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The Error Analysis for the Discontinuous Galerkin Finite Element Method of the Convection− Diffusion Problem

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Abstract

In this paper, we studied and analyzed of the discontinuous Galerkin finite element (DGFE) method of linear convection-diffusion problem. We have proved that the properties of the bilinear form $a(u, v)$, ($V-$elliptic and continuity). Considered the error estimate in the semi-discrete DGFE method where chopping with respect to space variables only is applied, whereas time remains continuous and proved the approximate solution is converges with error of $o(h)$. The theoretical results are illustrated by numerical experiments of steady-state DGFE method for (SIPG, NIPG, IIPG), when the penalty parameter σ is sufficiently large and showed the effect of penally parameter σ and the parameter ϵ to the numerical solutions.

keywords: convection-diffusion problem, Steady-state DGFE method, error analysis .

1 Introduction

The convection diffusion equations, are known to have many important applications such as fluid dynamics, heat and mass transfer, hydrolology and so on. The equation consists mainly of two parts. The convection terms with the convection (velocity) field and the diffusion terms with the diffusion coefficient, it is of mixed hyperbolic-parabolic type with more or less hyperbolic or parabolic character depending on the size of these two terms. In cases, when diffusion phenomena are small or negligible, the parabolic convection-diffusion model tends to a hyperbolic model. Therefore, the ratio of them plays an important role in the stability of the numerical solution. It is well known that the straight forward application of the Galerkin finite element method to singularly perturbed convection diffusion problems may lead to spurious oscillation in the approximate solution .i.e. for $h > 0, \frac{\alpha}{|b|h} << 1$, this condition can obviously occur as any combination of weak diffusion (small), strong convection (large) or as the result of a large domain. Several methods have been intensively studied to remove such a drawback, a popular idea is to add stabilization terms to the formulation of the problem. In actuality, this is mainly achieved by stabilized methods, such as upwinding techniques [1], [2], [3]and [4], Petrov-Galerkin approach [5], [6] and [7], artificial diffusivity method [8] and [9] . In the 1970s, researchers devised another way to solve these problems, called Discontinues Galerkin finite element (DGFE)method. The DGFE method approach approximates the approximate limits of the desired grid solution on finite elements without requiring any continuity requirements. The DGFE method makes utilize of the same function space as continues finite element and finite volume methods, but with relaxed continuity at inter element boundaries and can be considered the DGFE as a generalization for each of the FVM and FEM. In 1998, B.Cockburn [10], studied the Runge Kutta Discontinuous Galerkin (DG) method for numerically solving nonlinear hyperbolic systems and its extension for convection-dominated problems, the so-caused Local Discontinuous Galerkin method. In 1999, C. Bernardo [11], provided and analyzed by Runge Kutta Discerinuous Galerkin to solve numerical nonlinear deterministic systems, the basic method has been extended to convection problems that result in local intermittent Galerkin method. In 2007, X.Y. Xu and C.W. Shu [12] provided L^2-error estimates for the semi-discrete local discontinuous Galerkin methods for nonlinear convection-diffusion equations and KdV equations with smooth solutions. In this paper we will present the theory of analysis the DGFE method for space discretization of the linear convection- diffusion problem and analyze the error analysis in two cases semi-discrete DGFE method and fully discrete DGFE method and we discuss the numerical solution to L^2error and $L^\infty-$error of DGFE method for solving steady-state convection-diffusion problem for (SIPG, NIPG, IIPG), in two cases, the first case when the Penalty parameter is small, the second case when the Penalty parameter is large and discuss the stability when the Penalty parameter is small, and when the Penalty parameter is large. This paper is organized as follows. In Section 2 we present the convection-diffusion problem. Some important definitions are shown in section 3. In section 4 we consider Discontinuous Galerkin Finite Element spaces. The Discontinuous Variational formulation of the problem in section 5. The properties of the bilinear form are shown in section 6. In section 7 we analyze the error estimate of semi-discrete DGFE method.In section 8 we present the test problem. The numerical results are shown in section 9. The conclusions are shown in section 10.

2 The Convection-Diffusion Problem.

Let $\Omega \subset \mathbb{R}^d$ where $d = 2$ or 3 be a bounded polyhedral domain and $T > 0$. We consider the convection-diffusion problem: Find $u \in Q_T = \Omega \times (0, T) \longrightarrow \mathbb{R}$, [13], such that:

$$u_t - \alpha \Delta u + \boldsymbol{b} \cdot \nabla u = f \quad in \quad Q_T, \qquad (1)$$

$$u = u^D \quad on \quad \partial\Omega^D \times (0, T), \qquad (2)$$

$$\alpha \frac{\partial u}{\partial n} = u^N \quad on \quad \partial\Omega^N \times (0, T), \qquad (3)$$

$$u(x, 0) = u^0(x), \quad x \in \Omega. \qquad (4)$$

Assume that $\partial\Omega = \partial\Omega^D \cup \partial\Omega^N$

$$\boldsymbol{b} \cdot n \leq 0 \quad on \quad \partial\Omega^D, \qquad (5)$$

$$\boldsymbol{b} \cdot n \geq 0 \quad on \quad \partial\Omega^N; \forall t \in [0, T], \qquad (6)$$

where n is the unit outer normal to the boundary $\partial\Omega$ of Ω, $\partial\Omega^D$is the inflow boundary and $\partial\Omega^N$ is the outflow boundary.

2.1 Assumptions:

- $f \in C([0, T]; L^2(\Omega)), u^0, u, u_t \in L^2(\Omega)$,
- u^D is the trace of some $u \in C([0, T]; H^1(\Omega)) \cap L^\infty(Q_T)$ on $\partial\Omega^D(0, T)$
- $u^N \in C([0, T]; L^2(\partial\Omega_N))$.
- $|K|$ = the area of $K \in T_h$, and $\sigma = \frac{\sigma^0}{|e|^{\beta_0}}$, $\beta_0 \geq (d-1)^{-1}$, $\sigma^0 > 0$, $\epsilon \in \{-1, 0, 1\}$.

