Time Evolution of the Position Operators in a Bilayer Graphene

Munera W. Mahan[1*], M.J. Majid[2]

Abstract

In this work, the researchers mainly focus on the trembling motion which is known as Zitterbewegung in a bilayer grapheme. This is effectively achieved by means of the long-wave approximation. That is, the Heisenberg representation is ultimately employed in order to derive the analytical expression concerning the expectation value related to the position operator along the longitudinal and transversal orientation, which describes the motion concerning the electronic wave packet inside the bilayer graphene. Parameters' numbers are considered to explicate the packet of Gaussian wave, including the polarization of initial pseudo-spin as well as the wave number of the initial carrier number along with the localized wave packet's width along the longitudinal as well as transversal orientation. Consequently, the researchers show that the obvious oscillation in position operator can be effectively controlled not only by what is known as the initial parameters concerning the wave packet. Rather, it can mainly be controlled by selecting the localized quantum state's components. Furthermore, the interference's analysis between the conduction as well as valence bands concerning quantum states is really emphasized as the ability of what can be described as the transient's emergence, or in a sense, aperiodic temporal oscillations concerning the average value of position operator in the bilayer graphene.

Key Words: Bilayer Graphene, Zitterbewegung Effect, Wave Packet, Approximation of Long-Wave, States of Localized Quantum.

DOI Number: 10.14704/nq.2021.19.12.NQ21192 **NeuroQuantology 2021; 19(12):19-29**

Introduction

In the year 1930, Schrodinger obviously anticipated that a free particle is ultimately manifested and described by means of the equation of relativistic Dirac. This is ultimately characterized by a state of oscillatory motion. Such an oscillatory motion can be described as the zitterbewegung (ZB). Moreover, in the last few years, the phenomenon, which is known as Zitterbewegung (ZB), has ultimately attracted the researchers' attention due to the fact that it is viewed as one of the most important phenomena concerning the concept of relativistic quantum physics as well as in various systems which are ultimately described as low-dimensional and semiconducting (Zawadzki, et al., 2018). Furthermore, the theoretical investigation related to the localized as well as quantum states of what is known as free electron, which is ultimately known as solutions related to the equation of what is known as one particle relativistic Dirac, gives an ultimate rise to phenomena which is obviously associated with Zitterbewegung (ZB). This is a natural result of what is described as the interference between two things; the valence as well as conducting quantum states' bands concerning what is termed as the energy spectrum, consequently leads to the oscillation related to the average values of the coordinates as well as the electron velocities with times.

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Relevant conflicts of interest/financial disclosures: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.
Received: 09 October 2021 **Accepted:** 17 November 2021

The oscillations occur at a relatively low spatial scale in order concerning a Compton wavelength as well as scales of small time which are ultimately appropriate to the magnitude related to the energy gap $\Delta t \approx \hbar/(2mc^2) \approx 10^{-21}\,\mathrm{sec}$, m is related to the electron's mass. Moreover, c refers to the ultimate light's speed in vacuum. On the other hand, $\hbar$ *refers to* the constant of reduced Planck (Novoselov, et al., 2004).

However, with the discovery of what are rightly described as new carbon nanostructures' allotropes, like graphene as well as carbon nanotubes (Novoselov, et al., 2005, 197 & (Majid, et al., 2018, 159), which are ultimately characterized by what can be called as unusual physical, or in a sense, chemical properties. Moreover, they can ultimately be represented and explicated by what is termed as Dirac equation concerning the long-wave approximation. Besides, such structures ultimately allowed us to effectively predict the development of the nanodevices and nanoelectronics based on these nanoscales material (Yanyushkina, et al., 2012). The trembling motion's manifestation or ZB in the allotropes of carbon nanostructures drew attention of many researchers. Now, it is commonly known that the ZB can ultimately be noticed in any similar system that the concept of velocity operator does ultimately not commute with what is termed as the Hamiltonian like topological insulators, carbon nanotubes, Cooper pairs, heavy holes and graphene in quantum wells (Schliemann, et al., 2005).

Zawadzki conducted a study concerned with the ZB inside the narrow gap semiconductor (NGS). He used the similarity between the method of *k.p* concerning the bands of energy in (NGS) as well as the equation of free Dirac relativistic concerning electrons. Moreover, he ultimately found more obvious desirable amplitude as well as frequency concerning the oscillation than the amplitude and frequency related to a vacuum concerning a free electron (Zawadzki, et al., 2010). In this regard, another study, on the same topic, was conducted by Schliemann *et al.* They have studied the so-called (ZB) concerning an electron in (III-V) semiconductor of zincblende, known as QWs, in the availability of spin-orbit interaction. Such studies embody strong motivation for other theoretical studies on the same topic, that is, the electrons' ZB in systems of different condensed matter like crystalline solid[8], graphene (R. da Costa, et al., 2012), nanotubes of carbon (Rusin, 2014), Luttinger liquid (Demikhovskii, et al., 2010) superconductor Cserti, et al., 2006), ultra-cold atom (Yi-Cai Zhang, et al., 2013) as well as the topological insulators (Li-kun Shi, et al., 2013).

It was ultimately mentioned that the source of the minimal conductivity concerning graphene can be described on the basis of the peculiar phenomenon ZB (Katsnelson, 157). Furthermore, the ZB's problem concerning states of localized and electronic quantum, especially with what is described as cylindrical symmetry related to a nanotube of semiconductor carbon, in fact, was ultimately analyzed in Ref. (Rusin, et al., 2007). It was analyzed without as well as with field which is externally magnetic.

