D_q)\{x^{n+m+k}\}\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}\\
&\quad=T(y,1;D_q)\left\{x^k\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(xs)^m}{(q;q)_m}\right\}\\
&\quad=T(y,1;D_q)\left\{\frac{x^k}{(xs,xt;q)_\infty}\right\}\\
&\quad=\frac{(yt;q)_\infty}{(t,xt,xs;q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\frac{(y,xt;q)_{j+l}(xs;q)_j}{(yt;q)_{j+l}\,(q;q)_l}\,x^{k-j}\,s^l.
\end{aligned}$$

□

Setting $k=0$ in (3.7), we get the Rogers formula (3.6) for the homogeneous Rogers–Szegö polynomials $h_n(x,y|q)$.

4 Some other extended identities for $h_n(x,y|q)$

By using the Cauchy operator $T(a,b;D_q)$, we give some other extended identities for the homogeneous Rogers–Szegö polynomials $h_n(x,y|q)$.

By using identity (2.3), we give the following extended identity for $h_n(x,y|q)$.

Theorem 4.1 *We have*

$$\begin{aligned}
&\sum_{k=0}^{\infty} h_{m+k}(x,y|q)\,h_{n+k}(u,v|q)\,\frac{t^k}{(q;q)_k}\\
&\quad=\frac{(vt,uyt;q)_\infty}{(t,ut,uxt;q)_\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{m}\begin{bmatrix}n\\ i\end{bmatrix}\begin{bmatrix}m\\ j\end{bmatrix}\frac{(v,ut;q)_{i+l}(uxt;q)_{i+j}(y/x;q)_l(y;q)_j}{(uyt;q)_{i+j+l}(vt;q)_{i+l}(q;q)_l}\\
&\quad\times x^{l+m-j}u^{n-i}(tq^j)^l,
\end{aligned}\tag{4.1}$$

where $\max\{|t|,|xt|,|ut|,|xut|\}<1$.

Proof

$$\begin{aligned}
&\sum_{k=0}^{\infty} h_{m+k}(x,y|q)\,h_{n+k}(u,v|q)\,\frac{t^k}{(q;q)_k}\\
&\quad=T_u(v,1;D_q)\left\{u^n\sum_{k=0}^{\infty}h_{m+k}(x,y|q)\,\frac{(ut)^k}{(q;q)_k}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= T_u(v,1;D_q)\left\{u^n \frac{(uyt;q)_\infty}{(ut,uxt;q)_\infty}\sum_{j=0}^{m}\begin{bmatrix} m\\ j\end{bmatrix}\frac{(y,uxt;q)_j}{(uyt;q)_j}x^{m-j}\right\}\\
&= \sum_{j=0}^{m}\begin{bmatrix} m\\ j\end{bmatrix}(y;q)_j\, x^{m-j}\, T_u(v,1;D_q)\left\{u^n \frac{(uytq^j;q)_\infty}{(ut,uxtq^j;q)_\infty}\right\}\\
&= \sum_{j=0}^{m}\begin{bmatrix} m\\ j\end{bmatrix}(y;q)_j x^{m-j}\frac{(vt,uytq^j;q)_\infty}{(t,ut,uxtq^j;q)_\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{n}\begin{bmatrix} n\\ i\end{bmatrix}\\
&\quad\times \frac{(v,ut;q)_{i+l}(uxtq^j;q)_i(y/x;q)_l}{(vt,uytq^j;q)_{i+l}(q;q)_l}u^{n-i}\,(xtq^j)^l\\
&= \frac{(vt,uyt;q)_\infty}{(t,ut,uxt;q)_\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{m}\begin{bmatrix} n\\ i\end{bmatrix}\begin{bmatrix} m\\ j\end{bmatrix}\\
&\quad\times \frac{(v,ut;q)_{i+l}(uxt;q)_{i+j}(y/x;q)_l(y;q)_j\, x^{l+m-j}u^{n-i}(tq^j)^l}{(uyt;q)_{i+j+l}(vt;q)_{i+l}(q;q)_l}.
\end{aligned}
$$

□

Setting $n=m=0$ in (4.1), we get Mehler's formula (3.4) for $h_n(x,y|q)$. Setting $m=0$, we get the extended Mehler's formula (3.5) for $h_n(x,y|q)$.