3 Some important definitions

Definition 3.1. *Let Ω be an open set, with boundary represented by $\partial\Omega$. For $1 \leq s < \infty$, let $L^s(\Omega)$ denote the space of real-valued Lebesgue measurable function u defined on Ω such that*

$|u|^s$ is integrable on Ω with respect to the Lebesgue measurable in $\mathbb{R}^d$, $d \geq 1$. The Lebesgue space $L^s(\Omega)$ is defined by(see [14])

$$L^s(\Omega) = \{u : \int_\Omega |u|^s dx < \infty\}, \quad s \in [-1, \infty),$$

equipped with the norm,

$$||u||_{L^s(\Omega)} = (\int_\Omega |u|^s dx)^{1/s}.$$

For $s = 2$, we get $L^2(\Omega)$ the space of square integrable functions. For real-valued functions $u, v \in L^2(\Omega)$, we defined the L^2-inner product by

$$(u, v) = \int_\Omega u(x)v(x)dx.$$

For $s = \infty$, we get $L^\infty(\Omega)$ the space of all functions which are bounded for almost all $x \in \Omega$:

$$L^\infty(\Omega) = \{u : |u(x)| < \infty \quad for \quad almostall \quad x \in \Omega\},$$

equipped with norm, $||u||_{L^\infty(\Omega)} = \max_{x\in\Omega} |u(x)|$.

Definition 3.2. *Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a domain. The space $H^r(\Omega)$ of the function u on Ω defined by(see [15], [16])*

$$H^r(\Omega) = \{u : \partial u \in L^2(\Omega); \partial^\alpha u \in L^2(\Omega) \forall |\alpha| \leq r\},$$

is a Hilbert space with the inner product

$$(u, v)_{H^r(\Omega)} = \sum_{|\alpha|\leq r} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)},$$

associated norm

$$||u||_{H^r(\Omega)} = (\sum_{|\alpha|\leq r} (\partial^\alpha u, \partial^\alpha u)_{L^2(\Omega)})^{1/2},$$

$$||u||_{H^r(\Omega)} = (\sum_{|\alpha|\leq r} ||\partial^\alpha u||^2_{L^2(\Omega)})^{1/2},$$

and the semi-norm

$$|u|_{H^r(\Omega)} = (\sum_{|\alpha|=r} ||\partial^\alpha u||^2)^{1/2},$$

in the space $H^1_0(\Omega)$, beside its norm

$$||u||_{H^1_0(\Omega)} = (\int_\Omega (|u|^2 + |\nabla u|^2) d\Omega)^{1/2},$$

and the semi-norm

$$|u|_{H^1_0(\Omega)} = (\int_\Omega |\nabla u|^2 d\Omega)^{1/2}.$$

4 Discontinuous Galerkin Finite Element spaces:

4.1 Partition of the Domain:

Let τ_h be a partition of $\overline{\Omega}$ (the closure of the domain Ω) into a finite number of closed triangles with mutually disjoint interiors. We shall call τ_h a triangulation of Ω. We denote the conforming properties of τ_h used in the finite element method. This means that we admit the so-called hanging nodes. If two elements $K^i, K^j \in \tau_h$ contain a nonempty open part of their sides, we call them neighbors. Let ∂K denote the boundary of an element $K \in \tau_h$, if we set $\partial K^1 \cap$

∂K^2 has a positive $(d-1)-$ dimensional measure, we say that$E \in K$ is the edge of K, if it is a maximal connected open subset either of $K^1 \cap K^2$, where K^1 is a neighbor of K^2 or subset of $\partial K \cap \partial\Omega$. By $\partial\tau_h$ we denote the system of all edges of all elements $K \in \tau_h$. Further, we define the set of all inner and boundary edges by, [18]

$$\partial\tau_h^I = \{E \in \partial\tau_h, E \subset \Omega\},$$

$$\partial\tau_h^B = \{E \in \partial\tau_h, E \subset \partial\Omega\},$$

$$\Gamma^D = \{E \in \partial\tau_h^B, E \subset \partial\Omega^D\},$$

$$\Gamma^N = \{E \in \partial\tau_h^B, E \subset \partial\Omega^N\},$$

For $\varphi \in H^1(\Omega,\tau_h)$ Obviously $\partial\tau_h = \partial\tau_h^I \cup \partial\tau_h^B$, $\partial\tau_h^B = \Gamma^D \cup \Gamma^N$for each $E \in \partial\tau_h$. Introduce the following notation. Every side $E \in K$, E has arbitrary constant inside and outside element.The evaluation of a function v at side E on inside element is named $v^-(x) = v(x)|_{inside}$, $x \in E$. And outside element is named $v^+(x) = v(x)|_{outside}$, $x \in E$, where $v^-(x) = \lim_{\epsilon\longrightarrow 0}(x-\epsilon)$, $v^+(x) = \lim_{\epsilon\longrightarrow 0}(x+\epsilon)$, $\epsilon > 0$. The function v can be discontinuous on side E, it is necessary to quantify the size of discontinuity. Let for every $x \in E$, be $[v](x) = v^+(x) - v^-(x)$, defined as the jump of function v on side E. A function v is undefined at a discontinuity on side E, to close this gap in the definition the average v is used. Let for every $x \in E$, be $\{v\}(x) = \frac{1}{2}(v^+(x) + v^-(x))$, defined as the average of function v on side E.

Definition 4.1. *The discontinuous Galerkin method is based on the use of discontinuous approximations.This is the reason that over triangulation τ_h, for any $r \in \mathbb{N}$, we define the so-called broken Sobolev space(see [18]):*

$$H^r(\Omega,\tau_h) = \{v \in L^2(\Omega); v|_K \in H^r(K); \quad \forall K \in \tau_h\}.$$

For $v \in H^r(\Omega,\tau_h)$, we define the norm

$$||u||_{H^r(\Omega,\tau_h)} = (\sum_{K\in\tau_h} ||u||^2_{H^r(\Omega)})^{1/2},$$

and the semi -norm

$$|u|_{H^r(\Omega,\tau_h)} = (\sum_{K\in\tau_h} |u|^2_{H^r(\Omega)})^{1/2}.$$

Let $l \geq 0$ be an integer.We denote the space of discontinuous piecewise polynomial functions

$$S_h = \{v \in L^2(\Omega); v|_K \in P_l(K); \forall K \in \tau_h\},$$

where $P_l(K)$ denotes the space of all polynomials of degree $\leq l$ onK. We call the number l the degree of polynomial approximation. Obviously, $S_h \subset H^r(\Omega,\tau_h)$.