The ZB's dynamics, which were ultimately described by what is termed as packet of Gaussian wave, particularly in monolayer grapheme, was actually investigated, both analytically as well as at the numerical level, by (Demikhovskii, et al., 2008). The researchers effectively demonstrated that the wave packet's shape at arbitrary time relies on the pseudospin polarization. Furthermore, a general ZB's theory, in this regard, has ultimately been initiated and developed by (David, et al., 2010) as well as (Winkler, et al., 2007). In such studies, the researchers focused on considering an effective number of Hamiltonians that effectively represent different systems. They effectively studied different consequences related to (ZB) oscillations. In this context, (Gerritsma, et al., 2010) were successful in simulating the equation of 1+1 Dirac with the resulting (ZB). They did so by means of using cold ions that interact with beams of laser. Besides, an in-plane as well as magnetic field's effect on the (ZB) which is related to electrons concerning a well which is described as semiconducting quantum (QW) as well as in what can be termed as a quantum dot (QD), with the Rashba as well as interactions of Dresselhaus spin-orbit has ultimately been studied in Ref. (Biswas, et al., 2012) Besides, a motion of complex quantum, which is termed as "super zitterbewegung", concerning grapheme, has effectively and theoretically been studied by means of time-dependent and two-band Hamiltonian concerning the rotating wave approximation's framework (Rusin, et al., 2013). As for the representation of Heisenberg, the quantum's oscillation as well as its states' interference between the conduction as well as valence bands of graphene was ultimately investigated analytically by (Majid, et al., 2013, 141). In this context, the wave packets' dynamics, which were ultimately structured out of states of electronic surface (edge) as well as ZB, were in fact analytically as well as numerically

calculated concerning various values related to the parameters of Hamilton in topological insulator (Demikhovskii, 2014, 104). In this regard, (Biswas, et al., 2014) conducted a study concerning the dynamics of wave packet as well as the quantum oscillation related to heavy holes concerning III-V wells of semiconductor quantum, especially in the availability of a magnetic field which has a quantizing nature.

In this respect, the possibility that packets of non-stationary electron wave can be created in relation to zigzag carbon nanotubes (ZCNT), as they are illuminated by pulses of short laser, was studied by Rusin *et al*. Furthermore, (Ghosh, et al., 2015). clearly and theoretically explored the effect of ZB of a wave packet related to a zigzag graphene nanoribbon. Recently, (Lavor, et al., 2020) investigated the evolution of time of the packet concerning two-dimensional Gaussian wave with low-energy in ABC-stacked n-layer graphene. However, in the most of mentioned studies, the so-called phenomenon of ZB has ultimately a transient character which makes it impossible to be observed experimentally.

In the present work, firstly, the review of the previous works about the trembling motion or ZITTERBEWEGUNG (ZB) in carbonous material is presented. In the second section, the mathematical manipulations and derivations are represented. Moreover, the third section ultimately and effectively discusses the results concerning all possible parameters which can be used to control the phenomenon of ZITTERBEWEGUNG (ZB).

Theoretical Model

The effective Hamiltonian of the electrons closer to the *K* point for the 1st zone of Brillouin in the graphene of multilayers at the absence of external fields is given as follows (Majid, 2013):

$$H = \frac{v_F \hbar}{\gamma^{n-1}}\left(\sigma_x k_x^n + \sigma_y k_y^n\right) \qquad (1)$$

as *n* refers to the layers' number, γ indicates the parameter of interlayer hopping (Majid, 2013). Moreover, v_F ultimately indicates the Fermi velocity. Besides, $\hbar$ indicates the reduction of constant of Planck, k_x^n as well as k_y^n represent the two components of vector's wave $\vec{k}$ in the longitudinal as well as transversal orientation, σ_x as well as σ_y are the Pauli matrices.

The eigen energies as well as the corresponding eigen states related to the Hamiltonian in eq. (1) in the two-component pseudo-spinors' form (Majid, et al., 2018):

$$E_{kS}^n = S V_F^n \hbar^n k^n \big/ \gamma^{n-1} \qquad (2)$$

$$\left|\psi_{k,S}^n\right\rangle = \frac{1}{2\pi}\exp(i(k_x x + k_y y))\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ S\exp(in\theta)\end{pmatrix} \qquad (3)$$

The parameter of (S = ±1) represent the positive as well as negative term of the energy spectrum. Besides, θ represents the angle that takes place between the quasi-wave vector *k* as well as the OX axis. The probability density of the finding concerning the electron atomic sublattices *A* as well as *B* in the graphene has been expressed as the squared magnitudes of the first and second components of the two spinner of the wave function. The orthogonality condition concerning the eigen function is ultimately given by the Dirac δ-function as well as the Kronecker delta. It is represented as follows (Majid, et al., 2018):

$$\left\langle \psi_{k,S}^n \middle| \psi_{k',S'}^n \right\rangle = \delta(k-k')\,\delta_{SS'}\delta_{nn'} \qquad (4)$$

Let us assume a quantum state which is arbitrary and initial for the localized electron on the lattice of multilayers graphene, that represented as a superposition of eigenfunctions:

$$\left|\psi(0)\right\rangle = \sum_S \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_{k,S}^2 \left|\psi_{k,S}^2\right\rangle d^2k$$

$$a_{k,S}^2 = \left\langle \psi_{k,S}^2 \middle| \psi(0) \right\rangle \qquad (5)$$

where $a_{k,S}$ are the expansion coefficients of the initial quantum state $\left|\psi(0)\right\rangle$ in eigenstate, which can be expressed as follows:

$$a_{k,S}^2 = \frac{A}{2\pi\sqrt{2}}(\alpha + \beta S\exp(-2i\theta))\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2d^2} - \frac{y^2}{2d^2} + ik_o x\right)\exp(-i(k_x x + k_y y))\,dxdy \qquad (6)$$

