A Study of The Cohomology of The Orlik-Solomon Algebra As a Free Module For a Generic Arrangement

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Abstract:

This paper is devoted to study the cohomology of the Orlik-Solomon algebra $A(\mathcal{A})$ as a free module for an ℓ-generic r-arrangement $\mathcal{A}$ and $a = a_i - a_j$, $1 \leq i < j \leq \ell$. In particular, the dimension of $H^k(A(\mathcal{A}), a)$ has been determined for every $1 \leq k \leq r-1$.

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1. Introduction:

The first appearance of the study of hyperplane arrangements was by the researcher S. Robert in (1889, [1]). He defined the arrangement $\mathcal{A}$ as a finite set of lines in the real plane and the number of remaining areas of the plane after deleting those lines was calculated. By a hyperplane H in a finite dimensional vector space V over a field $K = \mathbb{R}$ or $\mathbb{C}$, we mean an affine subspace of codimension one (that is $\dim H = \dim V - 1$) and by a hyperplane arrangement $\mathcal{A}$, (or for shorten (an arrangement)) is a finite collection of hyperplanes in V. One of the most essential problems in the topological study in the field of arrangements is the determination of the topological invariants of the complement of an arrangement, $\mathrm{M}(\mathcal{A}) = \mathrm{V}\setminus \bigcup_{H\in\mathcal{A}} H$, in terms of combinatorics, i.e. those invariants, that can be determined by using the intersection lattice $L(\mathcal{A}) = \{X \subseteq V | X = \bigcap_{H\in B} H \text{ and } B \subseteq \mathcal{A}\}$ only, which is partially ordered by: $X \leq Y \Leftrightarrow Y \subseteq X$, that ordered the objects of $L(\mathcal{A})$ opposite of inclusion. As a best general reference here, we refer the reader to [2].

E. Fadell, R. Fox and L. Neuwirth, in (1962, [3] and [4]), studied the cohomological group of the complement of an arrangement in complex space. Perhaps, the first non-trivial result is due to Brieskorn, Orlik and Solomon who calculated the cohomology algebra of the complement in terms of generators and relations (see [5] and [6]). Orlik and Solomon theorem's state that the cohomology algebra of the complement of a complex hyperplane arrangement is isomorphic to an algebra (that named by their names and denoted by O-S algebra) that is combinatorial determined, in line for [6].

This paper is interested with Hattori class of arrangements (1975, [7]) as a subclass of the class of a hypersolvable arrangement which firstly introduced in (1998, [8]) and (2002, [9]) by Jambu and Papadima as a generalization of Stanley class of arrangements (1972, [10]). We emphasize that every hypersolvable arrangement $\mathcal{A}$ has a natural partition on the hyperplanes of $\mathcal{A}$, we call it a hypersolvable partition and denote it by HP due (2006, [11]),. As well as, a natural ordering was defined on the hyperplanes of $\mathcal{A}$ induced by a fixed HP, we call it a hypersolvable ordering of $\mathcal{A}$ and denoted it by $\trianglelefteq$. The advantage of giving any $\ell -$ generic $r -$arrangement $\mathcal{A}$, a fashion as a hypersolvable arrangement with HP $\Pi = (\Pi_1, \dots, \Pi_\ell)$ and exponent vector $w = (w_1, \dots, w_\ell) = (1, \cdots, 1)$, lies in the fact that we can use our knowledge about the structure of the O-S algebra $A(\mathcal{A})$ as a free module with the NBC monomial basis that related to the section of Π.

A subarrangement $C \subseteq \mathcal{A}$ is called a circuit if it is a minimal dependent subarrangement of $\mathcal{A}$. If H is the smallest hyperplane in C via a total ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, then $\bar{C} = C\setminus\{H\}$ is called a broken circuit of a circuit C. By NBC base B of $\mathcal{A}$, we mean a subarrangement of $\mathcal{A}$ which contains no broken circuit and such subarrangements must be independent and denoted by *k-NBC* base if $|B| = k$. Let $\mathcal{A}$ be an $\ell -$ generic $r -$arrangement and for $1 \leq k \leq r$, $NBC_k(\mathcal{A})$ be the set of all *k-NBC* bases of $\mathcal{A}$, then;

$$|NBC_k(\mathcal{A})| = \binom{\ell}{k} \text{ and } |NBC_r(\mathcal{A})| = \binom{\ell}{r} - \binom{\ell-1}{r} = \binom{\ell-1}{r-1}, \text{[11, 12].}$$

Moreover, for any partition $\Pi = (\Pi_1, \dots, \Pi_\ell)$ on the hyperplanes of an arrangement $\mathcal{A}$, we recall the definition of the partition $\mathcal{K}$-module to be $(\Pi)_* = (\Pi_1)_* \otimes \dots \otimes (\Pi_\ell)_*$, where for $1 \le k \le \ell$, $(\Pi_k)_*$ is the free $\mathcal{K}$-module with basis 1 and the elements of Π_k [2]. Similarly, we recall a fashion of the Orlik-Solomon algebra $A(\mathcal{A})$ as a free $\mathcal{K}$ −submodule of a hypersolvable arrangement $\mathcal{A}$, that was clarified by Al-Taai , Ali and Majeed (2009, [13]).

The study of the Orlik-Solomon algebra $A(\mathcal{A})$ is motivated by creating some topological invariants of the complement $\mathrm{M}(\mathcal{A})$ of an arrangement $\mathcal{A}$. For example, for a real r −arrangement in $\mathbb{R}^r$, the number of regions of the complement $\mathrm{M}(\mathcal{A})$ is, $\sum_{i=1}^{r} \dim A_i(\mathcal{A})$, see [14]. For a complex r −arrangement $\mathcal{A}$ in $\mathbb{C}^r$, the complement $\mathrm{M}(\mathcal{A})$ is a path-connected manifold which is a minimal CW space for some cases and for $1 \le k \le r$, $\dim A_k(\mathcal{A})$ represents the number of the $k-$ cells in each skeleton. Furthermore, for $a \in A_1$, one can define a local coefficient system $\mathcal{L}(a)$ and the connection between $H^*(\mathrm{M}(\mathcal{A}), \mathcal{L}(a))$ and the cohomology of the Orlik-Solomon algebra $H^*(A(\mathcal{A}); a)$ has been studied in many papers. In fact, there are many results relating to $\dim H^1(A(\mathcal{A}); a)$. In the case char $\mathcal{K} = 0$, it has been shown in [15] that $\dim H^1(A(\mathcal{A}); a)$ can be determined by a particular set of elements from $L(\mathcal{A})$. However, little is known about the higher dimension $\dim H^p(A(\mathcal{A}); a)$ for $p > 1$, [15, 16] . This paper contains four sections. The first one is devoted to introduce the preliminaries that we needed in this work. The motivation of each one of the other sections is to calculate the dimension of $H^k(A(\mathcal{A}); a)$ for $1 \le k \le r-1$.

2. PRELIMINARIES

In this section we briefly sketch the notions; O-S algebra, NBC module and partition module. Moreover, we will introduce a definition of the cohomology for O-S algebra. From now on, we will make assumption that $\mathcal{A} = \{H_1, \dots, H_n\}$ is an essential complex r-arrangement with an arbitrary total ordering $\trianglelefteq$ on the hyperplanes of it.

2.1 The Orlik-Solomon Algebra: [2]

Let $\{e_{H_1}, \dots, e_{H_n}\}$ be a set that one to one correspondence with $\mathcal{A}$ via the ordering $\trianglelefteq$ and let $\mathcal{K}$ be any commutative ring. The Orlik-Solomon algebra (or for shorten O-S algebra) $A_*(\mathcal{A})$ is defined to be the quotient of the exterior $\mathcal{K}$-algebra, $E_* = \wedge_{k\ge 0}(\bigoplus_{H\in\mathcal{A}} K e_H)$, by the homogeneous ideal $I_*(\mathcal{A})$ is generated by the relations, $\sum_{j=1}^{k}(-1)^{k-1} e_{H_{i_1}} \dots \widehat{e_{H_{i_j}}} \dots e_{H_{i_k}}$, for all $1 \le i_1 < \cdots < i_k \le n$ such that $\{H_{i_1}, \dots H_{i_k}\}$ is dependent subarrangement of $\mathcal{A}$, where the circumflex ˆ means $e_{H_{i_j}}$ is deleted.

Assume we have $\partial_*^E : E_* \to E_*$ to be a K-linear mapping defined as; $\partial_0^E(e_{\emptyset_\ell}) = 0$; $\partial_1^E(e_H) = 1$, for all $H \in \mathcal{A}$ and for $2 \le k \le r$, $\partial_k^E(e_C) = \sum_{j=1}^{k}(-1)^{k-1} e_{H_{i_1}} \dots \widehat{e_{H_{i_j}}} \dots e_{H_{i_k}}$, $C = \{H_{i_1}, \dots H_{i_k}\}$. Notice that (E_*, ∂_*^E) forms a chain complex, that is called an exterior complex and the chain complex $(A_*(\mathcal{A}), \partial_*^A)$ receives a structure as acyclic chain complex from the exterior complex (E_*, ∂_*^E), where $\partial_*^A = \psi_* \circ \partial_*^E$ and $\psi_* : E_* \to A_*(\mathcal{A})$ be the canonical chain map. The acyclic chain complex $(A_*(\mathcal{A}), \partial_*^A)$ is called the O-S complex.

2.2 The NBC module: [2]

For any commutative ring $\mathcal{K}$, the broken circuit module $\boldsymbol{NBC}_*(\mathcal{A})$ of the exterior $\mathcal{K}$-algebra E_* defined as; $\boldsymbol{NBC}_0(\mathcal{A}) = \mathcal{K}$ and for $1 \le k \le r$; $\boldsymbol{NBC}_k(\mathcal{A})$ is the free $\mathcal{K}$-module of E_k with (no broken circuit) monomials basis $\{e_C | C \in NBC_k(\mathcal{A})\}$, i.e.;

$$\boldsymbol{NBC}_k(\mathcal{A}) = \bigoplus_{C\in NBC_k(\mathcal{A})} \mathcal{K} e_C \text{ and } \boldsymbol{NBC}_*(\mathcal{A}) = \bigoplus_{k=0}^{r} \boldsymbol{NBC}_k(\mathcal{A}).$$

Notice that, the broken circuit subcomplex $(\boldsymbol{NBC}_*(\mathcal{A}), \partial_*^{NBC})$ inherits a structure as a cyclic chain complex from the exterior complex (E_*, ∂_*^E), where $\partial_*^{NBC} = \partial_*^E \circ i_*$ and $i_*: E_* \to \boldsymbol{NBC}_*(\mathcal{A})$ is the inclusion chain map. Moreover, the restriction of the canonical chain map $\psi_*: E_* \to \boldsymbol{A}_*(\mathcal{A})$ of the broken circuit module $\boldsymbol{NBC}_*(\mathcal{A})$, is a chain isomorphism, defined as;

$$\text{For } 1 \le k \le r,\ \psi_k(e_C) = e_C + I_k(\mathcal{A}) = a_C,\ C \in NBC_k(\mathcal{A}).$$

$$\begin{array}{ccccccccccccc}
0 \to & \boldsymbol{NBC}_r(\mathcal{A}) & \xrightarrow{\partial_r^{NBC}} & \boldsymbol{NBC}_{r-1}(\mathcal{A}) & \xrightarrow{\partial_{r-1}^{NBC}} \cdots \xrightarrow{\partial_2^{NBC}} & \boldsymbol{NBC}_1(\mathcal{A}) & \xrightarrow{\partial_1^{NBC}} & \boldsymbol{NBC}_0(\mathcal{A}) & \xrightarrow{\partial_0^{NBC}} 0 \\
& \psi_r \downarrow & & \psi_{r-1} \downarrow & & \psi_1 \downarrow & & \psi_0 \downarrow & \\
0 \to & A_r(\mathcal{A}) & \xrightarrow{\partial_r^{A}} & A_{r-1}(\mathcal{A}) & \xrightarrow{\partial_{r-1}^{A}} \cdots \xrightarrow{\partial_2^{A}} & A_1(\mathcal{A}) & \xrightarrow{\partial_1^{A}} & A_0(\mathcal{A}) & \xrightarrow{\partial_0^{A}} 0
\end{array}$$

Thus the O-S algebra has the following structure as a free $\mathcal{K}$-module:

$$A_*(\mathcal{A}) = \bigoplus_{k=0}^{r} A_k(\mathcal{A}) = \bigoplus_{k=0}^{r} \left(\bigoplus_{C\in NBC_k(\mathcal{A})} K a_C\right).$$

We emphasize that, the Poincare polynomials of an arrangement $\mathcal{A}$ and the O-S algebra $A_*(\mathcal{A})$ are equal, i.e. $P(\mathcal{A}, t) = P(A_*(\mathcal{A}), t)$. Therefore, for $1 \le k \le r$, the k^{th} Betti number b_k of the Poincare polynomial $P(\mathcal{A}, t)$ will be $b_k = |NBC_k(\mathcal{A})|$.