5 The Discontinuous Variational formulation of the problem.

The variational formulation of the equation (1), we multiply this equation by the test function $v \in V = H^1(\Omega, \tau_h)$, and integrating by parts over Ω, we give:

$$(f,v) = (u_t,v) + \sum_{K\in\tau_h} \alpha(\nabla u, \nabla v)_K$$

$$-\sum_{E\in\partial\tau_h} \int (\alpha\nabla u \cdot n)v ds + \sum_{E\in\partial\tau_h} \int (|\boldsymbol{b}\cdot n|u)v ds$$

$$-\sum_{K\in\tau_h} (\boldsymbol{b}\cdot\nabla u, v)_K,$$

where n denotes the outward normal to each element edge. The third and fourth terms on the right -hand side contain the integrals over the element edges, then we get

$$(f,v) = (u_t,v) + \sum_{K\in\tau_h} \alpha(\nabla u, \nabla v)_K$$

$$-\sum_{E\in\partial\tau_h} \int [(\alpha\nabla u \cdot n)v] ds + \sum_{E\in\partial\tau_h} \int [(|\boldsymbol{b}\cdot n|u)v] ds$$

$$-\sum_{K\in\tau_h} (\boldsymbol{b}\cdot\nabla u, v)_K.$$

Since $[uv] = \{u\}[v] + \{v\}[u]$, and u is continuous then $[u] = [a\nabla u \cdot n] = 0$, we get

$$(f,v) = (u_t,v) + \sum_{K\in\tau_h} \alpha(\nabla u, \nabla v)_K$$

$$-\sum_{E\in\partial\tau_h} \int \{\alpha\nabla u \cdot n\}[v] ds + \sum_{E\in\partial\tau_h} \int \{|\boldsymbol{b}\cdot n|u\}[v] ds$$

$$-\sum_{K\in\tau_h} (\boldsymbol{b}\cdot\nabla u, v)_K.$$

We note that the right- hand side of the above equation is still non-positivity and non-symmetric with respect to u and v, to rectify these properties. We add the terms [18]

$$\sigma \sum_{E\in\partial\tau_h} \int [u][v] ds \quad and \quad \epsilon \sum_{E\in\partial\tau_h} \int [u]\{\alpha\nabla v \cdot n\} ds,$$

we have,

$$(u_t,v) + \sum_{K\in\tau_h} \alpha(\nabla u, \nabla v)_K - \sum_{E\in\partial\tau_h} \int (\{\alpha\nabla u \cdot n\}[v]$$

$$-\epsilon[u]\{\alpha\nabla v \cdot n\}) ds + \sum_{E\in\partial\tau_h} \int \{|\boldsymbol{b}\cdot n|u\}[v] ds$$

$$-\sum_{K\in\tau_h} (\boldsymbol{b}\cdot\nabla u, v)_K + \sigma \sum_{E\in\partial\tau_h} \int [u][v] ds = (f,v).$$

$$(u_t,v) + \sum_{K\in\tau_h} \alpha(\nabla u, \nabla v)_K - \sum_{E\in\partial\tau_h} \int (\{\alpha\nabla u \cdot n\}[v]$$

$$-\epsilon[u]\{\alpha\nabla v \cdot n\}) ds + \sum_{E\in\partial\tau_h} \int \{|\boldsymbol{b}\cdot n|u\}[v] ds$$

$$-\sum_{K\in\tau_h} (\boldsymbol{b}\cdot\nabla u, v)_K + \sigma \sum_{E\in\partial\tau_h} \int [u][v] ds$$

$$= (f,v) + \sum_{E\in\Gamma^N} \int u^N v ds - \sum_{E\in\Gamma^D} \int \epsilon\alpha\nabla v \cdot n u^D ds$$

$$-\sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n| u^D v ds - \sigma \sum_{E\in\Gamma^D} \int u^D v ds. \quad (7)$$

Then the discontinuous variational formulation is, find $u \in V$ such that:

$$(u_t, v) + a(u, v) = l(v), \quad (8)$$

where,

$$a(u,v) = \sum_{K\in\tau_h} \alpha(\nabla u, \nabla v)_K - \sum_{E\in\partial\tau_h} \int (\{\alpha\nabla u\cdot n\}[v]$$

$$-\epsilon[u]\{\alpha\nabla v \cdot n\})ds + \sum_{E\in\partial\tau_h} \int \{|\boldsymbol{b}\cdot n|u\}[v]ds -$$

$$\sum_{K\in\tau_h} (\boldsymbol{b}\cdot\nabla u, v)_K + \sigma \sum_{E\in\partial\tau_h} \int [u][v]ds, \quad (9)$$

and,

$$l(v) = (f,v) + \sum_{E\in\Gamma^N} \int u^N v ds - \sum_{E\in\Gamma^D} \int \epsilon\alpha\nabla v\cdot n u^D ds$$

$$-\sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n| u^D v ds - \sigma \sum_{E\in\Gamma^D} \int u^D v ds. \quad (10)$$

The DGFE method is: find $u_h \in S_h \subset H^1(\Omega, \tau_h), \forall v \in S_h$, such that:

$$(u_{t,h}, v) + a(u_h, v) = l(v), \quad (11)$$

where,

$$a(u_h,v) = \sum_{K\in\tau_h} \alpha(\nabla u_h, \nabla v)_K - \sum_{E\in\partial\tau_h} \int (\{\alpha\nabla u_h\cdot n\}$$

$$[v] - \epsilon[u_h]\{\alpha\nabla v\cdot n\})ds + \sum_{E\in\partial\tau_h} \int \{|\boldsymbol{b}\cdot n|u_h\}[v]ds -$$

$$\sum_{K\in\tau_h} (\boldsymbol{b}\cdot\nabla u_h, v)_K + \sigma \sum_{E\in\partial\tau_h} \int [u_h][v]ds, \quad (12)$$

and,

$$l(v) = (f,v) + \sum_{E\in\Gamma^N} \int u^N v ds - \sum_{E\in\Gamma^D} \int \epsilon\alpha\nabla v\cdot n u^D ds$$

$$-\sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n| u^D v ds - \sigma \sum_{E\in\Gamma^D} \int u^D v ds. \quad (13)$$

6 Properties of the Bilinear form a (u,v).

($v-$elliptic). Assume that the penalty value σ is sufficiently large and that, $\beta_0 \geq (d-1)^{-1}$, there exists a positive constant ς independent of h such that,

$$a(u,u) \geq \varsigma ||u||^2_{H^1(\tau_h)}, \quad \forall u \in V. \quad (14)$$