One of the best ways to derive the equations of the expectation values concerning the position operators is the Heisenberg equation of motion, which is based on the Hamiltonian operator of electrons for multilayers graphene:

$$-i\hbar\,\dot{x}^n(t) = \left[H, x^n\right];\ -i\hbar\,\dot{y}^n(t) = \left[H, y^n\right] \qquad (7)$$

where $x(t)$ as well as $y(t)$ are the coordinate operators, $\hbar$ - the reduction of constant of Planck. By calculating the right term of equation (6), and considering the relations of commutation between the momentum as well as coordinate operators $\left[p_x, x(t)\right] = -i\hbar$, $\left[p_y, y(t)\right] = -i\hbar$, that leads to the following equations of the time evolution:

$$-(i\hbar)^n \dot{x}^n(t) = -(i\hbar)^n n!\sigma_x v_F^n \qquad (8)$$

$$-(i\hbar)^n \dot{y}^n(t) = -(i\hbar)^n n!\sigma_y v_F^n \qquad (9)$$

where $\sigma_x(t)$ as well as $\sigma_y(t)$ - the operators of Heisenberg that equal at the zero time to the Pauli matrices of $\sigma_x(0)$ as well as $\sigma_y(0)$ respectively. One can derive the Heisenberg operators for the position operators by integrating eq. (8) and eq.(9) and using the mathematical manipulations as follows:

$$x(t) = x(0) + \frac{(V_F^n)^2 p_x^n n!\, t}{\gamma^{n-1} H} + \frac{i\hbar^n V_F^n n!}{2\gamma^{n-1} H}\left(\sigma_x(0) - \frac{V_F^n p_x^n}{\gamma^{n-1} H}\right)(\exp(-2iHt/\hbar) - 1)$$

$$y(t) = y(0) + \frac{(V_F^n)^2 p_y^n n!\, t}{\gamma^{n-1} H} + \frac{i\hbar^n V_F^n n!}{2\gamma^{n-1} H}\left(\sigma_y(0) - \frac{V_F^n p_y^n}{\gamma^{n-1} H}\right)(\exp(-2iHt/\hbar) - 1) \qquad (10)$$

In our consideration, the bilayer graphene is being studied, therefore, the generalization in the equations can be set to the value *n*=2, as shown in Figure (1).

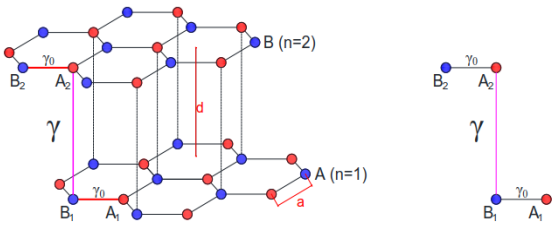


Figure 1. Schematic structure of bilayer graphene (AB Stacked)

We will assume a state which can be described as

$$\langle \psi_{k,+1,2} | x(t) | \psi_{k',+1,2} \rangle = \frac{1}{4\pi^2}\left(\left(\frac{2V_F^2 k_x^2 t}{|k^2|}\right) + \frac{i}{k^2}(\cos(2\theta) - \cos^2(\theta))\left(e^{-2iS'V_F^2\hbar^2 k^2 t/\hbar\gamma} - 1\right)\right)$$

$$\langle \psi_{k,-1,2} | x(t) | \psi_{k',-1,2} \rangle = \frac{1}{4\pi^2}\left(\left(-\frac{2V_F^2 k_x^2 t}{|k^2|}\right) + \frac{i}{k^2}(\cos(2\theta) - \cos^2(\theta))\left(e^{2iS'V_F^2\hbar^2 k^2 t/\hbar\gamma} - 1\right)\right) \qquad (15)$$

$$\langle \psi_{k,+1,2} | x(t) | \psi_{k',-1,2} \rangle = \frac{1}{4\pi^2}\left(\left(\frac{-\sin(2\theta)}{k^2}\right)\left(e^{2iV_F^2 k^2 t/\gamma} - 1\right)\right)$$

$$\langle \psi_{k,-1,2} | x(t) | \psi_{k',+1,2} \rangle = \frac{1}{4\pi^2}\left(\left(\frac{-\sin(2\theta)}{k^2}\right)\left(e^{-2iV_F^2 k^2 t/\gamma} - 1\right)\right)$$

The total average value related to the longitudinal component concerning the position operator is given by:

$$\bar{x}(t) = \sum_{S,S'=\pm 1} x_{S,S'}(t)$$

and correspondingly for the position operators *y(t)*

$$\langle \psi_{k,+1,2} | y(t) | \psi_{k',+1,2} \rangle = \frac{1}{4\pi^2}\left(\left(\frac{2V_F^2 k_y^2 t}{|k^2|}\right) + \frac{i}{k^2}(\sin(2\theta) - \sin^2(\theta))\left(e^{-2iS'V_F^2\hbar^2 k^2 t/\hbar\gamma} - 1\right)\right)$$

$$\langle \psi_{k,-1,2} | y(t) | \psi_{k',-1,2} \rangle = \frac{1}{4\pi^2}\left(\left(-\frac{2V_F^2 k_y^2 t}{|k^2|}\right) + \frac{i}{k^2}(\sin(2\theta) - \sin^2(\theta))\left(e^{2iS'V_F^2\hbar^2 k^2 t/\hbar\gamma} - 1\right)\right) \qquad (16)$$

localized quantum. This can be expressed at initial moment as a wave packet which given in the following form (Majid, et al., 2018):

$$|\psi(0)\rangle = \frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}\sqrt{\pi}\, d}\exp\left(-\frac{x^2}{2d^2} - \frac{y^2}{2d^2} + ik_{xo}x + ik_{yo}y\right)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \qquad (11)$$

where the α and β are the parameters which define the relation between the pseudospin polarization *i.e.* (the pseudo-spinors in the eigenstate), and *d* is the width of the localized packet. As it is known, in the Heisenberg picture, the average value any physical quantity A at time t can be described as follows:

$$\overline{A}(t) = \langle \psi(0) | A(t) | \psi(0) \rangle \qquad (12)$$

as the operator *A(t)* ultimately proves the following equation $-i\hbar\, \dot{A}(t) = [H, A(t)]$.