2.3 Partition module: [2]

Let $\Pi = (\Pi_1, \dots, \Pi_\ell)$ be a partition on an r-arrangement $\mathcal{A}$ and let $\mathcal{K}$ be any commutative ring. A partition $\mathcal{K}$-module is defined to be $(\Pi)_* = (\Pi_1)_* \otimes \dots \otimes (\Pi_\ell)_*$, where for $1 \le k \le \ell$, $(\Pi_k)_*$ is the free $\mathcal{K}$-module with basis 1 and the elements of Π_k. For each $B = \{H_{i_1}, \dots H_{i_k}\} \in S_k(\Pi)$ the set of all k −sections of Π, i.e. $H_{i_m} \in \Pi_{i_m}$, $1 \le i_1 < \cdots < i_k \le \ell$ and $1 \le m \le k$, define; $q_B = x_1 \otimes \dots \otimes x_\ell \in (\Pi)_*$ as;

$$x_j = \begin{cases} H_j & if\ j = i_m \text{ for some } 1 \le m \le k \\ 1 & if\ j \ne i_m \text{ for all } 1 \le m \le k \end{cases}$$

We agree that each of $q_{\emptyset_\ell} = 1 \otimes \dots \otimes 1$ and q_B is homogeneous of degree k. Denote $(\Pi)_k$ the k^{th}-homogeneous part of $(\Pi)_*$. Therefore,

$$(\Pi)_* = \bigoplus_{k=0}^{\ell} (\Pi)_k = \bigoplus_{k=0}^{\ell} \left(\bigoplus_{B\in S_k(\Pi)} \mathcal{K} q_B\right)$$

and $\{q_B | B \in S_k(\Pi)\}$ forms a basis for the free $\mathcal{K}$-module $(\Pi)_*$. Furthermore, $\{q_{\{H\}} | H \in \Pi_k\}$ forms a basis for the free $\mathcal{K}$-module $(\Pi_k)_*$, $1 \le k \le \ell$. Define a $\mathcal{K}$-linear mapping $\partial_*^\Pi: (\Pi)_* \to (\Pi)_*$ as; $\partial_0^\Pi(q_{\{\}}) = 0$, $\partial_1^\Pi(q_H) = 1$, for all $H \in \mathcal{A}$ and for $2 \le k \le \ell$, $\partial_k^\Pi(q_B) = \sum_{j=1}^{k} (-1)^{k-1} \widehat{q_B}_j$, where $B = \{H_{i_1}, \dots H_{i_k}\} \in S_k(\Pi)$, $q_B = x_1 \otimes \dots \otimes x_\ell$, and $\widehat{q_B}_j = x_1 \otimes \dots \otimes \widehat{H_{\iota_j}} \otimes \dots \otimes x_\ell$ by means of $\widehat{H_{\iota_j}} = 1$. ∂_*^π is a differentiation on $(\Pi)_*$ and the chain complex $((\Pi)_*, \partial_*^\pi)$ is called the partition complex. For $1 \le k \le \ell$, define the a map

$\tilde{\varphi}_k: \{q_B | B \in S_k(\Pi)\} \to A_*(\mathcal{A})$, as $\varphi_k(q_B) = a_B = e_B + I_k(\mathcal{A})$, $B \in S_k(\Pi)$. Let $\varphi_k: (\Pi)_k \to A_k(\mathcal{A})$ be the unique $\mathcal{K}$-linear map that extend this assignment. Accordingly, there is a unique K-chain mapping $\varphi_*: (\Pi)_* \to A_*(\mathcal{A})$ between acyclic chain complexes.

2.4 Theorem [13]

Let $\mathcal{A}$ be an $\ell-$ generic $r-$arrangement. Then;

1. for $1 \le k \le r-1$, be $NBC_k(\mathcal{A}) = S_k(\Pi)$ and $NBC_r(\mathcal{A}) = S_r(\Pi)\backslash (S_r(\Pi) \cap BC_r(\mathcal{A}))$, where $BC_r(\mathcal{A})$is the set of all $r-$broken circuits of $\mathcal{A}$ and;
2. The O-S algebra has a fashion as a submodule of the partition module (Π) with, $b_k(A(\mathcal{A})) = |S_k(\Pi)| = \binom{\ell}{k}$ and $b_r(A(\mathcal{A})) = \binom{\ell-1}{r-1}$.

2.5 Cohomology of O-S algebra: [16]

Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. Multiplication by a giving the differential $d_k: A_k(\mathcal{A}) \xrightarrow{a\cdot} A_{k+1}(\mathcal{A})$ forms a complex $(A(\mathcal{A}), a)$. The cohomology of this complex is said to be a cohomology of the O-S algebra and it is denoted by $H^*(A(\mathcal{A}), a)$**.**

2.6 Theorem: [16]

Let $\mathcal{A}$ be a central hyperplane arrangement. Let $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. If $\sum_{i=1}^{n} \lambda_i a_i \neq 0$, then $H^*(A(\mathcal{A}), a) = 0$.

2: THE STRUCTURE OF $H^1(A(\mathcal{A}), a)$:

From now on we make the assumption that, $\mathcal{A}$ will be an $\ell-$generic $r-$arrangement with HP $\Pi = (\Pi_1, \dots, \Pi_\ell)$ and exponent vector $w = (1, \cdots, 1)$ and $\ell(\mathcal{A}) = |\mathcal{A}|$.

3.1 Lemma:

If $a = a_1 - a_i$, for $2 \le i \le \ell$, then:

1. $d_1(a_1) = d_1(a_i) = a_1 a_i$ and;
2. $\dim(Im\, d_1) = \ell - 1$.

Proof:

Firstly, we study the homomorphism, $d_1: A_1(\mathcal{A}) \xrightarrow{a} A_2(\mathcal{A})$. Since $\mathcal{A}$ is $\ell-$generic $r-$arrangement, then $E_1 = \langle e_1, e_2, \cdots, e_\ell \rangle \cong A_1(\mathcal{A})$.

For 1:

$$d_1(a_1) = a_1 a_1 - a_1 a_i = -a_1 a_i \text{ and;}$$
$$d_1(a_i) = a_i a_1 - a_i a_i = -a_1 a_i. \text{ So, } d_1(a_1) = d_1(a_i).$$

For 2:

To prove 2, we study $d_1(a_j)$, for $2 \le j \ne i \le \ell$. From the definition of $d_1: A_1 \to A_2$;

$$d_1(a_j) = a_j.a = a_j a_1 - a_j a_i = -a_1 a_j - a_j a_i = \begin{cases} -a_1 a_j - a_j a_i; & 1 < j < i \le \ell \\ -a_1 a_j + a_i a_j; & 1 < i < j \le \ell \end{cases} \dots (2.1.1).$$

Notice that, since $\mathcal{A}$ is $\ell-$generic $r-$arrangement, then;

$$E_2 = \langle e_{i_1} e_{i_2} \rangle | 1 \le i_1 < i_2 \le \ell \rangle \cong A_2(\mathcal{A}).$$

So, for $1 < i < j \leq \ell$;

$$\partial_2(a_1 a_i a_j) = a_i a_j - a_1 a_j + a_1 a_i \Rightarrow a_i a_j = \partial_2(a_1 a_i a_j) - a_1 a_i + a_1 a_j \ldots (2.1.2)$$

Substitute (2.1.1) in (2.1.2), we have;

$$d_1(a_j) = a_1 a_j - \partial_2(a_1 a_i a_j) + a_1 a_i - a_1 a_j = -\partial_2(a_1 a_i a_j) + a_1 a_i.$$

$$\text{Then } \partial_2(a_1 a_i a_j) = d_1(a_j) - a_1 a_i.$$

Since $e_1 e_i e_j$ is an NBC-monomial, hence $d_1(e_j) - e_1 e_i \notin I_2$; So, $d_1(a_j) - a_1 a_i \neq 0_{A_2}$. Then $d_1(a_j) \neq a_1 a_i$ and the number of such cases is $\ell - i$.

Similarly, for $1 < j < i \leq \ell$;

$$\partial_2(a_1 a_j a_i) = a_j a_i - a_1 a_i + a_1 a_j \neq 0_{A_2}. \text{ So,}$$

$$\partial_2(a_1 a_j a_i) = d_1(a_j) + a_1 a_i \neq 0$$

Since $e_1 e_j e_i$ is an NBC-monomial in E_2 hence $\partial_2(e_1 e_j e_i) \notin I_2$, then $d_1(a_j) \neq -a_1 a_i$ and the number of such cases is $i - 1$.

So, the number of monomial in the bases of $Im\ d_1$ is $\ell - 1$. ■

3.2 Lemma:

If $a = a_i - a_j$ for all $1 \leq i < j \leq \ell$, then;

1. $d_1(a_i) = d_1(a_j) = -a_i a_j$, and;
2. $\dim(Im\ d_1) = \binom{\ell-1}{1}$.

Proof:

For 1:

$$d_1(a_i) = a_i . a = a_i a_i - a_i a_j = -a_i a_j \text{ and;}$$

$$d_1(a_j) = a_j . a = a_j a_i - a_j a_j = -a_i a_j. \text{ Thus, } d_1(a_i) = d_1(a_j).$$

For 2:

Now, to find the $\dim(Im\ d_1)$, we looking closely to the following cases;

$$d_1(a_k) = a_k . a = a_k a_i - a_k a_j = \begin{cases} a_k a_i - a_k a_j; & 1 \leq k < i < j \leq \ell \\ -a_i a_k - a_k a_j; & 1 < i < k < j \leq \ell \\ -a_i a_k + a_j a_k; & 1 < i < j < k \leq \ell \end{cases} \ldots (2.2.1).$$

Since $E_2 = \langle e_{i_1} e_{i_2} \rangle | 1 \leq i_1 < i_2 \leq \ell \rangle \cong A_2(\mathcal{A})$, hence $d_1(a_k)$ is written as a linear combination of NBC monomials. By studying all the cases above, we pointing that there are no $1 \leq k < l \leq \ell$ such that $d_1(a_k) = d_1(a_l)$ and by simple calculation of the number of the generators of $Im(d_1)$ we have $\dim(Im\ d_1) = \binom{\ell-1}{1}$.■

3.3 Proposition:

If $a = a_1 - a_j$, $2 \leq j \leq \ell$, then:

1. $d_2(a_1 a_j) = 0_{A_3}$;
2. $\dim(\ker d_2) = \dim(Im\ d_1)$;
3. $H^1(A(\mathcal{A}); a) = 0$;

4. If $r > 3$, then $\dim(Im\, d_2) = \binom{\ell}{2} - \binom{\ell-2}{1}$ and;
5. If $r = 3$, then $\dim(Im\, d_2) = \binom{\ell-2}{1}$.

Proof:

Firstly, we will study the homomorphism $d_2: A_2 \to A_3$ that is defined as;

$$d_2(a_{k_1}a_{k_2}) = a_{k_1}a_{k_2}a_1 - a_{k_1}a_{k_2}a_j, \text{ where } 2 \le j \le \ell \text{ and } 1 \le k_1 < k_2 \le \ell$$

We have many cases to study as follows;

Case (1):- For $k_1 = 1$ and $k_2 = j$,

$$d_2(a_1a_j) = a_1a_ja_1 - a_1a_ja_j = 0_{A_3}. \text{ Thus, } a_1a_j \in \ker d_2.$$

Case (2):- If $k_1 = 1$ and $1 < k_2 < j < \ell$, thus;

$$d_2(a_1a_{k_2}) = a_1a_{k_2}a_1 - a_1a_{k_2}a_j = -a_1a_{k_2}a_j.$$

As well as; $d_2(a_{k_2}a_j) = a_{k_2}a_ja_1 - a_{k_2}a_ja_j = a_1a_{k_2}a_j$,

i.e. $d_2(a_1a_{k_2}) = -d_2(a_{k_2}a_j) \neq 0_{A_3}$.

Case (3):- If $k_1 = 1$ and $1 < j < k_2 < \ell$, then;

$$d_2(a_1a_{k_2}) = a_1a_ja_{k_2}. \text{ Consequently, } d_2(a_ja_{k_2}) = a_1a_ja_{k_2}.$$

So, $d_2(a_1a_{k_2}) = d_2(a_ja_{k_2}) \neq 0_{A_3}$.

Case(4):- If $1 < k_1 < k_2 < j \le \ell$, then $d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} - a_{k_1}a_{k_2}a_j$ ...(2.3.1).

Since $\mathcal{A}$ is ℓ −generic r −arrangement, here we have two possible cases in this case, are listed below:

- If $r > 3$, we have $d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} - a_{k_1}a_{k_2}a_j \neq O_{A_3}$ is written as a linear combination of NBC- monomials, or;
- If $r = 3$, we have $\partial_4(e_1e_{k_1}e_{k_2}e_j) \in I_3$. Thus; $a_{k_1}a_{k_2}a_j - a_1a_{k_2}a_j + a_1a_{k_1}a_j - a_1a_{k_1}a_{k_2} = 0_{A_3}$. So, $a_{k_1}a_{k_2}a_j = a_1a_{k_2}a_j - a_1a_{k_1}a_j + a_1a_{k_1}a_{k_2}$ … (2.3.2). By substituting (2.3.2) in (2.3.1), we have $d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} - a_1a_{k_2}a_j + a_1a_{k_1}a_j - a_1a_{k_1}a_{k_2} = a_1a_{k_1}a_j - a_1a_{k_2}a_j \neq 0_{A_3}$ is written as a linear combination of NBC- monomials.

Case (5):- If $1 < k_1 < j < k_2 \le \ell$, then $d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} + a_{k_1}a_ja_{k_2}$ .. (2.3.3), in this case there are two possible cases as follows;

- If $r > 3$, then $e_{k_1}e_{k_2}e_j$ is an NBC-monomial of E_3, since $\{H_{k_1}, H_{k_2}, H_j\}$ is an NBC base of $\mathcal{A}$. Thus; $d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} + a_{k_1}a_ja_{k_2} \neq 0_{A_3}$ is written as a linear combination of NBC- monomials, or;
- If $r = 3$, we have $e_{k_1}e_{k_2}e_j$ is a broken circuit monomial of E_3 , since $\{H_{k_1}, H_{k_2}, H_j\}$ is a broken circuit of $\mathcal{A}$. We know that, $\partial_4(e_1e_{k_1}e_je_{k_2}) \in I_3$. Thus;

$$a_{k_1}a_ja_{k_2} - a_1a_ja_{k_2} + a_1a_{k_1}a_{k_2} - a_1a_{k_1}a_j = 0_{A_3} \dots (2.3.4).$$

By substituting (2.3.4) in (2.3.3), we have;

$$d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} + a_1a_ja_{k_2} - a_1a_{k_1}a_{k_2} + a_1a_{k_1}a_j = a_1a_{k_1}a_j + a_1a_ja_{k_2} \neq 0_{A_3};$$

is written as a linear combination of NBC- monomials.