Proof. put $v = u$ in the equation (9) we get

$$a(u,u) = \sum_{K\in\tau_h} \alpha(\nabla u, \nabla u)_K - \sum_{E\in\partial\tau_h} \int (\{\alpha\nabla u\cdot n\}[u]$$

$$-\epsilon[u]\{\alpha\nabla u \cdot n\})ds + \sum_{E\in\partial\tau_h} \int (\{|\boldsymbol{b}\cdot n|u\}[u])ds$$

$$-\sum_{K\in\tau_h} (\boldsymbol{b}\cdot\nabla u, u)_K + \sigma \sum_{E\in\partial\tau_h} \int [u][u]ds,$$

$$a(u,u) = \sum_{K\in\tau_h} \alpha(\nabla u, \nabla u)_K$$

$$+(\epsilon - 1) \sum_{E\in\partial\tau_h} \int (\{\alpha\nabla u\cdot n\}[u])ds$$

$$+\sum_{E\in\partial\tau_h} \int (\{|\boldsymbol{b}\cdot n|u\}[u])ds$$

$$-\sum_{K\in\tau_h}(\boldsymbol{b}\cdot\nabla u,u)_K+\sigma\sum_{E\in\partial\tau_h}\int[u][u]ds=\sum_{i=1}^{4}W_i. \quad (15)$$

To estimate W_1

$$W_1=\sum_{K\in\tau_h}\alpha(\nabla u,\nabla u)_K=\sum_{K\in\tau_h}||\alpha^{\frac{1}{2}}\nabla u||^2_{L^2(K)}. \quad (16)$$

For W_2

$$W_2=(\epsilon-1)\sum_{E\in\partial\tau_h}\int(\{\alpha\nabla u\cdot n\}[u])ds,$$

let $e=(\epsilon-1)$,

by the Cauchy inequality

$$W_2=e\sigma^{\frac{-1}{2}}\sigma^{\frac{1}{2}}\sum_{E\in\partial\tau_h}\int\{\alpha\nabla u\cdot n\}[v]ds$$

$$\leq e\sum_{E\in\partial\tau_h}(\int\sigma^{-1}(\{\alpha\nabla u\cdot n\})^2ds)^{\frac{1}{2}}$$

$$(\int\sigma^{1}[u]^2ds)^{\frac{1}{2}}$$

$$W_2\leq e(\sum_{E\in\partial\tau_h}\int\sigma^{-1}(\{\alpha\nabla u\cdot n\})^2ds)^{\frac{1}{2}}$$

$$(\sum_{E\in\partial\tau_h}\int\sigma^{1}[u]^2ds)^{\frac{1}{2}},$$

by using young inequality, we get

$$W_2\leq\frac{\delta}{2}(\sum_{E\in\partial\tau_h}\int\sigma^{-1}(\{\alpha\nabla u\cdot n\})^2ds)^{\frac{1}{2}}$$

$$\frac{e^2}{2\delta}(\sum_{E\in\partial\tau_h}\int\sigma^{1}[u]^2ds)^{\frac{1}{2}}. \quad (17)$$

To estimate W_3

$$W_3=\sum_{E\in\partial\tau_h}\int(\{|\boldsymbol{b}\cdot n|u\}[u])ds-\sum_{K\in\tau_h}(\boldsymbol{b}\cdot\nabla u,u)_K$$

$$=w_3^1+w_3^2. \quad (18)$$

To estimate w_3^1

$$w_3^1=\sum_{E\in\partial\tau_h}\int(\{|\boldsymbol{b}\cdot n|u\}[u])ds\leq\sum_{E\in\partial\tau_h}|\boldsymbol{b}\cdot n|$$

$$||[u]||_{L^2(E)}||u||_{L^2(E)}=\sum_{E\in\partial\tau_h}|\boldsymbol{b}\cdot n|\sigma^{\frac{1}{2}}$$

$$||[u]||_{L^2(E)}\sigma^{\frac{-1}{2}}||u||_{L^2(E)},$$

since

$$\sigma^{\frac{-1}{2}}||u||_{L^2(E)}\leq\frac{1}{2}h^{\frac{1}{2}}(||(u)_{K_1}||_{L^2(E)}+||(u)_{K_2}||_{L^2(E)}),$$

from the trace and Poincare inequality [19], we have

$$\frac{1}{2}h^{\frac{1}{2}}(||(u)_{K_1}||_{L^2(E)}+||(u)_{K_2}||_{L^2(E)})\leq\frac{1}{2}h^{\frac{1}{2}}G(h_{K_1}^{\frac{-1}{2}}$$

$$||(u)_{K_1}||_{L^2(K_1)}+h_{K_2}^{\frac{-1}{2}}||(u)_{E_2}||_{L^2(K_2)})$$

$$=\tfrac{1}{2}G(||(u)_{K_1}||_{L^2(K_1)}+||(u)_{K_2}||_{L^2(K_2)})\leq$$

$$\tfrac{1}{2}G(||u||_{L^2(K)}+||u||_{L^2(K)})=G||u||_{L^2(K)}$$

$$\leq G||u||_{H^1(K)},$$

then

$$w_3^1\leq\sum_{E\in\partial\tau_h}G|\boldsymbol{b}\cdot n|\sigma^{\frac{1}{2}}||[u]||_{L^2(E)}||u||_{H^1(K)}$$

$$\leq\vartheta\sum_{E\in\partial\tau_h}\sigma^{\frac{1}{2}}||[u]||_{L^2(E)}||u||_{H^1(K)}$$

$$\leq \frac{\delta}{2}\sum_{E\in\partial\tau_h}\sigma||[u]||^2_{L^2(E)}+\frac{\vartheta^2}{2\delta}||u||^2_{H^1(K)}. \quad (19)$$

Where, $\vartheta \geq G|\boldsymbol{b}\cdot n|$,

for w_3^2, by Schwartz and Young inequality, we have

$$w_3^2 = -\sum_{K\in\tau_h}(\boldsymbol{b}\cdot\nabla u,u)_K \leq \sum_{K\in\tau_h}|\boldsymbol{b}|||\nabla u||_{L^2(E)}||u||_{L^2(E)}$$

$$\leq \frac{\delta}{2}||u||^2_{L^2(\tau_h)}+\frac{\boldsymbol{b}^2}{2\delta}||\nabla u||^2_{L^2(\tau_h)}$$

$$\leq \varrho(||u||^2_{L^2(\tau_h)}+||\nabla u||^2_{L^2(\tau_h)})$$

$$=\varrho||u||^2_{H^1(\tau_h)},$$

hence

$$w_3^2 \leq \varrho||u||^2_{H^1(\tau_h)}. \quad (20)$$

Where, $\varrho=\max\{\frac{\delta}{2},\frac{\boldsymbol{b}^2}{2\delta}\}$.