Using Eqs. (3), (5) and (12), after straightforward calculations, we can obtain the average values for the *x* as well as *y* components concerning the position operator as follows:

$$\overline{x(t)} = \sum_{S,S'=\pm 1}\int_k a_{k,S}^{*n} a_{k',S'}^{n'} \langle \psi^n_{k,S} | x(t) | \psi^n_{k',S'} \rangle\, dk \qquad (13)$$

$$\overline{y(t)} = \sum_{S,S'=\pm 1}\int_k a_{k,S}^{*n} a_{k',S'}^{n'} \langle \psi^n_{k,S} | y(t) | \psi^n_{k',S'} \rangle\, dk \qquad (14)$$

Since, The evaluation of the matrix elements of the position operators *x(t)*are based on the eigenstate in eq.(3) as well as the operators in eq.(13) along with eq.(14) for all cases in band index (*S*=±)of bilayers graphene:

$$\left\langle \psi_{k,+1,2} \left| y(t) \right| \psi_{k',-1,2} \right\rangle = \frac{1}{4\pi^2}\left(\left(\frac{\cos(2\theta)}{k^2}\right)\left(e^{2i\,V_F^2k^2t/\gamma}-1\right)\right)$$

$$\left\langle \psi_{k,-1,2} \left| y(t) \right| \psi_{k',+1,2} \right\rangle = \frac{1}{4\pi^2}\left(\left(\frac{\cos(2\theta)}{k^2}\right)\left(e^{-2i\,V_F^2k^2t/\gamma}-1\right)\right)$$

The total average value related to the transversal component concerning the position operator is given by:

$$\bar{y}(t) = \sum_{S,S'=\pm 1} \bar{y}_{S,S'}(t)$$

These above derivations allow us to express the final formalism of the values of expectation related to position operator in the *x* as well as *y* directions as follows:

$$\begin{aligned}\bar{x}(t) = w\pi\Bigg(& \left(8V_F^2t\right)\alpha\beta\int_{-\infty}^{\infty} e^{-k^2d^2}\left(\left(I_2(x)+\frac{1}{2}I_0(x)+\frac{1}{2}I_4(x)\right)\right)\,k\,dk \\ & +2i\left(\alpha^2+\beta^2\right)\int_{-\infty}^{\infty}\frac{e^{-k^2d^2}}{k}\left(I_2(x)-I_0(x)\right)\left(\cos(2\,V_F^2\hbar^2k^2t/\hbar\gamma)-1\right)dk \\ & -4\alpha\beta\int_{-\infty}^{\infty}\frac{e^{-k^2d^2}}{k}\left(\left(\frac{1}{2}I_0(x)+\frac{1}{2}I_4(x)-I_2(x)\right)\sin(2\,V_F^2\hbar^2k^2t/\hbar\gamma)\right)\,dk \\ & +4\alpha\beta\int_{-\infty}^{\infty}\frac{e^{-k^2d^2}}{k}\left\{\left(I_0(x)-I_4(x)\right)\right\}\left(\sin(2\,V_F^2\hbar^2k^2t/\hbar\gamma)\right)\,dk\Bigg)\end{aligned} \tag{17}$$

$$\begin{aligned}\bar{y}(t) = w\pi\Bigg(& \left(8V_F^2t\right)\alpha\beta\int_{-\infty}^{\infty} e^{-k^2d^2}\left(\left(I_2(x)-\frac{1}{2}I_0(x)-\frac{1}{2}I_4(x)\right)\right)\,k\,dk \\ & -2i\left(\alpha^2+\beta^2\right)\int_{-\infty}^{\infty}\frac{e^{-k^2d^2}}{k}\left(I_0(x)-I_2(x)\right)\left(\cos(2\,V_F^2\hbar^2k^2t/\hbar\gamma)-1\right)dk \\ & -4\alpha\beta\int_{-\infty}^{\infty}\frac{e^{-k^2d^2}}{k}\left(\left(I_2(x)-\frac{1}{2}I_0(x)-\frac{1}{2}I_4(x)\right)\sin(2\,V_F^2\hbar^2k^2t/\hbar\gamma)\right)\,dk \\ & +2\left(\alpha^2-\beta^2\right)\int_{-\infty}^{\infty}\frac{e^{-k^2d^2}}{k}\left\{I_2\left(2kk_{xo}d^2\right)\right\}\left(\cos(2\,V_F^2\hbar^2k^2t/\hbar\gamma)-1\right)\,dk\Bigg)\end{aligned} \tag{18}$$

where $w = \left(\frac{Ad^2}{\sqrt{2}}\right)^2\left(\frac{1}{4\pi^2}\right)e^{-(k_{x0}^2+k_{y0}^2)d^2}$

Numerical Results as Well as Discussion

The researchers have previously investigated the phenomena related to trembling motion of (ZB) inside what is obviously known as the bilayer graphene and the dependence of time concerning the expectation values related to the position operators in the longitudinal as well as transversal orientations. Our numerical calculations include the variation in the pseudo-spin polarization and comparing between them. The relation between the values related to the initial pseudo-spin polarization is α and β is very important to examine the trend of the wave packet in the quantum state at the initial time localized on the sub-lattice *A* or *B* of atomic lattice in first layer of the bilayer graphene.

The dependence of time in relation to the expectation values concerning the position operators are calculated numerically. In our analysis, it is obviously agreed upon considering the wave packet parameters as the following values in the numerical calculations: v_f=h=1, to simplify, the vector of the initial transversal wave $\boldsymbol{k}_{yo}$=0. Firstly, we adopt the relation between the values concerning the initial polarization of pseudo-spin is α=1 as well as β=0. Thus, this indicates that the quantum state concerning the initial time localized upon the sub-lattice (*A*) of atomic lattice for the first layer of grapheme. The time dependence is related

to the average values for the transversal position operator $\bar{y}(t)$ is represented in Figure (2), concerning different values related to the vector of the initial wave $\boldsymbol{k_{xo}}$. It seems clear that there is no oscillation even at excess the value of the initial longitudinal wave vector k_{xo} for the width of the wave packet less or equal to d=5 nm as shown in Figure (2) (a). The ZB's behavior concerning the average value's oscillation of position operator related to the transversal direction of $\bar{y}(t)$ is represented in Figure (2) (b), (c) and (d). The amplitude of oscillation began high. After that, it decreases. Besides, the oscillations concerning various values of k_{xo} ultimately reflect similar shapes and period, but at the same time the reflect different amplitudes. It is possible to state that the oscillation has the property transient. We can notice from Figure(2)(b) that the oscillation decreases with the increase in the value of k_{xo}, correspondingly, for the same values of k_{xo}, it is found that the oscillation is directly proportional to the width of wave packet d as illustrated in Figure (2) (c) and (d). This can be interpreted that the increasing of the wave packet width d leads to an increase the probability of interference the positive and negative term of Dirac equation, i.e. the probability of the interference of the two subpackets of the splited initial wave packet on the graphene lattice. On the other hand, increasing the initial wave vector k_{xo} leads to decreasing the probability of the interference mentioned above.