Case (6):- If $1 < j < k_1 < k_2 < j \le \ell$, then;

$$d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} - a_ja_{k_1}a_{k_2} \dots (2.3.5),$$

We have two possible cases can be obtained from this case;

- If $r > 3$, we have $e_j\, e_{k_1}e_{k_2}$ is an NBC monomial of E_3, since $\{H_j, H_{k_1}, H_{k_2}\}$ is an NBC base of $\mathcal{A}$. Thus, $d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} - a_ja_{k_1}a_{k_2} \neq 0_{A_3}$ is written as a linear combination of NBC- monomials, or;
- If $r = 3$, then $e_je_{k_1}e_{k_2}$ is a broken circuit monomial of E_3 , since $\{H_j, H_{k_1}, H_{k_2}\}$ is a broken circuit of $\mathcal{A}$. Thus; $\partial_4(e_1e_je_{k_1}e_{k_2}) \in I_3$ and;

$$a_ja_{k_1}a_{k_2} - a_1a_{k_1}a_{k_2} + a_1a_ja_{k_2} - a_1a_ja_{k_1} = 0_{A_3} \;\dots (2.3.6)$$

By substituting (2.3.6) in (2.3.5), we have, $d_2(a_{k_1}a_{k_2}) = -a_1a_ja_{k_2} + a_1a_ja_{k_1}$ is written as a linear combination of NBC- monomials.

Now, we clarify our assertion:

For 1:- It is clear from case (1).

For 2 and 3:- From our study of the 6th cases that we discussed above, we have just one $2-NBC$-monomial a_1a_j that satisfied; $d_2(a_1a_j) = 0_{A_3}$. But $a_1a_j \in Im\, d_1$ and since $Im\, d_1 \subseteq \ker\, d_2$, hence $Im\, d_1 = \ker\, d_2$. Which implies that $H^1(A(\mathcal{A}); a) = 0$.

For 4:- The number of the images of the NBC monomials that are need to remove it from the $\binom{\ell}{2} - NBC$ monomial came from case (1) and the repetition that came from cases (2) and (3) and the number of such cases is equal to $\ell - 2$. Thus;

$$\dim(Im\, d_2) = \binom{\ell}{2} - \ell + 2 = \binom{\ell}{2} - \binom{\ell-2}{1}.$$

For 5:- If $r = 3$, then $\dim(Im\, d_2) = \ell - 2 = \binom{\ell-2}{1}$ is the number of the $2-NBC$ monomial that begin with a_1 and contain no a_j. ■

3.4 Proposition:

If $a = a_i - a_j$, $1 < i < j < \ell$, then:-

1. $d_2(a_ia_j) = 0_{A_3}$;
2. $\ker d_2 = Im\, d_1$;
3. $H^1(A(\mathcal{A}); a) = 0$;
4. If $r > 3$, $\dim(Im\, d_2) = \binom{\ell}{2} - \binom{\ell-2}{1}$ and;
5. If $r = 3$, $\dim(Im\, d_2) = \binom{\ell-2}{1}$.

Proof:

We will study the homomorphism $d_2: A_2 \to A_3$ to serve our aim, as; $d_2(a_{k_1}a_{k_2}) = a_{k_1}a_{k_2}a_i - a_{k_1}a_{k_2}a_j$; where $1 \le k_1 < k_2 \le \ell$. We have the following possible cases for choosing $k_1$and k_2;

Case (1):- If $k_1 = i$ and $k_2 = j$, then $d_2(a_ia_j) = a_ia_ja_i - a_ia_ja_j = 0_{A_3}$. Thus, $a_ia_j \in \ker d_2$. In fact $a_ia_j \in \mathrm{Im}\, d_1$.

Case (2):- If $k_1 = i$ and $k_2 \neq j$ and $1 \le k_2 \le \ell$ then;

$$d_2(a_i a_{k_2}) = a_i a_{k_2} a_i - a_i a_{k_2} a_j = \begin{cases} a_{k_2} a_i a_j; & k_2 < i < j \\ -a_i a_{k_2} a_j; & i < k_2 < j \\ a_i a_j a_{k_2}\,; & i < j < k_2 \end{cases} = \begin{cases} -d_2(a_{k_2} a_j); & k_2 < i < j \\ -d_2(a_{k_2} a_j); & i < k_2 < j \\ -d_2(a_j a_{k_2}); & i < j < k_2 \end{cases}.$$

We have two possible cases can be obtained from this case;

- If $r > 3$, then $d_2(a_i a_{k_2})$ is written as a linear combination of 3-NBC monomials.
- If $r = 3$, as we know, either $\partial_4(e_1 e_{k_2} e_i e_j) \in I_3$, $\partial_4(e_1 e_i e_{k_2} e_j) \in I_3$ or $\partial_4(e_1 e_i e_j e_{k_2}) \in I_3$ depends of $k_2 < i < j$ or $i < k_2 < j$ or $i < j < k_2$ respectively. So;

$$\text{either, } a_{k_2} a_i a_j - a_1 a_i a_j + a_1 a_{k_2} a_j - a_1 a_{k_2} a_i = 0_{A_3} \text{ or;}$$
$$a_i a_{k_2} a_j - a_1 a_{k_2} a_j + a_1 a_i a_j - a_1 a_i a_{k_2} = 0_{A_3} \text{ or;}$$
$$a_i a_j a_{k_2} - a_1 a_j a_{k_2} + a_1 a_i a_{k_2} - a_1 a_i a_j = 0_{A_3}\text{ . Thus;}$$

$$d_2(a_i a_{k_2}) = \begin{cases} a_1 a_i a_j - a_1 a_{k_2} a_j + a_1 a_{k_2} a_i; & k_2 < i < j \\ -a_1 a_{k_2} a_j + a_1 a_i a_j - a_1 a_i a_{k_2}; & i < k_2 < j \\ a_1 a_j a_{k_2} - a_1 a_i a_{k_2} - a_1 a_i a_j; & i < j < k_2 \end{cases}$$
$$= \begin{cases} -d_2(a_1 a_{k_2}) - d_2(a_1 a_i); & k_2 < i < j \\ d_2(a_1 a_{k_2}) + d_2(a_1 a_i); & i < k_2 < j \\ d_2(a_1 a_{k_2}) + d_2(a_1 a_i); & i < j < k_2 \end{cases}.$$

is written as a linear combination of 3-NBC monomials.

Case (4):- If $k_1 \neq i$ or j, $k_2 \neq i$ or j and $1 \leq k_1 < k_2 \leq \ell$, we have;

$$d_2(a_{k_1} a_{k_2}) = a_{k_1} a_{k_2} a_i - a_{k_1} a_{k_2} a_j = \begin{cases} a_{k_1} a_{k_2} a_i - a_{k_1} a_{k_2} a_j; & k_1 < k_2 < i < j \\ -a_{k_1} a_i a_{k_2} - a_{k_1} a_{k_2} a_j; & k_1 < i < k_2 < j \\ a_i a_{k_1} a_{k_2} - a_{k_1} a_{k_2} a_j; & i < k_1 < k_2 < j \\ a_i a_{k_1} a_{k_2} + a_{k_1} a_j a_{k_2}; & i < k_1 < j < k_2 \\ a_i a_{k_1} a_{k_2} - a_j a_{k_1} a_{k_2}; & i < j < k_1 < k_2 \end{cases} \ldots$$

(2.4.1)

If $r > 3$, then any $3-$monomial $a_{i_1} a_{i_2} a_{i_3}$ is an NBC monomial and $d_2(a_{k_1} a_{k_2})$ is written as a linear combination of 3-NBC monomials.

- If $r = 3$, we have $\partial_4(a_1 a_{k_1} a_{k_2} a_j) = 0_{A_3}$, since $\{H_1, H_{k_1}, H_{k_2}, H_j\}$ is dependent. So, $a_{k_1} a_{k_2} a_j - a_1 a_{k_2} a_j + a_1 a_{k_1} a_j - a_1 a_{k_1} a_{k_2} = 0_{A_3}$ ... (2.4.2). So we have the following cases:-

Case (4.a):- If $k_1 < k_2 < i$, then $\partial_4(a_1 a_{k_1} a_{k_2} a_i) = 0_{A_3}$, since $\{H_1, H_{k_1}, H_{k_2}, H_i\}$ is dependent. So $a_{k_1} a_{k_2} a_i - a_1 a_{k_2} a_i + a_1 a_{k_1} a_i - a_1 a_{k_1} a_{k_2} = 0_{A_3}$.. (2.4.3). From the equations (2.4.1), (2.4.2) and (2.4.3) above we have;

$$d_2(a_{k_1} a_{k_2}) = a_1 a_{k_2} a_i - a_1 a_{k_1} a_i + a_1 a_{k_1} a_{k_2} - a_1 a_{k_2} a_j + a_1 a_{k_1} a_j - a_1 a_{k_1} a_{k_2} =$$
$$-d_2(a_1 a_{k_1}) + d_2(a_1 a_{k_2}).$$

Case (4.b):- If $k_1 < i < k_2$, then;

$$\partial_4(a_1 a_{k_1} a_i a_{k_2}) = a_{k_1} a_i a_{k_2} - a_1 a_i a_{k_2} + a_1 a_{k_1} a_{k_2} - a_1 a_{k_1} a_i = 0_{A_3} \text{ ... (2.4.4).}$$

Since $\{H_1, H_{k_1}, H_i, H_{k_2}\}$ is dependent. From the equations (2.4.1), (2.4.2) and (2.4.4) we have;

$$d_2(a_{k_1}a_{k_2}) = a_{k_1}a_ia_{k_2} - a_{k_1}a_{k_2}a_j = -a_1a_ia_{k_2} + a_1a_{k_1}a_{k_2} - a_1a_{k_1}a_i - a_{k_1}a_{k_2}a_j + a_1a_{k_1}a_j - a_1a_{k_1}a_{k_2} = -d_2(a_1a_{k_1}) + d_2(a_1a_{k_2}).$$

Case (4.c):- If $i < k_1 < k_2$, then,

$$\partial_4(a_1a_ia_{k_1}a_{k_2}) = a_ia_{k_1}a_{k_2} - a_1a_{k_1}a_{k_2} + a_1a_ia_{k_2} - a_1a_ia_{k_1} = 0_{A_3} \text{ ... (2.4.5).}$$

Since $\{H_1, H_i, H_{k_1}, H_{k_2}\}$ is dependent subarrangement. From the equations (2.4.1), (2.4.2) and (2.4.5) we have;

$$d_2(a_{k_1}a_{k_2}) = a_ia_{k_1}a_{k_2} - a_{k_1}a_{k_2}a_j$$
$$= a_1a_{k_1}a_{k_2} - a_1a_ia_{k_2} + a_1a_ia_{k_1} - a_1a_{k_2}a_j + a_1a_{k_1}a_j - a_1a_{k_1}a_{k_2}$$
$$= a_1a_ia_{k_1} + a_1a_{k_1}a_j - a_1a_ia_{k_2} - a_1a_{k_2}a_j = d_2(a_1a_{k_1}) + d_2(a_1a_{k_2}).$$

- If $r = 3$ and $k_1 = 1$, then $d_2(a_1a_{k_2}) = \begin{cases} a_1a_{k_2}a_i - a_1a_{k_2}a_j & if \quad 1 < i < k_2 < j \\ a_1a_{k_2}a_i + a_1a_ja_{k_2} & if \quad 1 < i < j < k_2 \end{cases}$ written as a linear combination of NBC- monomials.

Case (5):-

- If $r > 3$, we have

$$d_2(a_{k_1}a_{k_2}) = a_{k_1}a_{k_2}a_i - a_{k_1}a_{k_2}a_j = \begin{cases} a_ia_{k_1}a_{k_2} + a_{k_1}a_ja_{k_2} & i < k_1 < j < k_2 \\ a_ia_{k_1}a_{k_2} - a_ja_{k_1}a_{k_2} & i < j < k_1 < k_2 \end{cases} \text{...(2.4.6)}$$

- If $r = 3$, we need to write $d_2(a_{k_1}a_{k_2})$ as a linear combination of NBC monomials. Since $\{H_1, H_i, H_{k_1}, H_{k_2}\}$ is dependent, then;

$$\partial_4(a_1a_ia_{k_1}a_{k_2}) = a_ia_{k_1}a_{k_2} - a_1a_{k_1}a_{k_2} + a_1a_ia_{k_2} - a_1a_ia_{k_1} = 0_{A_3} \text{.. (2.4.7),}$$

we have two cases as;

Case (5.1):- If $k_1 < j < k_2$, then $\partial_4(a_1a_ia_{k_1}a_{k_2}) = 0_{A_3}$ since $\{H_1, H_{k_1}, H_j, H_{k_2}\}$ is dependent. Thus, $a_{k_1}a_ja_{k_2} - a_1a_ja_{k_2} + a_1a_{k_1}a_{k_2} - a_1a_{k_1}a_j = 0_{A_3}$.. (2.4.8).

From the above equations (2.4.6), (2.4.7) and (2.4.8) we have;

$$d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} - a_1a_ia_{k_2} + a_1a_ia_{k_1} + a_1a_ja_{k_2} - a_1a_{k_1}a_{k_2} + a_1a_{k_1}a_j$$
$$= a_1a_ia_{k_1} + a_1a_{k_1}a_j - a_1a_ia_{k_2} + a_1a_ja_{k_2} = -d_2(a_1a_{k_1}) + d_2(a_1a_{k_2}).$$

Case (5.2):- If $j < k_1 < k_2$, then $\partial_4(a_1a_ja_{k_1}a_{k_2}) = 0_{A_3}$... (2.4.9), since $\{H_1, H_j, H_{k_1}, H_{k_2}\}$ is dependent subarrangement. Thus, we have;

$$d_2(a_{k_1}a_{k_2}) = a_1a_{k_1}a_{k_2} - a_1a_ia_{k_2} + a_1a_ia_{k_1} - a_1a_{k_1}a_{k_2} + a_1a_ja_{k_2} - a_1a_ja_{k_1}$$
$$= a_1a_ia_{k_1} - a_1a_ja_{k_1} - a_1a_ia_{k_2} + a_1a_ja_{k_2} = -a_1a_{k_1}a_i + a_1a_{k_1}a_j + a_1a_{k_2}a_i - a_1a_{k_2}a_j$$
$$= -d_2(a_1a_{k_1}) + d_2(a_1a_{k_2}).$$

Now, we verify our claim as follows:-

For 1:- See case 1.