To estimate W_4

$$W_4=\sum_{E\in\partial\tau_h}\int\sigma[u]^2ds \leq \sigma\sum_{E\in\partial\tau_h}(||[u]||_{L^2(E)})^2$$

$$=\sigma^2\sum_{E\in\partial\tau_h}(\sigma^{\frac{-1}{2}}||[u]||_{L^2(E)})^2,$$

since $\sigma^{\frac{-1}{2}}||[u]||_{L^2(E)} \leq G_t||u||_{L^2(\tau_h)}$, we have

$$W_4 \leq G_t||u||_{L^2(\tau_h)} \leq \sigma^2 G_t||u||^2_{H^1(\tau_h)}. \quad (21)$$

Substituting (16), (17), (18) and (21) in (15) we have,

$$a(u,u)=\sum_{K\in\tau_h}||\alpha^{\frac{1}{2}}\nabla u||^2_{L^2(K)}$$

$$+\frac{\delta}{2}(\sum_{E\in\partial\tau_h}\int\sigma^{-1}(\{\alpha\nabla u\cdot n\})^2ds)^{\frac{1}{2}}$$

$$+\frac{e^2}{2\delta}(\sum_{E\in\partial\tau_h}\int\sigma^1[u]^2ds)^{\frac{1}{2}}+\varrho||u||^2_{H^1(\tau_h)}+\sigma^2G_t||u||^2_{H^1(\tau_h)}$$

$$+\frac{\delta}{2}\sum_{E\in\partial\tau_h}\sigma||[u]||^2_{L^2(E)}+\frac{\vartheta^2}{2\delta}||u||^2_{H^1(K)}$$

$$a(u,u)\geq\mu(\sum_{K\in\tau_h}||\alpha^{\frac{1}{2}}\nabla u||^2_{L^2(K)}$$

$$+(\sum_{E\in\partial\tau_h}\int\sigma^{-1}(\{\alpha\nabla u\cdot n\})^2ds)^{\frac{1}{2}}$$

$$+\sum_{E\in\partial\tau_h}\sigma||[u]||^2_{L^2(E)}+||u||^2_{H^1(K)}+(\sum_{E\in\partial\tau_h}\int\sigma^1[u]^2ds)^{\frac{1}{2}})$$

$$+\varrho||u||^2_{H^1(\tau_h)}+\sigma^2G_t||u||^2_{H^1(\tau_h)},$$

where, $\mu=\min\{\frac{\delta}{2},\frac{\vartheta^2}{2\delta},1,\frac{e^2}{2\delta}\}$.

$$a(u,u)\geq p||u||^2_{H^(1^‘)(\tau_h)}+q||u||^2_{H^(1^‘)(\tau_h)},$$

then, $a(u,u)\geq\varsigma||u||^2_{H^1(\tau_h)}$,

where $q\leq(\varrho+\sigma^2G_t)$, and $\varsigma\leq(p+q)$. □

(countinuity): a bilinear form defined on V space equipped with norm $||\cdot||_V$ is continuous if there is a positive constant ε such that:

$$|a(u,v)|\leq\varepsilon||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)}, \quad \forall u,v\in V. \quad (22)$$

Proof. from the equation(9) we have,

$$|a(u,v)|=|\sum_{K\in\tau_h}\alpha(\nabla u,\nabla v)_K-\sum_{E\in\partial\tau_h}\int(\{\alpha\nabla u\cdot n\}[v]$$

$$-\epsilon\{\alpha\nabla v\cdot n\}[u])ds+\sum_{E\in\partial\tau_h}\int(\{|\boldsymbol{b}\cdot n|u\}[v])ds-\sum_{K\in\tau_h}(\boldsymbol{b}\cdot\nabla u,v)_K$$

$$+\sum_{E\in\partial\tau_h}\int\sigma[u][v]ds|,$$

$$|a(u,v)| \leq \sum_{K\in\tau_h} |\alpha(\nabla u,\nabla v)_K| + \sum_{E\in\partial\tau_h}\int |(\{\alpha\nabla u{\cdot}n\}[v]| + |\epsilon\{\alpha\nabla v{\cdot}n\}[u])|ds + \sum_{E\in\partial\tau_h}\int |(\{|\boldsymbol{b}{\cdot}n|u\}[v])|ds$$

$$+\sum_{K\in\tau_h} |(\boldsymbol{b}\cdot\nabla u,v)_K| + \sum_{E\in\partial\tau_h}\int |\sigma[u][v]ds|$$

$$= \sum_{i=1}^{5} W_i. \qquad (23)$$

To estimate W_1,

$$W_1 = \sum_{K\in\tau_h} (|\alpha(\nabla u,\nabla v)_K| + |(\boldsymbol{b}\cdot\nabla u,v)_K|$$

$$\leq \sum_{K\in\tau_h} (|\alpha|_{L^\infty}||\nabla u||_{L^2(K)}||\nabla v||_{L^2(K)} + |\boldsymbol{b}|_{L^\infty}||\nabla u||_{L^2(K)}||v||_{L^2(K)})$$

$$\leq g\sum_{K\in\tau_h} ||\nabla u||_{L^2(K)}(||\nabla v||_{L^2(K)} + ||v||_{L^2(K)})$$

$$\leq g\sum_{K\in\tau_h} (||\nabla u||_{L^2(K)} + ||u||_{L^2(K)})(||\nabla v||_{L^2(K)} + ||v||_{L^2(K)})$$

$$= g||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)}. \qquad (24)$$

Where, $g = \max\{|\alpha|_{L^\infty}, |\boldsymbol{b}|_{L^\infty}\}$.