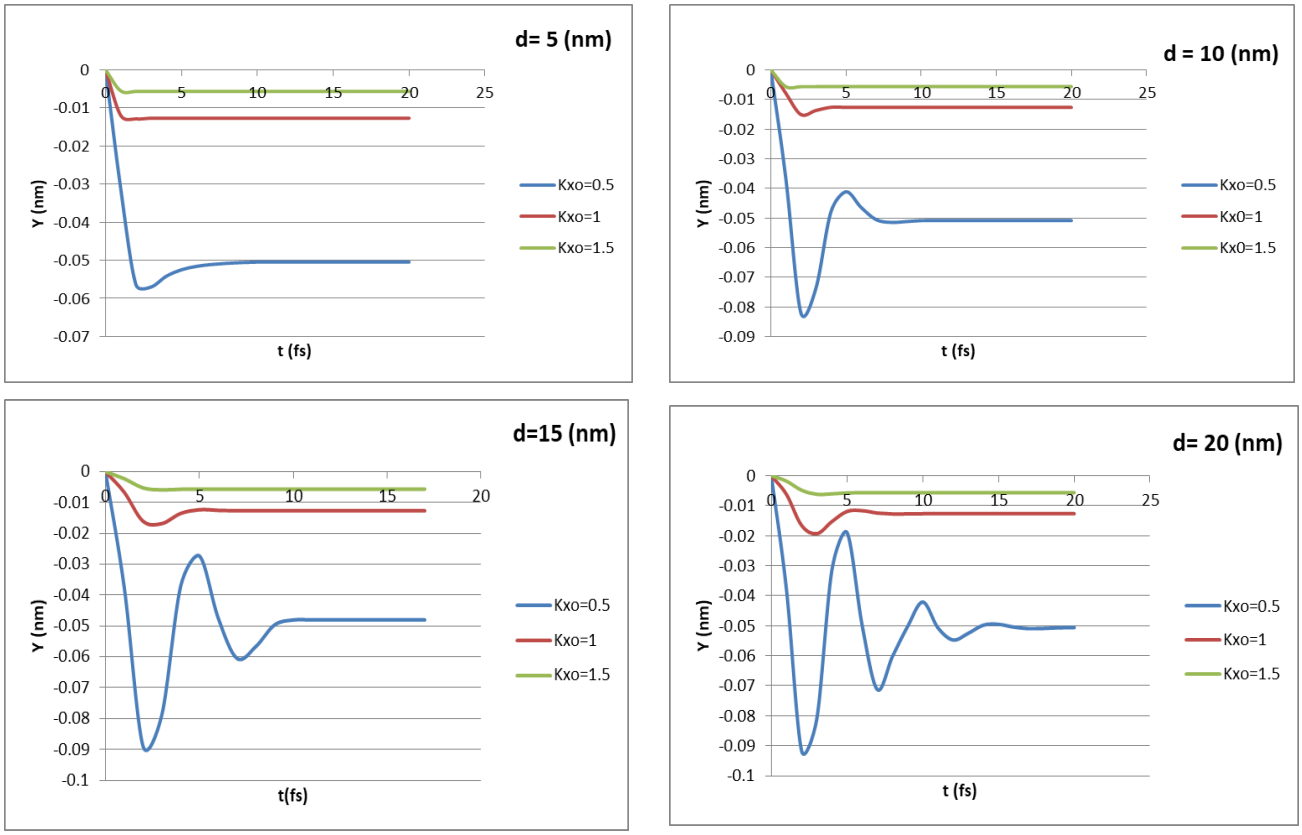


Figure 2. Time dependence concerning the average value of $\bar{y}(t)$ for α=1 and β=0, (a) in transversal orientation, k_{yo}=0, (a) d=5 nm, (b) d=10 nm, (c) d=15 nm, (d) d=20 nm

It has ultimately been taken into account that new parameters of the pseudo-spin polarization, in this case, is α=1 and β=1, where the electronic state localized at the initial moment concerning atomic sub-lattices of A as well as B simultaneously. In this context, figure (3) represents the data of the numerical calculations of the average value coordinate $\bar{x}(t)$ according to the formula (17) for various values of the parameters k_{xo} and d. As follows from the Figure (3), that the time dependence of the longitudinal position operator is linear, at long times, especially at the early stages of evolution, it has an aperiodic oscillatory character. Subsequently, it has not affected by the increase in the width concerning the wave packet or the increase in the initial value of the wave number k_{xo}.

This means that the phenomenon of trembling motion or the zitterbewegung cannot be achieved in the longitudinal direction as has been observed in the transversal direction.