For 2:- From case(1), we have just one $\ell - NBC$ monomial satisfied $d_2(a_ia_j) = 0_{A_3}$, which is a coboundary. Since $a_ia_j \in Im\ d_1 \subseteq \ker d_2$, hence $\ker d_2 = Im\ d_1$.

For 3:- It is clear that, $H^1(A(\mathcal{A}); a) = 0$.

For 4:- If $r > 3$, the repetition that we discussed them in the cases 2 and 3 and from case 1 is $\ell - 2$; $\dim(Im\, d_2) = \binom{\ell}{2} - \ell + 2$.

For 5:- If $r = 3$, from our discussion of all cases. If $r = 3$, it is clear that the collection $B = \{d_2(a_1 a_{k_2}) : 2 \leq k_2 \leq \ell \text{ and } k_2 \neq i \text{ or } j\} \cup \{d_2(a_1 a_{k_1})\}$play as a basis for $Im\, d_2$, so; $\dim d_2 = |B| = (\ell - 3) + 1 = \ell - 2 = \binom{\ell-2}{1}$. ■

4.THE STRUCTURE OF $\mathbf{H^2(A(\mathcal{A}), a)}$ AND $\mathbf{H^3(A(\mathcal{A}), a)}$:

4.1 Corollary:

If $r = 3$ and $a = a_1 - a_j$ for $2 \leq j \leq \ell$; then $dim\, H^2(A(\mathcal{A}), a) = \binom{\ell-1}{2} - \binom{\ell-2}{1}$.

Proof:

Since $d_3 : A_3 \to 0$ is the zero homomorphism. Then $\ker d_3 = A_3$ and $\dim \ker d_3 = \binom{\ell-1}{2}$. Since $Im\, d_2 \subseteq \ker d_3$ and from proposition (2.3), we have;

$$dim\, H^2(A(\mathcal{A}), a) = \dim \ker d_3 - \dim Im\, d_2$$
$$= \dim A_3 - \dim Im\, d_2 = \binom{\ell-1}{2} - \ell + 2 = \binom{\ell-1}{2} - \binom{\ell-2}{1}. \ ■$$

4.2* Corollary*:

If $r = 3$ and $a = a_i - a_j$ for $1 < i < j \leq \ell$, we have $\dim H^2(A(\mathcal{A}), a) = \binom{\ell-1}{2} - \binom{\ell-2}{1}$.

Proof:

Since $d_3 : A_3 \to 0$ is the zero homomorphism, then $\ker d_3 = A_3$ and since $Im\, d_2 \subseteq \ker d_3$, then;

$$\dim H^2(A(\mathcal{A}), a) = \dim(\ker d_3 \,/\, Im\, d_2)$$
$$= \dim A_3 - \dim Im\, d_2 = \binom{\ell-1}{2} - \ell + 2 = \binom{\ell-1}{2} - \binom{\ell-2}{1} \ ■$$

4.3 Proposition:

If $r > 3$ and $a = a_1 - a_j$, $2 \leq j \leq \ell$. Then:-

1. $d_3(a_1 a_m a_j) = d_3(a_1 a_j a_m) = 0_{A_4}$, if either $1 < m < j \leq \ell$ or $1 < j < m \leq \ell$.
2. $\ker d_3 = Im\, d_2$.
3. $H^2(A(\mathcal{A}), a) = 0$.
4. If $r > 4$, $\dim(Im\, d_3) = \binom{\ell}{3} - \binom{\ell-2}{2}$.
5. If $r = 4$, $\dim(Im\, d_3) = \binom{\ell}{3} - \binom{\ell-2}{2}$.

Proof:

Firstly, we will study the homomorphism $d_3 : A_3 \to A_4$. As we know the NBC-monomial basis for A_3 is; $B_3 = \{a_{m_1} a_{m_2} a_{m_3} : \{H_{m_1}, H_{m_2}, H_{m_3}\} \subseteq \mathcal{A} \text{ and } 1 \leq m_1 < m_2 < m_3 \leq \ell\}$ and that will enable us to exercise d_3. So, $d_3(a_{m_1} a_{m_2} a_{m_3}) = a_{m_1} a_{m_2} a_{m_3} a_i - a_{m_1} a_{m_2} a_{m_3} a_j$. To serve our aim we will discuss all the possible cases.

__Case (1):-__ If $m_1 = 1$ and m_2 or $m_3 = j$. Then as we mentioned it previously, $d_3(a_1 a_j a_{m_3}) = d_3(a_1 a_{m_2} a_j) = 0_{A_4}$.

__Case (2):-__ If $m_1 = 1$ and $m_2, m_3 \neq j$ then;

$$d_3(a_1a_{m_2}a_{m_3}) = a_1a_{m_2}a_{m_3}a_i = \begin{cases} a_1a_{m_2}a_{m_3}a_j & m_2 < m_3 < j \le \ell \\ -a_1a_{m_2}a_ja_{m_3} & m_2 < j < m_3 \le \ell \\ a_1a_ja_{m_2}a_{m_3} & j < m_2 < m_3 \le \ell \end{cases}$$

$$= \begin{cases} -d_3(a_{m_2}a_{m_3}a_j) & m_2 < m_3 < j \le \ell \\ d_3(a_{m_2}a_ja_{m_3}) & m_2 < j < m_3 \le \ell \\ -d_3(a_ja_{m_2}a_{m_3}) & j < m_2 < m_3 \le \ell \end{cases} \ne 0_{A_4}.$$

Notice that $d_3(a_1a_{m_2}a_{m_3})$ is an $4-NBC$ monomial for any $r \ge 4$. The number of such repetitions is equal to the number of the $3-NBC$ monomials that begin with a_1 and the number of such monomials equal to the number of $2-NBC$ monomials that not begin with 1 and not equal to j. Thus we have $\binom{\ell-2}{2}$ repetition in this case.

Case (3):- If $1 \le m_1 < m_2 < m_3 \le \ell$ and $m_n \ne i$ or j for $n = 1,2,3$. Then we have the following cases:-

Case (3.a):- If $1 < m_1 < m_2 < m_3 < j \le \ell$, then;

$$d_3(a_{m_1}a_{m_2}a_{m_3}) = a_{m_1}a_{m_2}a_{m_3}a_1 - a_{m_1}a_{m_2}a_{m_3}a_j$$
$$= -a_1a_{m_1}a_{m_2}a_{m_3} - a_{m_1}a_{m_2}a_{m_3}a_j \text{ ... (3.3.1)}$$

- If $r > 4$, then $d_3(a_{m_1}a_{m_2}a_{m_3}) \ne 0_{A_4}$ can be written as a linear combination of $4-NBC$monomials.
- If $r = 4$, then we have $\partial_5(e_1e_{m_1}e_{m_2}e_{m_3}e_j) \in I_4$; $a_{m_1}a_{m_2}a_{m_3}a_j - a_1a_{m_2}a_{m_3}a_j + a_1a_{m_1}a_{m_3}a_j - a_1a_{m_1}a_{m_2}a_j + a_1a_{m_1}a_{m_2}a_{m_3} \ne 0_{A_4}$ .. (3.3.2)

  From the equations (3.3.1) and (3.3.2) above, we have;

$$d_3(a_{m_1}a_{m_2}a_{m_3}) = -a_1a_{m_1}a_{m_2}a_{m_3} - a_1a_{m_2}a_{m_3}a_j + a_1a_{m_1}a_{m_3}a_j$$
$$-a_1a_{m_1}a_{m_2}a_j + a_1a_{m_1}a_{m_2}a_{m_3} = -a_1a_{m_2}a_{m_3}a_j + a_1a_{m_1}a_{m_3}a_j - a_1a_{m_1}a_{m_2}a_j$$
$$= -d_3(a_1a_{m_2}a_{m_3}) + d_3(a_1a_{m_1}a_{m_3}) - d_3(a_1a_{m_1}a_{m_2}) \ne 0_{A_4}.$$

Case (3.b):- We have three cases:

Case (3.b.1):- If $1 < m_1 < m_2 < j < m_3 \le \ell$; then $d_3(a_{m_1}a_{m_2}a_{m_3}) = a_{m_1}a_{m_2}a_{m_3}a_1 - a_{m_1}a_{m_2}a_{m_3}a_j = -a_1a_{m_1}a_{m_2}a_{m_3} + a_{m_1}a_{m_2}a_ja_{m_3}$ .. (3.3.3).

- If $r > 3$, then $d_3(a_{m_1}a_{m_2}a_{m_3}) \ne 0_{A_4}$ can be written as a linear combination of $4-NBC$ −monomials.
- If $r = 3$, then we have $\partial_5(e_1e_{m_1}e_{m_2}e_je_{m_3}) \in I_4$. So;

  $a_{m_1}a_{m_2}a_ja_{m_3} - a_1a_{m_2}a_ja_{m_3} + a_1a_{m_1}a_ja_{m_3} - a_1a_{m_1}a_{m_2}a_{m_3} + a_1a_{m_1}a_{m_2}a_j = 0_{A_4}$.. (3.3.4). From (3.3.3) and (3.3.4) we have;

$$d_3(a_{m_1}a_{m_2}a_{m_3}) = -a_1a_{m_1}a_{m_2}a_{m_3} + a_1a_{m_2}a_ja_{m_3} - a_1a_{m_1}a_ja_{m_3} + a_1a_{m_1}a_{m_2}a_{m_3} - a_1a_{m_1}a_{m_2}a_j = a_1a_{m_2}a_ja_{m_3} - a_1a_{m_1}a_ja_{m_3} - a_1a_{m_1}a_{m_2}a_j = -d_3(a_1a_{m_2}a_{m_3}) + d_3(a_1a_{m_1}a_{m_3}) - d_3(a_1a_{m_1}a_{m_2}).$$

Case (3.b.2):- If $1 < m_1 < j < m_2 < m_3 \le \ell$, then;

$d_3(a_{m_1}a_{m_2}a_{m_3}) = a_{m_1}a_{m_2}a_{m_3}a_1 - a_{m_1}a_{m_2}a_{m_3}a_j = -a_1a_{m_1}a_{m_2}a_{m_3} - a_{m_1}a_ja_{m_2}a_{m_3}$ .. (3.3.5).

- If $r > 3$, then $d_3(a_{m_1}a_{m_2}a_{m_3}) \neq 0_{A_4}$ written as a linear combination of the $4-NBC$-monomials.
- If $r = 3$, then we have $\partial_5(e_1e_{m_1}e_je_{m_2}e_{m_3}) \in I_4$. So;
  $a_{m_1}a_ja_{m_2}a_{m_3} - a_1a_ja_{m_2}a_{m_3} + a_1a_{m_1}a_{m_2}a_{m_3} - a_1a_{m_1}a_ja_{m_3} + a_1a_{m_1}a_ja_{m_2} = 0_{A_4}$... (3.3.6). From (3.3.5) and (3.3.6), we have;
  $$d_3(a_{m_1}a_{m_2}a_{m_3}) = -a_1a_{m_1}a_{m_2}a_{m_3} - a_1a_ja_{m_2}a_{m_3} + a_1a_{m_1}a_{m_2}a_{m_3} - a_1a_{m_1}a_ja_{m_3} + a_1a_{m_1}a_ja_{m_2} = -a_1a_ja_{m_2}a_{m_3} - a_1a_{m_1}a_ja_{m_3} + a_1a_{m_1}a_ja_{m_2} = -d_3(a_1a_{m_2}a_{m_3}) + d_3(a_1a_{m_1}a_{m_3}) - d_3(a_1a_{m_1}a_{m_2}) \neq O_{A_4}.$$

Case (3.b.3):- If $1 < j < m_1 < m_2 < m_3 \leq \ell$, then $d_3(a_{m_1}a_{m_2}a_{m_3}) = a_{m_1}a_{m_2}a_{m_3}a_1 - a_{m_1}a_{m_2}a_{m_3}a_j = -a_1a_{m_1}a_{m_2}a_{m_3} + a_ja_{m_1}a_{m_2}a_{m_3}$ .. (3.3.7).

- If $r > 3$, $d_3(a_{m_1}a_{m_2}a_{m_3}) \neq O_{A_4}$ can be written as a linear combination of the $4-NBC$ monomials.
- If $r = 3$, then we have $\partial_5(e_1e_je_{m_1}e_{m_2}e_{m_3}) \in I_4$. So;
- $a_ja_{m_1}a_{m_2}a_{m_3} - a_1a_{m_1}a_{m_2}a_{m_3} + a_1a_ja_{m_2}a_{m_3} - a_1a_ja_{m_1}a_{m_3} + a_1a_ja_{m_1}a_{m_2} = O_{A_4}$... (3.3.8). From (3.3.7) and (3.3.8), we have;
  $$d_3(a_{m_1}a_{m_2}a_{m_3}) = -a_1a_{m_1}a_{m_2}a_{m_3} + a_1a_{m_1}a_{m_2}a_{m_3} - a_1a_ja_{m_2}a_{m_3} + a_1a_ja_{m_1}a_{m_3} + a_1a_ja_{m_1}a_{m_2} = -a_1a_ja_{m_2}a_{m_3} + a_1a_ja_{m_1}a_{m_3} - a_1a_ja_{m_1}a_{m_2} = -d_3(a_1a_{m_2}a_{m_3}) + d_3(a_1a_{m_1}a_{m_3}) - d_3(a_1a_{m_1}a_{m_2}) \neq O_{A_4}.$$

To prove our claim;

For (1): see case (1).