To estimate W_2

$$W_2 = \sum_{E\in\partial\tau_h}\int |\{\alpha\nabla u\cdot n\}[v]|ds$$

$$= \sigma^{\frac{1}{2}}\sigma^{\frac{-1}{2}}\sum_{E\in\partial\tau_h}\int |\{\alpha\nabla u\cdot n\}[v]|ds,$$

by the Cauchy inequality

$$W_2 \leq \sum_{E\in\partial\tau_h} (\int \sigma^{-1}(\{\alpha\nabla u{\cdot}n\})^2 ds)^{\frac{1}{2}}(\int \sigma^1[v]^2 ds)^{\frac{1}{2}} \leq (\sum_{E\in\partial\tau_h}\int \sigma^{-1}(\{\alpha\nabla u{\cdot}n\})^2 ds)^{\frac{1}{2}}(\sum_{E\in\partial\tau_h}\int \sigma^1[v]^2 ds)^{\frac{1}{2}}$$

$$= |\alpha|\sigma G_t^2||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)}. \qquad (25)$$

To estimate W_3

$$W_3 = \sum_{E\in\partial\tau_h}\int |\epsilon\{\alpha\nabla v\cdot n\}[u]|ds$$

$$= \sigma^{\frac{-1}{2}}\sigma^{\frac{1}{2}}|\epsilon|\sum_{E\in\partial\tau_h}\int |\{\alpha\nabla v\cdot n\}[u]|ds,$$

by the Cauchy inequality

$$W_3 \leq |\epsilon|\sum_{E\in\partial\tau_h} (\int \sigma^{-1}(\{\alpha\nabla v\cdot n\})^2 ds)^{\frac{1}{2}} (\int \sigma^1[u]^2 ds)^{\frac{1}{2}},$$

$$= |\epsilon||\alpha|\sigma G_t^2||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)}. \qquad (26)$$

To estimate W_4

$$W_4 = \sum_{E\in\partial\tau_h}\int |\{|\boldsymbol{b}\cdot n|u\}[v]|ds$$

$$\leq \sigma G_t^2||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)}. \qquad (27)$$

To estimate W_5

$$W_5 = \sum_{E\in\partial\tau_h}\int |\sigma|[u][v]|ds \leq |\sigma|\sum_{E\in\partial\tau_h} ||[u]||_{L^2(E)} ||[v]||_{L^2(E)} \leq |\sigma|^2\sum_{E\in\partial\tau_h} \sigma^{\frac{-1}{2}}||[u]||_{L^2(E)}\sigma^{\frac{-1}{2}} ||[v]||_{L^2(E)},$$

since, $\sigma^{\frac{-1}{2}}||[u]||_{L^2(E)} \leq G_t||u||_{L^2(K)}$ and $\sigma^{\frac{-1}{2}}||[v]||_{L^2(E)} \leq G_t||v||_{L^2(K)}$, then,

$$W_5 \leq \sigma^2 G_t^2 ||u||_{L^2(\tau_h)} ||v||_{L^2(\tau_h)}. \quad (28)$$

Where G_t is a constant function [20]. Subsriting (24), (25), (26), (27) and (28) in (23), we have

$$|a(u,v)| \leq g||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)} + |\alpha|\sigma G_t^2||u||_{H^1(\tau_h)}$$

$$||v||_{H^1(\tau_h)} + |\epsilon||\alpha|\sigma G_t^2||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)}$$

$$+\sigma G_t^2||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)} + \sigma^2 G_t^2||u||_{L^2(\tau_h)}||v||_{L^2(\tau_h)}$$

$$= (g + |\alpha|\sigma G_t^2 + |\epsilon||\alpha|\sigma G_t^2 + \sigma G_t^2 + \sigma^2 G_t^2)||u||_{L^2(\tau_h)}$$

$$||v||_{L^2(\tau_h)} \leq \varepsilon ||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)},$$

then

$$|a(u,v)| \leq \varepsilon ||u||_{H^1(\tau_h)}||v||_{H^1(\tau_h)},$$

where, $\varepsilon \geq (g + |\alpha|\sigma G_t^2 + |\epsilon||\alpha|\sigma G_t^2 + \sigma G_t^2 + \sigma^2 G_t^2)$.

□

7 The Error Estimate of Semi-discrete DGFE Method:

In this section, we will introduce the error estimates of semi-discrete DGFE method for convection-diffusion problem (1)-(4). Namely, the error $u - u_h$, will be estimated in the $L^2(\Omega)$−norm.

Theorem 7.1. *Let $u \in L^2(H^1(\Omega))$ be the exact solution of (8), and $u_h \in S_h \subset H^1(\Omega, \tau_h)$ be the approximate solution of (11), and σ sufficiently large then there exists a constant C such that:*

$$||u - u_h||_{L^2(\Omega)} \leq Ch(||u||_{L^2(H^1)} + (||u_t||_{L^2(0,T;H^1(\Omega))}$$

$$+||u||_{L^2(0,T;H^1(K))})).$$

Proof. Let $\widetilde{u}$ be the L^2- Projection, and

$$e = u - u_h = u - \widetilde{u} + \widetilde{u} - u_h = \zeta - \chi,$$

then

$$||u - u_h||_{L^2(\Omega)} \leq ||\zeta||_{L^2(\Omega)} + ||\chi||_{L^2(\Omega)}. \quad (29)$$

From, [21], we have

$$||\zeta||_{L^2(\Omega)} \leq ch||u||_{L^2(H^1)}. \quad (30)$$

To estimate χ,
let $v = \chi$, we have,

$$(\chi_t, \chi) + a(\chi, \chi) = (\zeta_t, \chi) + a(\zeta, \chi). \quad (31)$$

Since,

$$(\chi_t, \chi) = \frac{1}{2}\frac{d}{dt}||\chi||^2_{L^2(\Omega)}. \quad (32)$$

By subtracting (1) from (4), we have,

$$((u - u_h)_t, v) + a(u - u_h, v) = ((\zeta - \chi)_t, v)$$

$$+a(\zeta-\chi,v)=0,$$

then,

$$(\zeta_t,v)+a(\zeta,v)=(\chi_t,v)+a(\chi,v),$$

From Lemmas (6), (6), by Schwartz and Young inequality of the equation (31), we have,

$$\frac{1}{2}\frac{d}{dt}||\chi||^2_{L^2(\Omega)}+\varsigma||\chi||^2_{L^2(K)}\leq\frac{\delta}{2}c^2h^2||u_t||^2_{L^2(H^1)}$$

$$+\frac{1}{2\delta}||\chi||^2_{L^2(K)}+\frac{\delta}{2}c^2h^2||u||^2_{L^2(H^1)}+\frac{\mu^2}{2\delta}||\chi||^2_{L^2(K)},$$

then,

$$\frac{1}{2}\frac{d}{dt}||\chi||^2_{L^2(\Omega)}+(\varsigma-\frac{1}{2\delta}-\frac{\mu^2}{2\delta})||\chi||^2_{L^2(K)}$$

$$\leq\frac{\delta}{2}c^2h^2||u_t||^2_{L^2(H^1)}+\frac{\delta}{2}c^2h^2||u||^2_{L^2(H^1)},$$

then,

$$\frac{1}{2}\frac{d}{dt}||\chi||^2_{L^2(\Omega)}+R||\chi||^2_{L^2(K)}\leq\frac{\delta}{2}c^2h^2||u_t||^2_{L^2(H^1)}$$