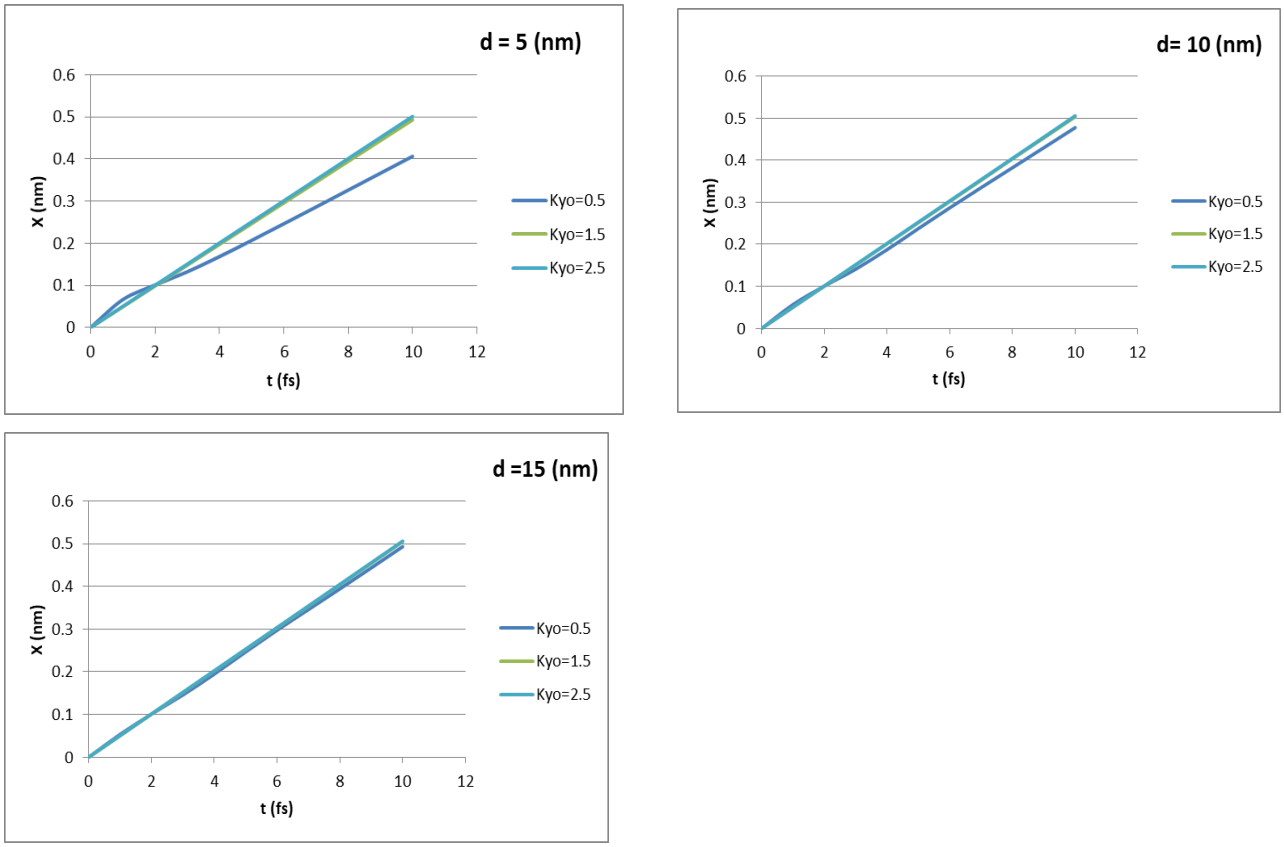
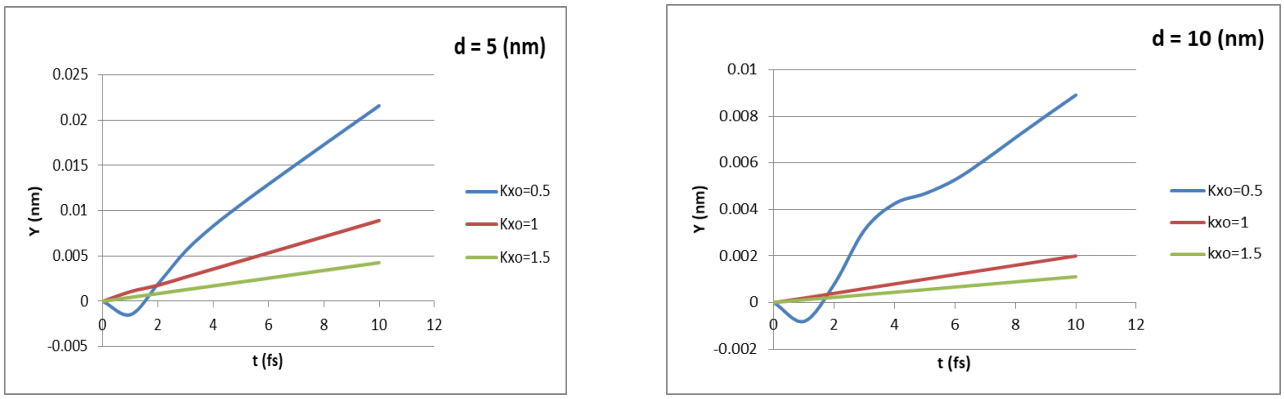


Figure 3. Time dependence concerning the average value $\bar{X}(t)$ for α=1 and β=1, in a longitudinal orientation, k_{xo}=0, (a) d=5 nm, (b) d=10 nm, (c) d=15 nm.

Figure (4) illustrates the dependence of time concerning the average values related to the transversal position operator $\bar{y}(t)$ at α=β=1. It seems clear from Figure (4) (*a*),(*b*),(c) and *(d)* that the oscillation begins at small values of k_{xo} and small times, then the oscillation disappears for the large values which indicates a decline in the phenomenon of ZB. This is achieved even when the values concerning the wave packet's width are increased. But, the effect of increasing the width of wave packet appears clearly on the increase in the fluctuation of the transversal position operator $\bar{y}(t)$, which means an increase in the probability of the occurrence of the phenomenon of ZB, as an illustrated in Figure(4)(d).