For (2): From case (1) above and lemma (3), we have $a_1a_ma_j, a_1a_ja_n \in \ker d_3$ and $a_1a_ma_j, a_1a_ja_n \in \operatorname{Im} d_2$ for all $11 < m < j < n \leq \ell$. Since $Im\, d_2 \subseteq \ker d_3$, so $Im\, d_2 = \ker d_3$.

For (3): Our claim is done from case (2); i.e. $H^2(A(\mathcal{A}), a) = 0$.

For (4): If $r > 4$, then from all cases that studied in our proof; we have $\dim Im\, d_3 = \binom{\ell}{3} - (\binom{\ell}{2} + 2\ell - 3) = \binom{\ell}{3}\binom{\ell-2}{2}$.

For (5): If $r = 4$, then from all the cases we studied above we have $\dim Im\, d_3 = \binom{\ell}{2} - (\ell-1) - (\ell-2) = \binom{\ell}{2} - 2\ell + 3 = \binom{\ell-2}{2}$, which is depend on $4-NBC$ bases that begins by a_1 that equal to the number $2-NBC$ bases which is not contains the number of $3-NBC$- monomial that it contained in $\ker d_3$, that is $d_3(a_1a_ja_m) = d_3(a_1a_ma_j) = 0_{A_4}$. ■

4.4 Corollary :

If $r = 4$ and $a = a_1 - a_j$, $2 \leq j \leq \ell$. Then, $\dim H^3\,(A(\mathcal{A}), a) = \binom{\ell-1}{3} - \binom{\ell-2}{2}$.

Proof:

Since $d_4 : A_4 \to 0$ is the zero homomorphism, then $\ker d_4 = A_4$ and $\dim \ker d_4 = \binom{\ell-1}{3}$. By applying Proposition (3.1), we have;

$$\dim H^3(A(\mathcal{A}),a) = \dim(\ker d_4/Im\, d_3) = \dim\ (\ker d_4) - \dim(Im\, d_3) = \binom{\ell-1}{3} - \binom{\ell}{2} + 2\ell - 3 = \binom{\ell-1}{3} - \binom{\ell-2}{2}. \blacksquare$$

4.5 Proposition:

If $r > 3$ and $a = a_i - a_j$, $1 < i < j \leq \ell$. Then

1. $d_3(a_{m_1}a_ia_j) = d_3(a_ia_{m_2}a_j) = d_3(a_ia_ja_{m_3}) = O_{A_4}$, for each $1 \leq m_1 < i < m_2 < j < m_3 \leq \ell$
2. $Im\, d_2 = \ker d_3$.
3. $H^2(A(\mathcal{A}),a) = 0$.
4. If $r > 3$, $\dim(Im\, d_3) = \binom{\ell}{3} - \binom{\ell-2}{2}$.
5. If $r = 3$, $\dim(Im\, d_3) = \binom{\ell-2}{2}$.
6. If $r = 3$, then $\dim H^3(A(\mathcal{A}),a) = \binom{\ell-1}{3} - \binom{\ell-2}{2}$.

Proof:

To prove our claim, we need to study the homomorphism $d_3: A_3 \to A_4$, which is defined by; $d_3(a_{m_1}a_{m_2}a_{m_3}) = a_{m_1}a_{m_2}a_{m_3}a_i - a_{m_1}a_{m_2}a_{m_3}a_j$, for all $3-NBC$-monomials $a_{m_1}a_{m_2}a_{m_3} \in A_3$, since the collection $\{a_{m_1}a_{m_2}a_{m_3}: \{H_{m_1}, H_{m_2}, H_{m_3}\} \in NBC_3(\mathcal{A})$, where $1 \leq m_1 < m_2 < m_3 \leq \ell\}$ play as a basis for A_3 as a K −module. So, we will discuss the following possible cases:-

Case (1):- If $\{m_1, m_2, m_3\} = \{i, j, m\}$, then we have;

$$d_3(a_{m_1}a_{m_2}a_{m_3}) = \begin{cases} d_3(a_ma_ia_j), & 1 \leq m < i < j \leq \ell \\ d_3(a_ia_ma_j), & 1 \leq i < m < j \leq \ell \\ d_3(a_ia_ja_m), & 1 \leq i < j < m \leq \ell \end{cases} = O_{A_4},$$

for this case, the number of such choices is $\binom{\ell-2}{1} = \ell - 2$.

Case (2):- If m_1 or m_2 or $m_3 = i$ and $\neq j$, then;

$$d_3(a_{m_1}a_{m_2}a_{m_3}) = \begin{cases} d_3(a_{m_1}a_{m_2}a_i); & 1 \leq m_1 < m_2 < i < j \leq \ell \\ d_3(a_{m_1}a_ia_{m_3}); & 1 \leq m_1 < i < m_3 \leq \ell \\ d_3(a_ia_{m_2}a_{m_3}); & 1 \leq i < m_2 < m_3 \leq \ell \end{cases}$$

$$= \begin{cases} -a_{m_1}a_{m_2}a_ia_j; & 1 \leq m_1 < m_2 < i < j \leq \ell \\ -a_{m_1}a_ia_{m_3}a_j; & 1 \leq m_1 < i < m_3 \leq \ell \\ -a_ia_{m_2}a_{m_3}a_j; & 1 \leq i < m_2 < m_3 \leq \ell \end{cases}$$

$$= \begin{cases} -a_{m_1}a_{m_2}a_ia_j; & 1 \leq m_1 < m_2 < i < j \leq \ell \\ \begin{cases} -a_{m_1}a_ia_{m_3}a_j; & 1 \leq m_1 < i < m_3 < j \leq \ell \\ a_{m_1}a_ia_ja_{m_3}; & 1 \leq m_1 < i < j < m_3 \leq \ell \end{cases} \\ \begin{cases} -a_ia_{m_2}a_{m_3}a_j; & 1 \leq i < m_2 < m_3 < j \leq \ell \\ a_ia_{m_2}a_ja_{m_3}; & 1 \leq i < m_2 < j < m_3 \leq \ell \\ -a_ia_ja_{m_2}a_{m_3}; & 1 \leq i < j < m_2 < m_3 \leq \ell \end{cases} \end{cases}$$

$$= \begin{cases} -d_3(a_{m_1}a_{m_2}a_j); & 1 \le m_1 < m_2 < i < j \le \ell \\ \begin{cases} -d_3(a_{m_1}a_{m_3}a_j); & 1 \le m_1 < i < m_3 < j \le \ell \\ d_3(a_{m_1}a_j a_{m_3}); & 1 \le m_1 < i < j < m_3 \le \ell \end{cases} \\ \begin{cases} -d_3(a_{m_2}a_{m_3}a_j); & 1 \le i < m_2 < m_3 < j \le \ell \\ d_3(a_{m_2}a_j a_{m_3}); & 1 \le i < m_2 < j < m_3 \le \ell \\ -d_3(a_j a_{m_2}a_{m_3}); & 1 \le i < j < m_2 < m_3 \le \ell \end{cases} \end{cases}$$

- If $r = 4$, we will discuss all cases given in equations $(2.)$:-

Case (2.1):- If $1 \le m_1 < m_2 < i < j \le \ell$, then $\{H_1, H_{m_1}, H_{m_2}, H_i, H_j\}$ is a dependent subarrangement of $\mathcal{A}$. Thus, $\partial_5(e_1e_{m_1}e_{m_2}e_ie_j) \in I_4$ and;

$$d_3(a_{m_1}a_{m_2}a_i) = -a_1a_{m_2}a_ia_j + a_1a_{m_1}a_ia_j - a_1a_{m_1}a_{m_2}a_j + a_1a_{m_1}a_{m_2}a_i$$
$$= d_3(a_1a_{m_2}a_i) - d_3(a_1a_{m_1}a_i) + d_3(a_1a_{m_1}a_{m_2}).$$

Case (2.2):- If $1 \le m_1 < i < m_3 < j \le \ell$, then the subarrangement $\{H_1, H_{m_1}, H_i, H_{m_3}, H_j\}$ of $\mathcal{A}$ is a dependent subarrangement. Thus, $\partial_5(e_1e_{m_1}e_ie_{m_3}e_j) \in I_4$ and;

$$d_3(a_{m_1}a_ia_{m_3}) = -a_1a_{m_2}a_ia_j + a_1a_{m_1}a_ia_j - a_1a_{m_1}a_{m_2}a_j + a_1a_{m_1}a_{m_2}a_i =$$
$$d_3(a_1a_ia_{m_3}) + d_3(a_1a_{m_1}a_i) - d_3(a_1a_{m_1}a_{m_3}).$$

Case (2.3):- If $1 \le m_1 < i < j < m_3 \le \ell$, then the subarrangement $\{H_1, H_{m_1}, H_i, H_j, H_{m_3}\}$ of $\mathcal{A}$ is a dependent subarrangement. Thus, $\partial_5(e_1e_{m_1}e_ie_je_{m_3}) \in I_4$ and;

$$d_3(a_{m_1}a_ia_{m_3}) = a_1a_{m_2}a_ia_j - a_1a_{m_1}a_ia_j + a_1a_{m_1}a_{m_2}a_j - a_1a_{m_1}a_{m_2}a_i = d_3(a_1a_ia_{m_3}) +$$
$$d_3(a_1a_{m_1}a_i) - d_3(a_1a_{m_1}a_{m_3}).$$

Case (2.4):- If $1 \le i < m_2 < m_3 < j \le \ell$, then the subarrangement $\{H_1, H_i, H_{m_2}, H_{m_3}, H_j\}$ of $\mathcal{A}$ is a dependent subarrangement. Thus, $\partial_5(e_1e_ie_{m_2}e_{m_3}e_j) \in I_4$ and;

$$d_3(a_{m_1}a_ia_{m_3}) = -a_ia_{m_2}a_{m_3}a_j = -a_1a_{m_2}a_{m_3}a_j + a_1a_ia_{m_3}a_j - a_1a_ia_{m_2}a_j +$$
$$a_1a_ia_{m_2}a_{m_3} = d_3(a_1a_ia_{m_3}) + d_3(a_1a_ia_{m_2}) - d_3(a_1a_{m_2}a_{m_3}).$$

Case (2.5):- If $1 \le i < m_2 < j < m_3 \le \ell$, then the subarrangement $\{H_1, H_i, H_{m_2}, H_j, H_{m_3}\}$ of $\mathcal{A}$ is a dependent subarrangement. Thus, $\partial_5(e_1e_ie_{m_2}e_je_{m_3}) \in I_4$ and;

$$d_3(a_{m_1}a_ia_{m_3}) = a_ia_{m_2}a_ja_{m_3} = a_1a_{m_2}a_ja_{m_3} - a_1a_ia_ja_{m_3} + a_1a_ia_{m_2}a_{m_3} - a_1a_ia_{m_2}a_j =$$
$$d_3(a_1a_ia_{m_3}) - d_3(a_1a_ia_{m_2}) + d_3(a_1a_{m_2}a_{m_3}).$$

Case (2.6):- If $1 \le i < j < m_2 < m_3 \le \ell$, then the subarrangement $\{H_1, H_i, H_j, H_{m_2}, H_{m_3}\}$ of $\mathcal{A}$ is a dependent subarrangement. Thus, $\partial_5(e_1e_ie_je_{m_2}e_{m_3}) \in I_4$ and;

$$d_3(a_{m_1}a_ia_{m_3}) - a_1a_ja_ja_{m_3} + a_1a_{m_1}a_{m_2}a_{m_3} - a_1a_ia_ja_{m_3} + a_1a_ia_ja_{m_2} = d_3(a_1a_ia_{m_3}) -$$
$$d_3(a_1a_ia_{m_2}) + d_3(a_1a_{m_2}a_{m_3}).$$

For this case, the number of such repetition is equal to the number of our choices of 2 −indeices equal i and not equal j from ℓ choices which is equal to $\binom{\ell-2}{3}$.

Case(3):- If m_1 or m_2 or $m_3 \ne i$ or j, then;

$$d_3(a_{m_1}a_{m_2}a_{m_3}) = a_{m_1}a_{m_2}a_{m_3}a_i - a_{m_1}a_{m_2}a_{m_3}a_j$$

$$= \begin{cases} a_{m_1}a_{m_2}a_{m_3}a_i - a_{m_1}a_{m_2}a_{m_3}a_j; & 1\le m_1<m_2<m_3<i<j\le \ell \\ -a_{m_1}a_{m_2}a_i a_{m_3} - a_{m_1}a_{m_2}a_{m_3}a_j; & 1\le m_1<m_2<i<m_3<j\le \ell \\ -a_{m_1}a_{m_2}a_i a_{m_3} + a_{m_1}a_{m_2}a_j a_{m_3}; & 1\le m_1<m_2<i<j<m_3\le \ell \\ a_{m_1}a_i a_{m_2}a_{m_3} - a_{m_1}a_{m_2}a_{m_3}a_j; & 1\le m_1<i<m_2<m_3<j\le \ell \\ a_{m_1}a_i a_{m_2}a_{m_3} + a_{m_1}a_{m_2}a_j a_{m_3}; & 1\le m_1<i<m_2<j<m_3\le \ell \\ -a_{m_1}a_i a_{m_2}a_{m_3} - a_{m_1}a_j a_{m_2}a_{m_3}; & 1\le m_1<i<j<m_2<m_3\le \ell \\ -a_i a_{m_1}a_{m_2}a_{m_3} - a_{m_1}a_{m_2}a_{m_3}a_j; & 1\le i\le m_1<m_2<m_3<j\le \ell \\ -a_i a_{m_1}a_{m_2}a_{m_3} + a_{m_1}a_{m_2}a_j a_{m_3}; & 1\le i\le m_1<m_2<j<m_3\le \ell \\ -a_i a_{m_1}a_{m_2}a_{m_3} - a_{m_1}a_j a_{m_2}a_{m_3}; & 1\le i\le m_1<j<m_2<m_3\le \ell \\ -a_i a_{m_1}a_{m_2}a_{m_3} + a_j a_{m_1}a_{m_2}a_{m_3}; & 1\le i<j\le m_1<m_2<m_3\le \ell \end{cases}.$$

- If $r > 4$, then every subarrangement $\{H_{n_1}, H_{n_2}, H_{n_3}, H_{n_4}\}$ with $1 \le n_1 < n_2 < n_3 < n_4 \le \ell$ forms a $4 - NBC$ basis of $\mathcal{A}$, hence $d_3\left(a_{m_1}a_{m_2}a_{m_3}\right)$ can be written as a linear combination of NBC-monomial. So, there is no repetition among them and the number of such cases is equal to all the choices of the indices $m_1 < m_2 < m_3$ from ℓ choices such that m_1, m_2 and m_3 are not i or j, that is $\binom{\ell-2}{3}$.
- If $r = 4$, then for every cases above we will rewrite $d_3\left(a_{m_1}a_{m_2}a_{m_3}\right)$ as a linear combination of NBC-monomials by using the fact that every subarrangement $\{H_1, H_{n_1}, H_{n_2}, H_{n_3}\}$ with $1 \le n_1 < n_2 < n_3 \le \ell$ forms $4 - NBC$ base for $\mathcal{A}$, and the fact that every $\{H_1, H_i, H_{m_1}, H_{m_2}, H_{m_3}\}$ and $\{H_1, H_j, H_{m_1}, H_{m_2}, H_{m_3}\}$ are $5-$dependent subarrangement of $\mathcal{A}$ for all $1 \le m_1 < m_2 < m_3 \le \ell$ and not equal to i or j.