By using (2.4), we give the following identity for $h_n(x,y|q)$:

Theorem 4.2 *We have*

$$
\begin{aligned}
&\sum_{n,m,k=0}^{\infty} h_{n+m+k}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{v^k}{(q;q)_k}\\
&= \frac{(yt;q)_\infty}{(t,xt,xs,xv;q)_\infty}\sum_{i,j=0}^{\infty}\frac{(y,xt;q)_{i+j}(xs;q)_i}{(yt;q)_{i+j}}\frac{v^i}{(q;q)_i}\frac{s^j}{(q;q)_j},
\end{aligned}
\tag{4.2}
$$

where $\max\{|s|,|t|,|xs|,|xt|,|xv|\}<1$.

Proof

$$
\begin{aligned}
&\sum_{n,m,k=0}^{\infty} h_{n+m+k}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{v^k}{(q;q)_k}\\
&= \sum_{n,m,k=0}^{\infty} T(y,1;D_q)\{x^{n+m+k}\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{v^k}{(q;q)_k}\\
&= T(y,1;D_q)\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(xs)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(xv)^k}{(q;q)_k}\right\}\\
&= T(y,1;D_q)\left\{\frac{1}{(xt,xs,xv;q)_\infty}\right\}
\end{aligned}
$$

$$= \frac{(yt;q)_\infty}{(t,xt,xs,xv;q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(y,xt;q)_{i+j}(xs;q)_i}{(yt;q)_{i+j}}\frac{v^i}{(q;q)_i}\frac{s^j}{(q;q)_j}.$$

□

Setting $v = 0$ in (4.2), we get the Rogers formula (3.6) for $h_n(x, y|q)$.

The following identity for $h_n(x, y|q)$ will be derived by using (2.5) as follows:

Theorem 4.3 *We have*

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} h_{m+k}(x,y|q)\,h_{n+k}(u,v|q)\,\frac{t^m}{(q;q)_m}\frac{s^k}{(q;q)_k}$$
$$= \frac{(ys,xvs;q)_\infty}{(s,xs,xt,xus;q)_\infty}\sum_{l,j=0}^{\infty}\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}\frac{(y,xs;q)_{j+l}(xus;q)_{i+l}(v/u;q)_j(v;q)_i}{(ys;q)_{j+l}(xvs;q)_{i+j+l}(q;q)_j(q;q)_l}$$
$$\times\, u^{j+n-i}(sq^i)^j t^l, \tag{4.3}$$

where $\max\{|s|, |xs|, |us|, |xt|, |xus|\} < 1$.

Proof

$$\sum_{m,k=0}^{\infty} h_{m+k}(x,y|q)\,h_{n+k}(u,v|q)\,\frac{t^m}{(q;q)_m}\frac{s^k}{(q;q)_k}$$
$$= T_x(y,1;D_q)\left\{\sum_{m=0}^{\infty}\frac{(xt)^m}{(q;q)_m}\sum_{k=0}^{\infty}h_{n+k}(u,v|q)\frac{(xs)^k}{(q;q)_k}\right\}$$
$$= \sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}(v;q)_i\,u^{n-i}\;T_x(y,1;D_q)\left\{\frac{(xvsq^i;q)_\infty}{(xs,xusq^i,xt;q)_\infty}\right\}$$
$$= \sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}(v;q)_i\,u^{n-i}\frac{(ys,xvsq^i;q)_\infty}{(s,xs,xusq^i,xt;q)_\infty}$$
$$\times\sum_{l,j=0}^{\infty}\frac{(y,xs;q)_{j+l}\,(v/u;q)_j\,(xusq^i;q)_l t^l(usq^i)^j}{(ys,xvsq^i;q)_{j+l}(q;q)_l(q;q)_j}$$
$$= \frac{(ys,xvs;q)_\infty}{(s,xs,xt,xus;q)_\infty}$$
$$\times\sum_{l,j=0}^{\infty}\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}\frac{(y,xs;q)_{j+l}(xus;q)_{i+l}(v/u;q)_j(v;q)_i}{(ys;q)_{j+l}(xvs;q)_{i+j+l}(q;q)_j(q;q)_l}$$
$$\times\, u^{j+n-i}(sq^i)^j t^l.$$

□

Setting $n = t = 0$ in (4.3), we get Mehler's formula (3.4) for $h_n(x, y|q)$.