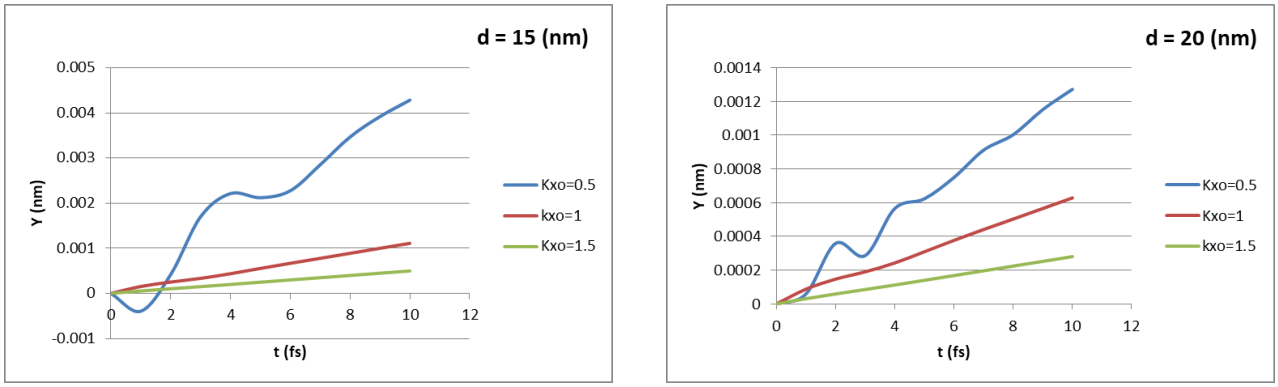


Figure 4. Time dependence concerning the average value $\bar{y}(t)$ for α=1 and β=1, in the transversal orientation, k_{yo}=0, (a) d=5 nm, (b) d=10 nm, (c) d=15 nm, (d) d=20 nm

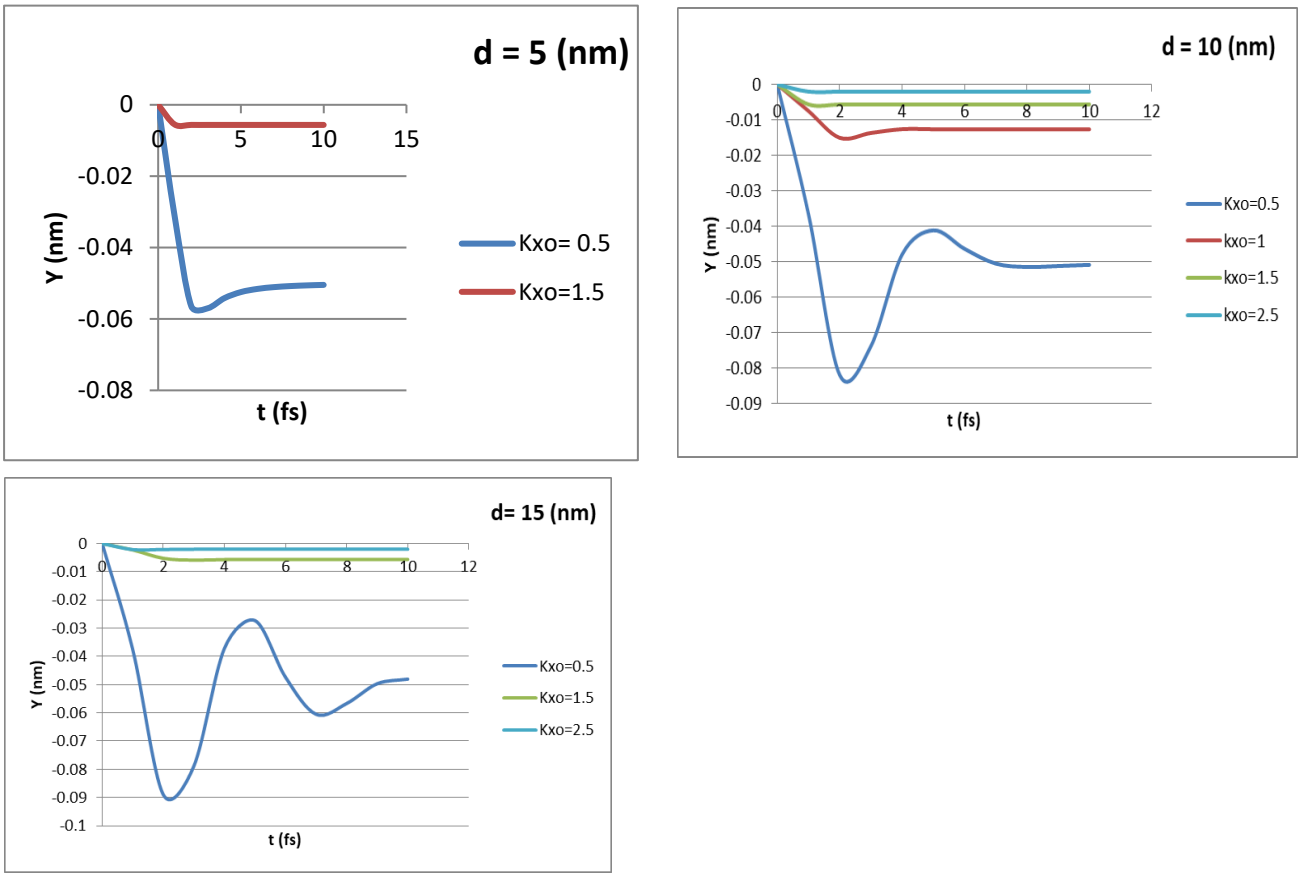


Figure 5. Time dependence of the average value $\bar{y}(t)$ for α=1 and β=I, in the transversal orientation, k_{yo}=0, (a) d=5 nm, (b) d=10 nm, (c) d=15 nm,

Figure (5) obviously reflects the impact of what is termed as the wave packet's width d upon the evolution of time concerning the average value related to the position operator for α=1 and β=I. The oscillations have similar behavior as in the previous case that is further explained in Figure (1), but with varying amplitude concerning transversal components concerning the position operator. The oscillation's amplitude decreases when the width ultimately decreases at a fixed value of k_{ox} for the transversal components concerning the position operator. But the wave packet width's reduction d ultimately reduces the possibility of its two branches' interference.