Notice that, if $m_1 = 1$ for the cases (3.1) to (3.6), then $d_3\left(a_{m_1}a_{m_2}a_{m_3}\right)$ can be written as a linear combination of NBC-monomail, so we shall assume $m_1 > 1$ for the other cases and by a simple calculations we get:

$$d_3\left(a_{m_1}a_{m_2}a_{m_3}\right) == d_3\left(a_1 a_{m_2}a_{m_3}\right) - d_3\left(a_1 a_{m_1}a_{m_3}\right) + d_3\left(a_1 a_{m_1}a_{m_2}\right).$$

Now we will prove our claim as follows:

For (1):- It is clear from Case (1).

For (2):- From the fact that $Im\, d_2 \subseteq \ker d_3$ and from case (1) we have each of $a_m a_i a_j = d_2(a_m a_i)$, $a_i a_m a_j = d_2(a_i a_m)$, , $a_i a_j a_m = d_2(i a_m) \in Im\, d_2$. So $\ker\, d_3 = Im\, d_2$.

For (3):- $H^2(A(\mathcal{A}), a) = \frac{\ker\, d_3}{Im\, d_2} = 0$.

For (4):- If $r > 4$, from our discussion in the cases 2 and 3 above, we have $Im\, d_3$ will be generated by all the $3 - NBC$- monomials $a_{m_1}a_{m_2}a_{m_3}$ such that; There are no two choices of them can be i and j; and by deleting the $3 - NBC$- monomials that satisfied two of a_{m_1}, a_{m_2} and a_{m_3} are not a_i or a_j and by deleting the $3 - NBC$-monomials that satisfied two of a_{m_1}, a_{m_2} and a_{m_3} are not a_i or a_j and the third is a_j, where such monomials represent their repetition that we discussed in case 2 above. So $\dim Im\, d_3 = \binom{\ell}{3} - \binom{\ell-2}{2}$, where the number of repetition is equal to $\binom{\ell}{2} - \binom{\ell-1}{2} - (\ell - 2) = \binom{\ell-2}{2}$.

For (5):- If $r = 4$, then $Im\ d_3$ has a basis, the set of all $3 - NBC$-monomials that begin with a_1 plays as a rule to construct it by removing that containing a_j, so $\dim Im\ d_3 = \binom{\ell-2}{2}$.

For (6):- If $r = 4$, then $d_4: A_4 \to 0$ has $\ker d_4 = A_4$ so $\dim ker\ d_4 = \binom{\ell-1}{3}$. From (5) above we have $H^3(A(\mathcal{A}), a) = \dim ker\ d_4 - \dim Im\ d_3 = \binom{\ell-1}{3} - \binom{\ell-2}{2}$. ■

5.A GENERALIZATION OF OUR WORK:

In this section, we will generalize our work as follows:

5.1 Theorem:

If $a = a_1 - a_j$, $2 \le j \le \ell$, then :

1. $H^{r-2}(A(\mathcal{A}), a) = 0$.
2. $\dim(Im\ d_{r-1}) = \binom{\ell-2}{r-2}$.
3. $\dim(H^{r-1}(A(\mathcal{A}), a) = \binom{\ell-1}{r-1} - \binom{\ell-2}{r-2}$.

Proof:

The $\mathcal{K}$ $-$homomorphism $d_{r-1}: A_{r-1} \to A_r$ is defined as;

$d_{r-1}(a_{m_1} \cdots a_{m_{r-1}}) = a_{m_1} \cdots a_{m_{r-1}} a_1 - a_{m_1} \cdots a_{m_{r-1}} a_j$;

for all NBC-monomial related to $(r-1) -$ section $\{H_{m_1}, \cdots, H_{m_{r-1}}\}$ of Π. We will study all the possible cases for our choices to $m_1 < m_2 < \cdots < m_{r-1} \le \ell$, as follows:-

Case (1):- If $m_1 = 1$ and $m_k = j$ for some $2 \le k \le r-1$, then;

$$d_{r-1}(a_1 \cdots a_{m_{r-1}}) = a_1 \cdots a_{m_{r-1}} a_1 - a_1 \cdots a_j \cdots a_{m_{r-1}} a_j = 0_{A_r} - 0_{A_r} = 0_{A_r}.$$

Now, we can prove (1) above as; if $\{H_{m_1}, \cdots, H_{m_{r-1}}\}$, then $a_1 \cdots a_{m_{r-1}} \in \ker d_{r-1}$. We notice that $d_{r-1}(a_1 \cdots \widehat{a_j} \cdots a_{m_{r-1}} a_j) = (-1)^{m+1} a_1 \cdots a_j \cdots a_{m_{r-1}}$, where m is a positive integer represents the number of transposition that we needed to rearrange $a_{m_1} \cdots a_{m_{r-1}}$ with $m_1 = 1 < m_2 < \cdots < j < \cdots m_{r-1} \le \ell$. Thus, $\ker d_{r-1} = \operatorname{Im} d_{r-2}$ and our claim in (1) above is done, that is $H^{r-2}(A(\mathcal{A}), a) = \frac{\ker d_{r-1}}{\operatorname{Im} d_{r-2}} = 0$.

Case (2):- If $m_1 = 1$ and $m_k \ne j$ for all $2 \le k \le r-1$, then we have the following:- $d_{r-1}(a_{m_1} \cdots a_{m_{r-1}}) = (-1)^{n+1} a_1 \cdots a_j \cdots a_{m_{r-1}} = (-1)^{n+1}(-1)^r d_{r-1}(a_{m_2} \cdots\ a_j \cdots a_{m_{r-1}})$, where n is a positive integer represents the number of the transpositions that we needed to reordered $a_1 \cdots a_{m_{r-1}} \cdots a_j$ to be $a_1 \cdots a_j \cdots a_{m_{r-1}}$ with $m_1 < m_2 < \cdots < j < \cdots m_{r-1}$. We notice that $a_1 \cdots a_j \cdots a_{m_{r-1}}$ is an NBC-monomial.

Case (3):- If $m_1 \ne i$ and $m_k \ne j$ for all $1 \le k \le r-1$, then, we need to compute the position of H_j in $H_1 \cdots H_{m_1} \cdots H_{m_{k-1}} H_j H_{m_k} \cdots H_{m_{r-1}}$. Since $\{H_1, H_{m_1}, \cdots, H_j, \cdots, H_{m_{r-1}}\}$ is an $(r+1)$ $-$dependent section for Π, hence $\partial_{r+1}(e_1 e_{m_1} \cdots e_j \cdots e_{m_{r-1}}) \in I_r$, so;

$$a_{m_1} \cdots a_j \cdots a_{m_{r-1}} - a_1 a_{m_2} \cdots a_j \cdots a_{m_{r-1}} + a_1 a_{m_1} a_{m_3} \cdots a_j \cdots a_{m_{r-1}} + \cdots + (-1)^{r+1} a_1 a_{m_2} \cdots a_j \cdots a_{m_{r-2}} = 0_{A_r}$$

we define the homomorphism $d_{r-1}: A_{r-1} \to A_r$ as follows:-

$$d_{r-1}(a_{m_1} \cdots a_{m_{r-1}}) = a_{m_1} \cdots a_{m_{r-1}} a_1 - a_{m_1} \cdots a_{m_{r-1}} a_j = (-1)^{r-1} a_1 a_{m_1} \cdots a_{m_{r-1}} + (-1)^{r-k+1} a_{m_1} \cdots a_j \cdots a_{m_{r-1}} \quad .. (4.1.1).$$

Thus;

$$(-1)^0 \widehat{a_1} a_{m_1} \cdots a_j \cdots a_{m_{r-1}} + (-1)^1 a_1 \widehat{a_{m_1}} \cdots a_j \cdots a_{m_{r-1}} + \cdots + (-1)^{k-1} a_1 \cdots \widehat{a_{m_{k-1}}} a_j \cdots a_{m_{r-1}} + (-1)^k a_1 \cdots a_{m_{k-1}} \widehat{a_j} a_{m_k} \cdots a_{m_{r-1}} + (-1)^k a_1 \cdots a_{m_{k-1}} a_j \widehat{a_{m_k}} a_{m_{k+1}} \cdots a_{m_{r-1}} + \cdots + (-1)^r a_1 \cdots a_j \cdots a_{m_{r-2}} \widehat{a_{m_{r-1}}} = 0_{A_r} \quad ... (4.1.2)$$

So, from the equation (4.1.2); we have

$$\widehat{a_1} a_{m_1} \cdots a_j \cdots a_{m_{r-1}} = (-1)^2 a_1 \widehat{a_{m_1}} \cdots a_j \cdots a_{m_{r-1}} + \cdots + (-1)^k a_1 \cdots \widehat{a_{m_{k-1}}} a_j a_{m_k} \cdots a_{m_{r-1}} + (-1)^{k+1} a_1 \cdots a_{m_{k-1}} \widehat{a_j} a_{m_k} \cdots a_{m_{r-1}} + \cdots + (-1)^{r+1} a_1 \cdots a_j \cdots a_{m_{r-2}} \widehat{a_{m_{r-1}}} \quad ... (4.1.3).$$

By substituting the equation (4.1.3) in the equation (4.1.2), we have;

$$d_{r-1}(a_{m_1} \cdots a_{m_{r-1}}) = (-1)^{r-1} a_1 \cdots a_{m_{k-1}} \widehat{a_j} a_{m_k} \cdots a_{m_{r-1}} + (-1)^{r-k+3} a_1 \widehat{a_{m_1}} a_{m_2} \cdots a_j \cdots a_{m_{r-1}} + (-1)^{r-k+4} a_1 a_{m_1} \widehat{a_{m_2}} \cdots a_j \cdots a_{m_{r-1}} + \cdots + (-1)^{r-k+1+k+1} a_1 \cdots a_{m_{k-1}} \widehat{a_j} a_{m_k} \cdots a_{m_{r-1}} + \cdots + (-1)^{r-k-1+r+1} a_1 \cdots a_j \cdots a_{m_{r-2}} \widehat{a_{m_{r-1}}}.$$

$$d_{r-1}(a_{m_1} \cdots a_{m_{r-1}}) = (-1)^{r-k+1} d_{r-1}(a_1 \widehat{a_{m_1}} a_{m_2} \cdots a_{m_{r-1}}) + (-1)^{r-k+4} d_{r-1}(a_1 a_{m_1} \widehat{a_{m_2}} a_{m_3} \cdots a_{m_{r-1}}) + \cdots + (-1)^{2r-k} d_{r-1}(a_1 a_{m_1} \cdots a_{m_{r-2}} \widehat{a_{m_{r-1}}}).$$

Now to prove our conjuctor:

For (2):- That is, $Im\, d_{r-1}$ will be generated by the $(r-1)-NBC$-monomials that begin with a_1 and contains no a_j and the number of such monomials is $\binom{\ell-2}{r-2}$, the number of all $(r-2)-NBC$-monomials that not contain each of a_1 and a_j, that is; $\dim Im\, d_{r-1} = \binom{\ell-2}{r-2}$.

For (3):- Since $d_r: A_r \to 0$ is the zero homomorphism, hence $\ker d_r = A_r$ and $\dim \ker d_r = \binom{\ell-1}{r-1}$. Therefore, $\dim H^{r-1}(A(\mathcal{A}), a) = \dim \ker d_r - \dim Im\, d_{r-1} = \binom{\ell-1}{r-1} - \binom{\ell-2}{r-2}$. ■

5.2 Theorem:

If $a = a_1 - a_j$, $2 < j \le \ell$, and for $3 \le k < r-1$, we have;

1. $H^{k-1}(A(\mathcal{A}), a) = 0$.

2. $\dim(Im\, d_k) = \binom{\ell}{k} - \binom{\ell-2}{k-1}$.