$$+\frac{\delta}{2}c^2h^2||u||^2_{L^2(H^1)},$$

where,$R\leq(\varsigma-\frac{1}{2\delta}-\frac{\mu^2}{2\delta})$. By integrating both sides in the last inequality from 0 to t, we have,

$$||\chi(t)||^2_{L^2(\Omega)}-||\chi^0||^2_{L^2(\Omega)}\leq\frac{\delta}{2}c^2h^2\int_0^t(||u_t||^2_{L^2(H^1)}$$

$$+||u||^2_{L^2(H^1)}),$$

since $\chi^0=0$, then

$$||\chi(t)||^2_{L^2(\Omega)}\leq\frac{\delta}{2}c^2h^2(||u_t||^2_{L^2(0,T;H^1(\Omega))}$$

$$+||u||^2_{L^2(0,T;H^1(K))}),$$

$$||\chi||_{L^2(\Omega)}\leq c\sqrt{\frac{\delta}{2}}h(||u_t||_{L^2(0,T;H^1(\Omega))}$$

$$+||u||_{L^2(0,T;H^1(K))}). \qquad (33)$$

Substituting (30) and (33) in (29), we have,

$$||u-u_h||_{L^2(\Omega)}\leq ch||u||_{L^2(H^1)}$$

$$+c\sqrt{\frac{\delta}{2}}h(||u_t||_{L^2(0,T;H^1(\Omega))}+||u||_{L^2(0,T;H^1(K))})$$

then

$$||u-u_h||_{L^2(\Omega)}\leq Ch(||u||_{L^2(H^1)}+(||u_t||_{L^2(0,T;H^1(\Omega))}$$

$$+||u||_{L^2(0,T;H^1(K))}))$$

where, $C\geq c+c\sqrt{\frac{\delta}{2}}$. □

8 The test problem.

In this section, we consider the Steady-State case for the convection-diffusion problem

$$-\alpha\Delta u+\boldsymbol{b}\cdot\nabla u+cu=f, \quad in \quad \Omega, \qquad (34)$$

with initial condition

$$u=0, \quad on \quad \partial\Omega.$$

Where $\Omega=[0,1]^2$, α is a diffusion coefficient, $\boldsymbol{b}$ is a known convection filed, and c is a known linear absorption term.
Let $c=0$, $f=(2\pi^2\sin(\pi x)\sin(\pi y)+\pi\cos(\pi x)\sin(\pi y)+\pi\cos(\pi y)\sin(\pi x))$, and the exact solution [19]

is $u = 2\sin(\pi x)\sin(\pi y)$.

The DGFE method for the problem (34)is: find $u_h \in S_h \subset H^1(\Omega, \tau_h), \forall v \in S_h$, such that:

$$a(u_h, v) = (f, v). \tag{35}$$

Where,

$$a(u_h, v) = \sum_{K\in\tau_h} \alpha(\nabla u_h, \nabla v)_K - \sum_{E\in\partial\tau_h} \int (\{\alpha\nabla u_h \cdot n\}[v]$$

$$-\epsilon[u_h]\{\alpha\nabla v \cdot n\})ds + \sum_{E\in\partial\tau_h} \int \{|\boldsymbol{b}\cdot n|u_h\}[v]ds -$$

$$\sum_{K\in\tau_h} (\boldsymbol{b}\cdot\nabla u_h, v)_K + \sigma \sum_{E\in\partial\tau_h} \int [u_h][v]ds.$$

Note that:- The parameter ϵ determines the type of DGFE method [22], which takes the values $\{-1, 1, 0\}$. If $\epsilon = -1$ gives symmetric interior penalty Galerkin (SIPG) method, if $\epsilon = 1$ gives non symmetric interior penalty Galerkin (NIPG) method and if $\epsilon = 0$ gives inconsistent interior penalty Galerkin (IIPG) method.

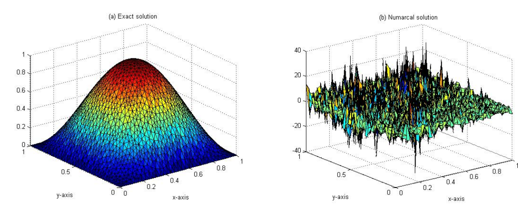
9 Numerical Results

In this section, we will discussion of numerical results for three cases of DGFE (SIPG, NIPG, IIPG),we find the errors $u - u_h$ of L^2-error and $L^\infty-$error of the equation (35) by using MATLAB software. We will discuss two test cases as follows:

Case (SIPG) first test: In this test when the diffusion coefficient α is taken from 0.00001 to 1 and the convection filed $\boldsymbol{b} = [1, 1]^2$, $h = \frac{1}{32}$, and the Penalty parameter ($\sigma = 3$), we see an oscillation occurs in the solution. See (Table 1) and Figures ((1) and (2)).

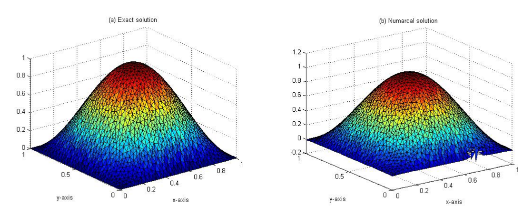
α	L^2-error	$L^\infty-$ error
0.00001	17.5223	38.9250
0.0001	36.3048	125.7973
0.001	2.6680	18.0626
0.01	1.1827	14.5484
0.1	0.7571	1.6832
1	0.0075	0.1364

Table 1: The L^2-error and $L^\infty-$error for SIPG method with $\sigma = 3$.



(a) (b)

Figure 1: The exact solution and numerical solution by DGFE method for SIPG with Penalty parameter $\sigma = 3$ and diffusion coefficient $\alpha = 0.00001$.



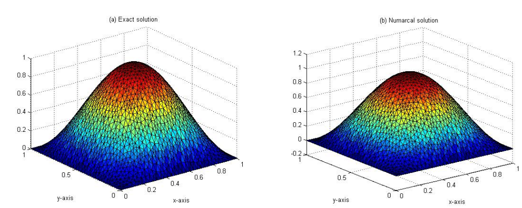
(a) (b)

Figure 2: The exact solution and numerical solution by DGFE method for SIPG with Penalty parameter $\sigma = 3$ and diffusion coefficient ($\alpha = 1$).