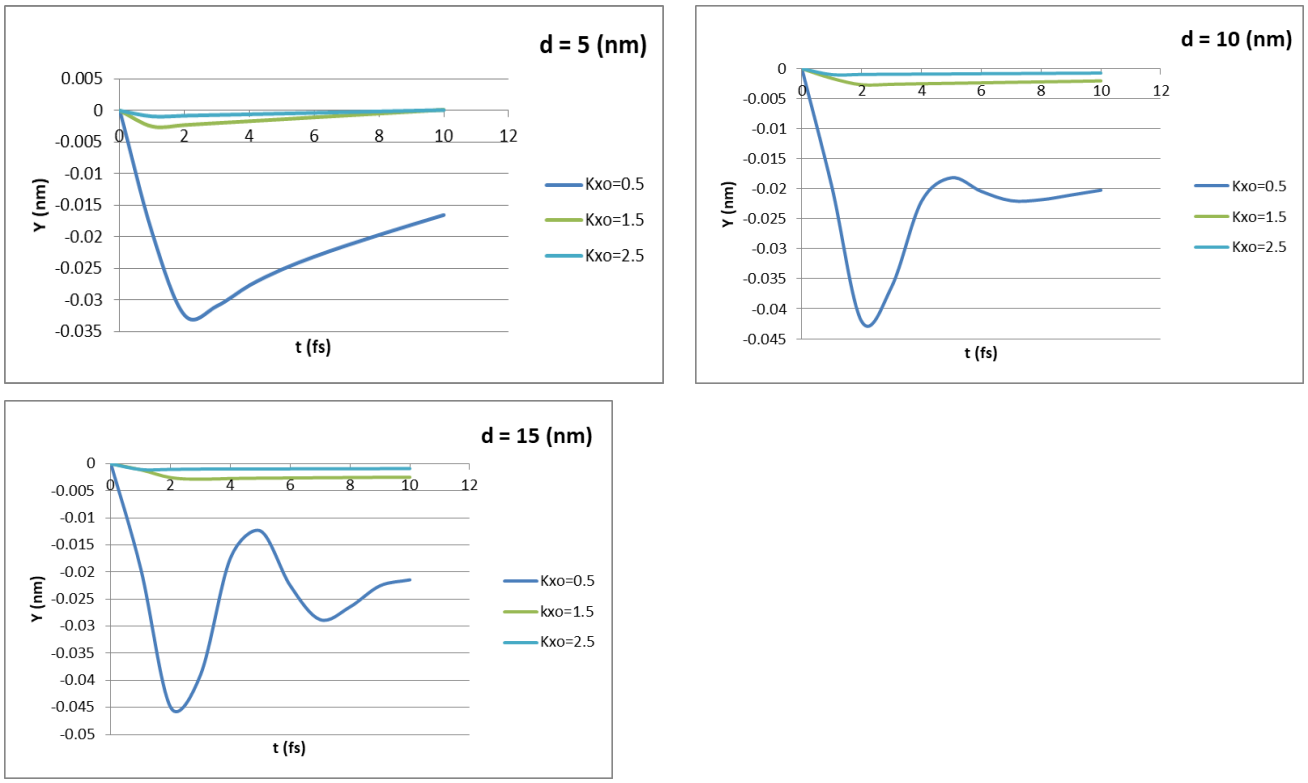


Figure 6. Dependence of time concerning the average value of $\bar{y}(t)$ for α=1 and β= exp(I*Pi/4), in transversal orientation, k_{yo}=0, (a) d=5 nm, (b) d=10 nm, (c) d=15 nm, (d) d=20 nm

Figure (6) (*a*) makes it clear that there is a strong evident concerning the behavior of oscillation in the transversal component concerning the average value related to the position operator for α=1 and $\beta = \exp(i\pi/4)$. Such an evident is represented by the fact that the transversal component is gradually decreases with time. Moreover, it has a transient behavior. Besides, its amplitude ultimately decreases with increasing k_{xo} for each value of *d*. By comparing Figure(5) (*a*) and Figure (6) (*a*), it is obviously noted that the oscillation's amplitude ultimately increases with inserting the phase between α and β. It may not ultimately get transient fluctuation concerning small values of k_{xo}.

The oscillation in time-dependent part concerning the position operator is based on the off-diagonal matrix elements related to the operator, then, the characteristic frequency of the localized quantum state can ultimately be expressed as follows:

$$\omega^{ZB} \approx 2V_F\sqrt{k_{xo}^2 + k_{yo}^2} \qquad (19)$$

Thus, as it is obviously mentioned above, one can obviously notice that the oscillations of ZB concerning the current density ultimately have a transient character. For example, they damped with time. Moreover, the decay of time can ultimately be estimated with looking at the kinematic characteristics concerning wave packets motion. This is represented by:

$$\Delta t_x^{ZB} \approx \frac{d\sqrt{k_{xo}^2 + k_{yo}^2}}{v_F k_{xo}},\ \Delta t_y^{ZB} \approx \frac{d\sqrt{k_{xo}^2 + k_{yo}^2}}{v_F k_{yo}} \qquad (20)$$

Based on the suggested values related to parameters of ZB, it becomes possible to ultimately estimate the frequency of ZB regarding the states of localized quantum wave packet of the order of $\omega^{ZB} \approx 10^{15}$ Hz. The decay times, along the axial direction, as well as spiral direction take the order of $\Delta t_x^{ZB} \approx 10^{-14}$ sec and $\Delta t_y^{ZB} \approx 10^{-15}$ sec respectively.

Conclusions

The researchers have ultimately performed theoretical calculations to investigate the ZB phenomena in the bilayer graphene and the effect of the variation in the essential parameters on the characteristics of the trembling motion.

Our present study emphasized that the ability of noticing the phenomena of ZB in the electronic quantum state which localized across the longitudinal as well as transversal orientation on the surface of the graphene directly and accurately depends on the selected parameters such as α, β, k_{xo}

and *d*. Concerning states of localized quantum that ultimately have oscillatory nature related to the dependence of time of the position operator reveal aperiodic damping, damping and linear behavior. Hence, this is mainly described as that the effect of ZB in the bilayer graphene is ultimately dominated for the position operator along the transversal direction greater than the longitudinal direction. This can ultimately be explicated just as an indication of ZB. That is, ZB is viewed as a robust as well as basic phenomenon. But it is not very sensitive to the details related to the employed model as well as the parameters of the wave packet. Thus, our results indicate that occurrence of the phenomenon of ZB has the same behavior as presented for single-layer graphene in the previous studies.

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*e*ISSN 1303-5150

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