Proof:-

We will use the induction to prove our claim and we will recall proposition (3.3) to show that our conjecture is true for $k = 3$. Assume our claim is true for $k-1$ and we will prove it for k. So, we shall begin with the definition of $d_k: A_k \to A_{k+1}$. Since the set of all $K-NBC$-monomials play as a basis of $A_k(\mathcal{A})$, so firstly we will use our ordering to arrange the position of H_j in the HP Π. To explain it's position in any NBC-base contain it, let $\{H_{m_1}, \cdots, H_{m_k}\}$ be an NBC base such that $H_{m_n} \neq H_j$ for all $1 \le n \le k$, then we can assume the position of H_j among $H_{m_1}, \cdots, H_{m_k}$ that respect the hypersolvable ordering that we defined on the hyperplanes of $\mathcal{A}$ is;

$$H_{m_1}, \dots, H_{m_n}, H_j, H_{m_n+1}, \dots, H_{m_k},$$

such that;

$$\underbrace{\underbrace{m_1 < m_2 < \cdots < m_n}_{n} < j}_{n+1} < \underbrace{m_{n+1} < \cdots < m_k}_{k-n}.$$

Thus, $d_k(a_{m_1} \cdots a_{m_k}) = a_{m_1} \cdots a_{m_k} a_1 - a_{m_1} \cdots a_{m_k} a_j$;

and we will study the possible cases of our choices of $m_1, \dots, m_k$, as follows:-

Case (1):- If $m_1 = 1$ and $m_n = j$ for some $1 \le n \le k$, then;

$$d_k(a_1 a_2 \cdots a_{m_k}) = a_1 \cdots a_{m_k} a_1 - a_1 \cdots a_2 \cdots a_{m_k} a_j = 0_{A_{k+1}} - 0_{A_{k+1}} = 0_{A_{k+1}}.$$

Now, we can prove (1) above as:- If $H_1, H_j \in \{H_{m_1}, \cdots, H_{m_k}\}$, then $a_1 \cdots a_{m_k} \in \ker d_k$. We notice that $d_{k-1}(a_1 \cdots \widehat{a_j} \cdots a_{m_k}) = -a_1 \cdots \widehat{a_j} \cdots a_{m_k} a_j = (-1)^{m+1} a_1 \cdots a_j \cdots a_{m_k}$. Thus $\ker d_k = Im\, d_{k-1}$ and our claim in (1) is done. That is, $H^{k-1}(A(\mathcal{A}), a) = \frac{\ker d_k}{Im d_{k-1}} = 0$.

Case (2):- If $m_1 = 1$ and $m_n \ne j$ for all $2 \le n \le k$, then we have the following:-

$$d_k(a_1 \cdots a_{m_k}) = (-1)^{n+1} a_1 \cdots a_j \cdots a_{m_k} = (-1)^{n+1}(-1)^k d_k(a_{m_2} \cdots a_j \cdots a_{m_k}),$$

where n is the positive integer represent the number of the transposition that we needed to reordered $a_1 \cdots a_{m_k} \cdots a_j$ to be $a_1 \cdots a_j \cdots a_{m_k}$ with $m_1 < m_2 < \cdots < j < \cdots < m_k$, we notice that $a_1 \cdots a_j \cdots a_{m_k}$ is an NBC-monomial.

Case (3):- If $m_1 \ne 1$ and $m_n \ne j$ for all $1 \le n \le k$, then we have the following:-

Let $\{H_{m_1}, \cdots, H_{m_k}\}$ be an NBC base such that $H_{m_n} \ne H_j$ for all $1 \le n \le k$, then we can assume the position of H_j among $H_{m_1}, \cdots, H_{m_k}$ that respect the hypersolvable ordering that we defined on the hyperplanes of $\mathcal{A}$. Thus;

$$d_k(a_{m_1} \cdots a_{m_k}) = a_{m_1} \cdots a_{m_k} a_1 - a_{m_1} \cdots a_{m_k} a_j = (-1)^k a_1 a_{m_1} \cdots a_{m_k} + (-1)^{n-k+1} a_{m_1} \cdots a_{m_n} a_j a_{m_n+1} \cdots a_{m_k}.$$

So, $d_k(a_{m_1} \cdots a_{m_k})$ can be written as a linear combination of $(k+1) - NBC$-bases.

For (2):- The $k - NBC$-monomial plays as a basis to $Im\, d_k$ with keeping in mind that we need to remove the $k - NBC$-monomials contain a_j. That is $\dim Im\, d_k = \binom{\ell}{k} - \binom{\ell-2}{k-1}$. ■

5.3 Theorem*:*

If $a = a_i - a_j$, where $1 < i < j \le \ell$, then;

1. $H^{r-2}(A(\mathcal{A}), a) = 0$.
2. $\dim(Im\, d_r) = \binom{\ell-2}{r-2}$.
3. $\dim(H^{r-1}(A(\mathcal{A}), a)) = \binom{\ell-1}{r-1} - \binom{\ell-2}{r-2}$.

Proof:-

To show our claim, we need to study the homomorphism $d_{r-1}: A_{r-1} \to A_r$. As we know, the family of all $(r-1)-NBC$ basis play as a role to construct a basis to A_{r-1}. Then, for all $(r-1)-NBC$-monomial $a_{m_1} \cdots a_{m_{r-1}}$, we define d_{r-1} as;

$$d_{r-1}(a_{m_1} \cdots a_{m_{r-1}}) = a_{m_1} \cdots a_{m_{r-1}} a_i - a_{m_1} \cdots a_{m_{r-1}} a_j.$$

we will study the possible cases for our choices to the indices $1 \leq m_1 < \cdots < m_{r-1} \leq \ell$ as follows:-

Case (1): If $m_{k_1} = i$ and $m_{k_1} = j$ for some $1 \leq k_1 < k_1 \leq r-1$, then;

$$d_{r-1}\left(a_{m_1} \cdots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \cdots a_{m_{k_2-1}} a_j a_{m_{k_2-1}} a_{m_{k_2+1}} \cdots a_{m_{r-1_r}}\right) =$$
$$a_{m_1} \cdots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \cdots a_{m_r-1} a_i - a_{m_1} \cdots a_{m_{k_2-1}} a_j a_{m_{k_2+1}} \cdots a_{m_{r-1}} a_j = 0_{A_r} - 0_{A_r} = 0_{A_r}.$$

For (1):-

Since $Im\, d_{r-1} \subseteq \ker d_r$ and from the fact that;

$$d_{r-2}\left(a_{m_1} \ldots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \ldots a_{m_{k_2-1}} \widehat{a_j} a_{m_{k_2+1}} \ldots a_{m_{r-1}}\right)$$
$$= (-1)\, a_{m_1} \ldots a_i \ldots a_{m_{k_2-1}} a_{m_{k_2+1}} \ldots a_{m_{r-1}} a_j$$
$$= (-1)^{r-m_{k_2}} a_{m_1} \ldots a_i \ldots a_{m_{k_2-1}} a_j a_{m_{k_2+1}} a_{m_{r-1}}.$$

Therefore, $a_{m_1} \ldots a_i \ldots a_j \ldots a_{m_{r-1}} \in Im\, d_{r-2}$ and $Im\, d_{r-2} = \ker d_{r-1}$. So;

$$H^{r-2}(A(\mathcal{A}), a) = \ker d_{r-1} / Im\, d_{r-2} = \frac{Z_{r-2}}{B_{r-2}} = 0.$$

<u>Case (2):-</u> If $m_{k_1} = i$ for some $1 \leq k_1 \leq r-1$ and there is no $m_k = j$ for all $k_1 \leq k \leq r-1$, then;

$$d_{r-1}(a_{m_1} \cdots a_{m_r-1}) = -a_{m_1} \ldots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \ldots a_{m_r-1} a_j =$$
$$(-1)^{r-k_2+1} \underbrace{a_{m_1} \ldots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \ldots a_{m_{k_2-1}}}_{k_2-1}\; a_j\; \underbrace{a_{m_{k_2+1}} \ldots a_{m_{r-1}}}_{r-k_2} \ldots (4.3.1).$$

where $k_2 = Min\{k | H_j \trianglelefteq H_{m_k}\}$. Thus;

$$d_{r-1}(a_{m_1} \cdots a_{m_r-1})$$
$$= (-1)^{r-k_2+1}(-1)^{r-k_1+1} d_{r-1}\left(a_{m_1} \ldots a_{m_{k_1-1}} \widehat{a_i} a_{m_{k_1+1}} \ldots a_{m_{k_2-1}} a_j a_{m_{k_2}} \ldots a_{m_{r-1}}\right)$$
$$= (-1)^{2r-k_1-k_2} d_{r-1}\left(a_{m_1} \ldots a_{m_{k_1-1}} a_{m_{k_1+1}} \ldots a_{m_{k_2-1}} a_j a_{m_{k_2}} \ldots a_{m_{r-1}}\right).$$

As we know, $a_{m_1} \ldots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \ldots a_{m_{k_2-1}} a_j a_{m_{k_2}} \ldots a_{m_{r-1}}$ is not $r-NBC$-monomial, so we need to write it as a linear combination of $r-NBC$-monomials. In fact;

$$\{H_1, H_{m_1}, \ldots, H_{m_{k_1-1}}, H_i, H_{m_{k_1+1}}, \ldots, H_{m_{k_2-1}}, H_j, H_{m_{k_2}}, \ldots, H_{m_{r-1}}\};$$

is a dependent subarrangement of $\mathcal{A}$ and;

$$\partial_{r+1}(e_1 e_{m_1} \ldots e_{m_{k_1-1}} e_i e_{m_{k_1+1}} \ldots e_{m_{k_2-1}} e_j e_{m_{k_2}} \ldots e_{m_{r-1}}) \in I_r. \text{ Thus;}$$
$$(-1)^0 \widehat{a_1} a_{m_1} \ldots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \ldots a_{m_{k_2-1}} a_j a_{m_{k_2}} \ldots a_{m_{r-1}} +$$
$$(-1)^1 a_1 \widehat{a_{m_1}} \ldots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \ldots a_{m_{k_2-1}} a_j a_{m_{k_2}} \ldots a_{m_{r-1}} + \cdots +$$
$$(-1)^{k_1} a_1 a_{m_1} \ldots a_{m_{k_1-1}} \widehat{a_i} a_{m_{k_1+1}} \ldots a_{m_{k_2-1}} a_j a_{m_{k_2}} \ldots a_{m_{r-1}} + \cdots +$$
$$(-1)^{k_2} a_1 a_{m_1} \ldots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \ldots a_{m_{k_2-1}} \widehat{a_j} a_{m_{k_2}} \ldots a_{m_{r-1}} +$$

$$(-1)^{k_2+1}a_1a_{m_1}\dots a_{m_{k_1-1}}a_ia_{m_{k_1+1}}\dots a_{m_{k_2-1}}a_j\widehat{a_{m_{k_2}}}\dots a_{m_{r-1}}+\dots+$$
$$(-1)^{r}a_1a_{m_1}\dots a_{m_{k_1-1}}a_ia_{m_{k_1+1}}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots \widehat{a_{m_{r-1}}}=$$
$$a_{m_1}\dots a_{m_{k_1-1}}a_ia_{m_{k_1+1}}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}}+$$
$$(-1)^2a_1\widehat{a_{m_1}}\dots a_{m_{k_1-1}}a_ia_{m_{k_1+1}}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}}+\dots+$$
$$(-1)^{k_1+1}a_1a_{m_1}\dots a_{m_{k_1-1}}\widehat{a_i}a_{m_{k_1+1}}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}}+\dots+$$
$$(-1)^{k_2+1}a_1a_{m_1}\dots a_{m_{k_1-1}}a_ia_{m_{k_1+1}}\dots a_{m_{k_2-1}}\widehat{a_j}a_{m_{k_2}}\dots a_{m_{r-1}}+\dots+$$
$$(-1)^{r+1}a_1a_{m_1}\dots a_{m_{k_1-1}}a_ia_{m_{k_1+1}}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots \widehat{a_{m_{r-1}}} \,..\, (4.3.2).$$

By substituting (4.3.2) in (4.3.1) we have:-

$$d_{r-1}\left(a_{m_1}\cdots a_{m_{r-1}}\right)=(-1)^{r-k_2+3}(-1)^{r-k_2}d_{r-1}\left(a_1\widehat{a_{m_1}}a_{m_2}\dots a_{m_{r-1}}\right)+$$
$$(-1)^{r-k_2+4}(-1)^{r-k_2}d_{r-1}\left(a_1a_{m_2}\widehat{a_{m_2}}a_{m_3}\dots a_{m_{r-1}}\right)+\dots+$$
$$(-1)^{r-k_2+k_1+2}(-1)^{r-k_2}d_{r-1}\left(a_1a_{m_2}\dots a_{m_{k_1-1}}\widehat{a_i}a_{m_{k_1+1}}\dots a_{m_{r-1}}a_j\right)+\dots+$$
$$(-1)^{2r-k_2+2}(-1)^{r-k_2}d_{r-1}\left(a_1a_{m_2}\dots a_{m_{r-2}}\widehat{a_{m_{r-1}}}a_j\right)=(-1)d_{r-1}\left(a_1\widehat{a_{m_1}}\dots a_{m_{r-1}}\right)+$$
$$(-1)^2d_{r-1}\left(a_1a_{m_2}\widehat{a_{m_2}}\dots a_{m_{r-1}}\right)+\dots+(-1)^{k_1-1}d_{r-1}\left(a_1a_{m_1}\dots\widehat{a_{k_1-1}}a_i\dots a_{m_{r-1}}\right)+$$
$$(-1)^{k_1+1}d_{r-1}\left(a_1a_{m_1}\dots a_i\widehat{a_{k_1+1}}\dots a_{m_{r-1}}\right)+\dots+(-1)^{r+2}d_{r-1}\left(a_1a_{m_1}\dots a_{m_{r-2}}\widehat{a_{m_{r-1}}}\right)+$$
$$(-1)^{k_1}a_1a_{m_1}\dots\widehat{a_i}\dots a_{m_{r-1}}a_j+(-1)^{k_1+1}a_1a_{m_1}\dots\widehat{a_j}\dots a_{m_{r-1}}a_i=$$
$$(-1)d_{r-1}\left(a_1\widehat{a_{m_1}}\dots a_{m_{r-1}}\right)+(-1)^2d_{r-1}\left(a_1a_{m_2}\widehat{a_{m_2}}\dots a_{m_{r-1}}\right)+\dots+$$
$$(-1)^{r+2}d_{r-1}\left(a_1a_{m_1}\dots a_{m_{r-2}}\widehat{a_{m_{r-1}}}\right)-d_{r-1}\left(a_1a_{m_1}\dots\widehat{a_i}\dots\widehat{a_j}\dots a_{m_{r-1}}\right).$$

Thus, $d_{r-1}\left(a_{m_1}\cdots a_{m_{r-1}}\right)$ can be written as a linear combination of $r-NBC$-monomials that related to the image of the homomorphism d_{r-1} of $(r-1)-NBC$-monomials that begin with a_1. Such NBC-monomials depend on the NBC-basis that not begin with H_1. As well as $m_k\neq j$ or $m_k\neq i$ and the number of such repetition is $\binom{\ell-2}{r-2}$.