Theorem 4.4 *We have*

$$\sum_{n,m=0}^{\infty} h_{n+m}(x,y|q)\,h_n(u,v|q)\,h_m(z,w|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\frac{(yt,xvt,xws;q)_\infty}{(t,xt,xut,xs,xzs;q)_\infty}$$
$$\times\sum_{j,k,l=0}^{\infty}\frac{(y,xt;q)_{j+k+l}(xut;q)_{k+l}(xs,w/z;q)_k(v/u;q)_j(ut)^j(zs)^k s^l}{(yt,xvt;q)_{j+k+l}(xws;q)_k(q;q)_j(q;q)_k(q;q)_l}, \tag{4.4}$$

where $\max\{|t|,|xt|,|ut|,|xut|,|xs|,|xzs|\}<1$.

Proof

$$\sum_{n,m=0}^{\infty} h_{n+m}(x,y|q)\,h_n(u,v|q)\,h_m(z,w|q)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$
$$=T_x(y,1;D_q)\left\{\sum_{n=0}^{\infty}h_n(u,v|q)\frac{(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}h_m(z,w|q)\frac{(xs)^m}{(q;q)_m}\right\}$$
$$=T_x(y,1;D_q)\left\{\frac{(xvt,xws;q)_\infty}{(xt,xut,xs,xzs;q)_\infty}\right\}$$
$$=\frac{(yt,xvt,xws;q)_\infty}{(t,xt,xut,xs,xzs;q)_\infty}$$
$$\times\sum_{j,k,l=0}^{\infty}\frac{(y,xt;q)_{j+k+l}(xut;q)_{k+l}(xs,w/z;q)_k(v/u;q)_j(ut)^j(zs)^k s^l}{(yt,xvt;q)_{j+k+l}(xws;q)_k(q;q)_j(q;q)_k(q;q)_l}.$$

□

Setting $s=0$ in (4.4), we get Mehler's formula (3.4) for $h_n(x,y|q)$.

Finally, we derive the following identity for $h_n(x,y|q)$ by using (1.22) and (2.5):

Theorem 4.5 *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}h_{n+k}(x,y|q)\,h_{m+k}(u,v|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{w^k}{(q;q)_k}$$
$$=\frac{(yw,xvw;q)_\infty}{(w,us,xt,xw,xuw;q)_\infty}$$
$$\times\sum_{i,j,l=0}^{\infty}\frac{(y,xw;q)_{i+l}(xuw;q)_{i+j}(v;q)_j(v/u;q)_l}{(yw;q)_{i+l}(xvw;q)_{i+j+l}(q;q)_i(q;q)_j(q;q)_l}t^i s^j (uwq^j)^l, \tag{4.5}$$

where $\max\{|s|,|w|,|us|,|xw|,|uw|,|xt|,|xuw|\}<1$.

Proof

$$
\begin{aligned}
&\sum_{n,m,k=0}^{\infty} h_{n+k}(x,y|q)\, h_{m+k}(u,v|q)\, \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \frac{w^k}{(q;q)_k} \\
&= \sum_{n,m,k=0}^{\infty} T_x(y,1;D_q)\{x^{n+k}\}\, T_u(v,1;D_q)\{u^{m+k}\} \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \frac{w^k}{(q;q)_k} \\
&= T_x(y,1;D_q)\, T_u(v,1;D_q) \left\{ \sum_{n=0}^{\infty} \frac{(xt)^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(us)^m}{(q;q)_m} \sum_{k=0}^{\infty} \frac{(xuw)^k}{(q;q)_k} \right\} \\
&= T_x(y,1;D_q) \left\{ \frac{1}{(xt;q)_\infty}\, T_u(v,1;D_q) \left\{ \frac{1}{(us,uxw;q)_\infty} \right\} \right\} \\
&= T_x(y,1;D_q) \left\{ \frac{1}{(xt;q)_\infty} \frac{(xvw;q)_\infty}{(xw,us,xuw;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} v, xuw \\ xvw \end{matrix} ; q, s \right) \right\} \\
&= \frac{1}{(us;q)_\infty} \sum_{j=0}^{\infty} \frac{(v;q)_j\, s^j}{(q;q)_j}\, T_x(y,1;D_q) \left\{ \frac{(xvwq^j;q)_\infty}{(xw,xuwq^j,xt;q)_\infty} \right\}.
\end{aligned}
$$

Applying (2.5), we get the desired identity. □

Setting $t = s = 0$ in (4.5), we get Mehler's formula (3.4) for $h_n(x,y|q)$.

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