<u>Case (3):-</u> If there is no $1\le n_1<n_2\le r-1$ such that; $a_{m_{n_1}}=a_i$ and $a_{m_{n_2}}=a_j$. Then, we will assume; $k_1=Min\{k|H_i\trianglelefteq H_{m_{k_1}}\}$ and $k_2=Min\{k|H_j\trianglelefteq H_{m_{k_2}}\}$, to define;

$$d_{r-1}\left(a_{m_1}\cdots a_{m_{r-1}}\right)=(-1)^{r-k_1+1}a_{m_1}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots a_{m_{r-1}}+$$
$$(-1)^{r-k_2+1}a_{m_1}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}}\dots(4.3.3);$$

in order to write $d_{r-1}\left(a_{m_1}\cdots a_{m_{r-1}}\right)$ as a linear combination $r-NBC$-monomials. Consequently, we recall the fact that each of; $\{H_1,H_{m_1},\dots,H_{m_{k_1-1}},H_i,H_{m_{k_1}},\dots,H_{m_{r-1}}\}$, and $\{H_1,H_{m_1},\dots,H_{m_{k_2-1}},H_j,H_{m_{k_2}},\dots,H_{m_{r-1}}\}$; are dependent subarrangements of $\mathcal{A}$, hence;

$$\partial_{r+1}(e_1e_{m_1}\dots e_{m_{k_1-1}}e_ie_{m_{k_1}}\dots e_{m_{r-1}}),\ \partial_{r+1}(e_1e_{m_1}\dots e_{m_{k_1-1}}e_ie_{m_{k_1}}\dots e_{m_{r-1}}),$$
$$\partial_{r+1}(e_1e_{m_1}\dots e_{m_{k_2-1}}e_je_{m_{k_2}}\dots e_{m_{r-1}})\in I_r.$$

Thus;

$$(-1)^0a_{m_1}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots a_{m_{r-1}}+(-1)^1a_1\widehat{a_{m_1}}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots a_{m_{r-1}}+$$
$$(-1)^2a_1a_{m_1}\widehat{a_{m_2}}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots a_{m_{r-1}}+\dots+(-1)^{k_1}a_1a_{m_1}\dots a_{m_{k_1-1}}\widehat{a_i}a_{m_{k_1}}\dots a_{m_{r-1}}+$$
$$(-1)^{k_1+1}a_1a_{m_1}\dots a_{m_{k_1-1}}a_i\widehat{a_{m_{k_1}}}\dots a_{m_{r-1}}+\dots+(-1)^{r}a_1a_{m_1}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots\widehat{a_{m_{r-1}}}.$$

$$a_{m_1}a_{m_2}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots a_{m_{r-1}} = (-1)^2a_1\widehat{a_{m_1}}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots a_{m_{r-1}} + (-1)^3a_1a_{m_1}\widehat{a_{m_2}}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots a_{m_{r-1}} + \dots + (-1)^{k_1+1}a_1a_{m_1}\dots a_{m_{k_1-1}}\widehat{a_i}a_{m_{k_1}}\dots a_{m_{r-1}} + \dots + (-1)^{r+1}a_1a_{m_1}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots\widehat{a_{m_{r-1}}} \cdots$$

(4.3.4). Therefore;

$$(-1)^0a_{m_1}a_{m_2}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}} + (-1)^1a_1\widehat{a_{m_1}}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}} + (-1)^2a_1a_{m_1}\widehat{a_{m_2}}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}} + \dots + (-1)^{k_2}a_1a_{m_1}\dots a_{m_{k_2-1}}\widehat{a_j}a_{m_{k_2}}\dots a_{m_{r-1}} + (-1)^{k_2+1}a_1a_{m_1}\dots a_{m_{k_2-1}}a_j\widehat{a_{m_{k_2}}}\dots a_{m_{r-1}} + \dots + (-1)^ra_1a_{m_1}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots\widehat{a_{m_{r-1}}}.$$

$$a_{m_1}a_{m_2}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}} = (-1)^2a_1\widehat{a_{m_1}}a_{m_2}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}} + \dots + (-1)^{k_2}a_1a_{m_1}\dots a_{m_{k_2-1}}\widehat{a_j}a_{m_{k_2}}\dots a_{m_{r-1}} + (-1)^{k_2+1}a_1a_{m_1}\dots a_{m_{k_2-1}}\widehat{a_j}a_{m_{k_2}}\dots a_{m_{r-1}} + \dots + (-1)^{r+1}a_1a_{m_1}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots\widehat{a_{m_{r-1}}}\dots \quad (4.3.5).$$

By substituting (4.3.4) and (4.3.5) in (4.3.3) it follows that;

$$d_{r-1}\big(a_{m_1}\cdots a_{m_{r-1}}\big) = (-1)^{r-k_1+3}a_1\widehat{a_{m_1}}a_{m_2}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots a_{m_{r-1}} + (-1)^{r-k_1+4}a_1a_{m_1}\widehat{a_{m_2}}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots a_{m_{r-1}} + \dots + (-1)^{r-k_1+1+k_1+1}a_1a_{m_1}a_{m_2}\dots a_{m_{k_1-1}}\widehat{a_i}a_{m_{k_1}}\dots a_{m_{r-1}} + \dots + (-1)^{2r-k_1+2}a_1a_{m_1}a_{m_2}\dots a_{m_{k_1-1}}a_ia_{m_{k_1}}\dots\widehat{a_{m_{r-1}}} + (-1)^{r-k_2+2}a_1\widehat{a_{m_1}}a_{m_2}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}} + (-1)^{r-k_2+3}a_1a_{m_1}\widehat{a_{m_2}}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots a_{m_{r-1}} + \dots + (-1)^{r-k_2+k_2+1}a_1a_{m_1}a_{m_2}\dots a_{m_{k_2-1}}\widehat{a_j}a_{m_{k_2}}\dots a_{m_{r-1}} + \dots + (-1)^{r-k_2+r+1}a_1a_{m_1}a_{m_2}\dots a_{m_{k_2-1}}a_ja_{m_{k_2}}\dots\widehat{m_{r-1}} = (-1)^{r+3}a_1\widehat{a_{m_1}}a_{m_2}\dots a_{m_{r-1}}a_i + (-1)^{r+2}a_1\widehat{a_{m_1}}a_{m_2}\dots a_{m_{r-1}}a_j + (-1)^{r+4}a_1a_{m_1}\widehat{a_{m_2}}\dots a_{m_{r-1}}a_i + (-1)^{r+3}a_1a_{m_1}\widehat{a_{m_2}}\dots a_{m_{r-1}}a_j + \dots + (-1)^{2r+2}a_1a_{m_1}a_{m_2}\dots\widehat{a_{m_{r-1}}}a_i + (-1)^{r+1}a_1a_{m_1}a_{m_2}\dots\widehat{a_{m_{r-1}}}a_j = (-1)^rd_{r-1}\big(a_1\widehat{a_{m_1}}\dots a_{m_{r-1}}\big) + (-1)^rd_{r-1}\big(a_1a_{m_1}\widehat{a_{m_2}}\dots a_{m_{r-1}}\big) + \dots + (-1)^{2r}d_{r-1}\big(a_1a_{m_1}\dots\widehat{a_{m_{r-1}}}\big).$$

Therefore, every coboundary of $A_r(\mathcal{A})$ can be written as a linear combination of r −coboundaries that begin with a_1. However, the number of such NBC-monomials depends of $(r-2)$- NBC-monomials that begin with a_1 and contains no a_j.

For (2):- From above we established $Im\ d_{r-1}$ structure with;

$$\dim Im\ d_{r-1} = \dim B_{r-1} = \binom{\ell-2}{r-2}.$$

For (3):- As well as; $H^{r-1}(A(\mathcal{A}),a) = \frac{\ker d_r}{Im\ d_{r-1}} = Z_{r-1}/B_{r-1}$ and;

$$\dim H^{r-1}\left(A(\mathcal{A}),a\right) = \dim Z_{r-1} - \dim B_{r-1} = \binom{\ell-1}{r-1} - \binom{\ell-2}{r-2}. \ \blacksquare$$

5.4 Theorem:

If $a = a_i - a_j$, where $1 < i < j \le \ell$, then, for $3 \le k < r-1$, we have:-

1. $H^{k-1}(A(\mathcal{A}),a) = 0.$
2. $\dim(Im\ d_k) = \binom{\ell}{k} - \binom{\ell-2}{k-1}$.

Proof:

We will prove our conjecture inductively. Recall the proposition (3.5) for $k = 3$. So, we shall assume our claim is true for $k - 1$ and we will prove it for k. In fact, the family of all $k - NBC$-bases of $\mathcal{A}$ plays as a role to construct a basis to $A_k(\mathcal{A})$. Hence we will study the homomorphism $d_k: A_k \to A_{k+1}$ as follows; for all $k - NBC$-monomial $a_{m_1}, \dots, a_{m_k}$ we have;

$$d_k(a_{m_1} \dots a_{m_k}) = a_{m_1} \dots a_{m_k} a_i - a_{m_1} \dots a_{m_k} a_j.$$

The image of d_k depends of the $k - NBC$-monomial. So, we will study all the possible cases of these NBC-monomials.

Case(1):- If there is $1 < k_1 < k_2 < r - 1$ such that $m_{k_1} = i$ and $m_{k_2} = j$, then;

$$d_k(a_{m_1} \dots a_{m_k}) = a_{m_1} \dots a_i \dots a_j \dots a_{m_k} a_i - a_{m_1} \dots a_i \dots a_j \dots a_{m_k} a_j = 0_{A_{k+1}} - 0_{A_{k+1}} = 0_{A_{k+1}}.$$

Inductively, we show that such $k - NBC$-monomial is k −cocycles. So, $a_{m_1} \dots a_i \dots a_j \dots a_{m_k} \in \ker d_k$. Moreover, $Im\, d_{k-1} \subseteq \ker d_k$ and $a_{m_1} \dots a_i \dots a_j \dots a_{m_k} \in \operatorname{Im} d_{k-1} = B_{k-1}$, since;

$$d_{k-1}(a_{m_1} \dots a_i \dots \widehat{a_j} \dots a_{m_k}) = (-1)^{k_1 - k_2} a_{m_1} \dots a_i \dots a_j \dots a_{m_k}.$$

Therefore, $\ker d_k = Im\, d_{k-1}$ ($Z_{k-1} = B_{k-1}$) and $H^{k-1}(A(\mathcal{A}), a) = 0$. That is (1) above is verified.

Case (2):- If there is $1 < k_1 \leq k$ such that $m_{k_1} = i$ and there is no $1 < n \leq k$ such that $m_n = j$; then, put $k_2 = Min\{n | H_i \trianglelefteq H_{m_n}\}$. Thus;

$$d_k\left(a_{m_1} \dots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \dots a_{m_k}\right) =$$
$$(-1)^{k-k_2} a_{m_1} \dots a_{m_{k_1-1}} a_i a_{m_{k_1+1}} \dots a_{m_{k_2-1}} a_j a_{m_{k_2}} \dots a_{m_k} =$$
$$(-1)^{k-k_2}(-1)^{k-k_1+2} d_k(a_{m_1} \dots a_{m_{k_1-1}} \widehat{a_i} a_{m_{k_1+1}} \dots a_{m_{k_2-1}} a_j a_{m_{k_2}} \dots a_{m_k}).$$

The important point here, $d_k(a_{m_1} \dots a_i \dots a_{m_k})$ is a $(k+1) - NBC$-monomial.

Case (3):- If there is no $1 \leq n_1 < n_2 \leq k$ such that $n_1 = i$ and $n_2 = j$, then put $k_1 = Min\{n | H_i \trianglelefteq H_{m_n}\}$, and $k_2 = Min\{n | H_j \trianglelefteq H_{m_n}\}$. Thus;

$$d_k(a_{m_1} \dots a_{m_k}) = (-1)^{k-k_1+1} a_{m_1} \dots a_{m_{k_1-1}} a_i a_{m_{k_1}} \dots a_{m_k} +$$
$$(-1)^{k-k_2+2} a_{m_1} \dots a_{m_{k_2-1}} a_j a_{m_{k_2}} \dots a_{m_k}.$$

That is, $d_k(a_{m_1} \dots a_{m_k})$ written as a linear combination of $(k+1) - NBC$-monomials.

For (2): Our study to the three cases above, leads to $Im\, d_k$ have a basis generated by $k - NBC$-monomials that represent repetition in case (2) and co-cycle in case (1). Therefore, $\dim(Im\, d_k) = \binom{\ell}{k} - \binom{\ell-2}{k-1}$. ■

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دراسة لكهمولوجيا الأورلاك-سولومون ألجبرا كموديول حر لترتيبة عامة

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المستخلص

كُرِسَ هذا البحث لدراسة كهمولوجيا الأورلاك-سولومون ألجبرا $A(\mathcal{A})$ كموديول حر لترتيبة ℓ- عامة، حيث $a = a_i - a_j$، $1 \le i < j \le \ell$. على وجه الخصوص. كما بعد $H^k(A(\mathcal{A}), a)$ تم حسابه لكل $1 \le k \le r - 1$.