Case (SIPG) second test: In this test when the diffusion coefficient α is taken from 0.00001 to 1 and the convection filed $\boldsymbol{b} = [1, 1]^2$, $h = \frac{1}{32}$, and the Penalty parameter ($\sigma = 2187$), we see we see an Oscillation disappears in the solution. See (Table 2) and Figure (3).

α	L^2-error	$L^\infty-$ error
0.00001	4.6e-03	10.5e-03
0.0001	1.7e-03	5.7e-03
0.001	8.4715e-04	1.6e-03
0.01	6.519e-04	9.4223e-04
0.1	5.3468e-04	8.4381e-04
1	5.2243e-04	8.3620e-04

Table 2: The L^2-error and $L^\infty-$error for SIPG method with $\sigma = 2187$.



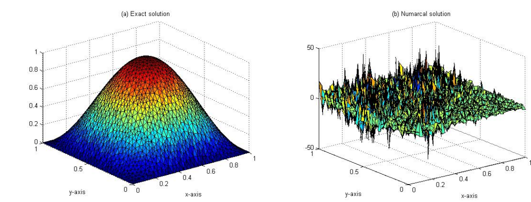
(a) (b)

Figure 3: The exact solution and numerical solution by DGFE method for SIPG with Penalty parameter $\sigma = 2187$ and diffusion coefficient ($\alpha = 0.00001, 0.0001, 0.001, 0.01,$ $0.1, 1$).

Case (NIPG) first test: In this test when the diffusion coefficient α is taken from 0.00001 to 1 and the convection filed $\boldsymbol{b} = [1, 1]^2$, $h = \frac{1}{32}$, and the Penalty parameter ($\sigma = 3$), we see an oscillation occurs in the solution. See (Table 3) and Figures ((4) and (5)).

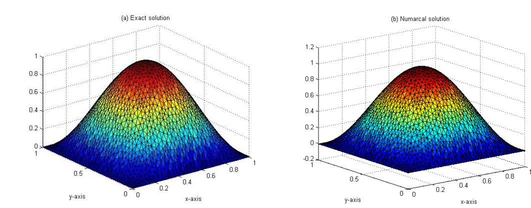
α	L^2-error	$L^\infty-$ error
0.00001	19.4537	44.5124
0.0001	211.5645	987.2391
0.001	1.7566	17.4378
0.01	0.8575	1.2855
0.1	0.8163	0.7092
1	0.0016	0.0020

Table 3: The L^2-error and $L^\infty-$error for NIPG method with $\sigma = 3$.



(a) (b)

Figure 4: The exact solution and numerical solution by DGFE method for NIPG with Penalty parameter $\sigma = 3$ and diffusion coefficient $\alpha = 0.00001$.



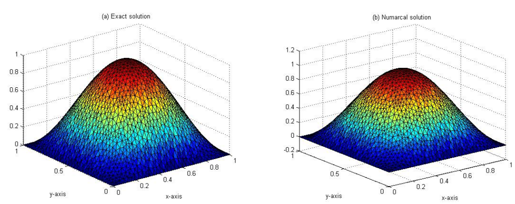
(a) (b)

Figure 5: The exact solution and numerical solution by DGFE method for NIPG with Penalty parameter $\sigma = 3$ and diffusion coefficient $\alpha = 1$.

Case (NIPG) second test: In this test when the diffusion coefficient α is taken from 0.00001 to 1 and the convection filed $\boldsymbol{b} = [1, 1]^2$, $h = \frac{1}{32}$, and the Penalty parameter ($\sigma = 2187$), we see we see an Oscillation disappears in the solution. See (Table 4) and Figure (6).

α	L^2-error	$L^\infty-$ error
0.00001	4.6e-03	10.5e-03
0.0001	1.7e-03	5.6e-03
0.001	8.6102e-04	1.6e-03
0.01	7.9842e-04	1.0e-03
0.1	6.772e-04	9.6485e-04
1	5.1974e-04	8.3327e-04

Table 4: The L^2-error and $L^\infty-$error for NIPG method with $\sigma = 2187$.



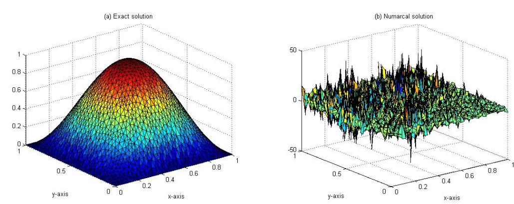
(a) (b)

Figure 6: The exact solution and numerical solution by DGFE method for NIPG with Penalty parameter $\sigma = 2187$ and diffusion coefficient $(\alpha = 0.00001, 0.0001, 0.001, 0.01,$
$0.1, 1)$.

Case (IIPG) first test: In this test when the diffusion coefficient α is taken from 0.00001 to 1 and the convection filed $\boldsymbol{b} = [1, 1]^2$, $h = \frac{1}{32}$, and the Penalty parameter $(\sigma = 3)$, we see an oscillation occurs in the solution. See (Table 5) and Figures ((7) and (8)).

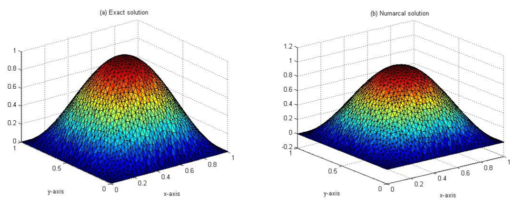
α	L^2-error	$L^\infty-$ error
0.00001	18.4537	41.1482
0.0001	4.5239	12.0043
0.001	1.2098	7.0194
0.01	0.8651	3.0365
0.1	0.7685	0.6633
1	0.0014	0.0019

Table 5: The L^2-error and $L^\infty-$error for NIPG method with $\sigma = 3$.



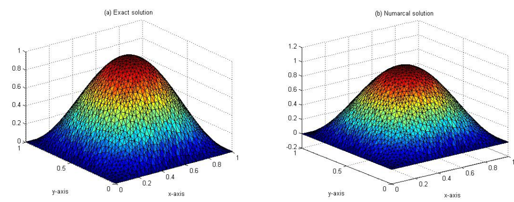
(a) (b)

Figure 7: The exact solution and numerical solution by DGFE method for IIPG with Penalty parameter $\sigma = 3$ and diffusion coefficient $(\alpha = 0.00001)$.



(a) (b)

Figure 8: The exact solution and numerical solution by DGFE method for IIPG with Penalty parameter $\sigma = 3$ and diffusion coefficient $(\alpha = 1)$.



(a) (b)

Figure 9: The exact solution and numerical solution by DGFE method for IIPG with Penalty parameter $\sigma = 2187$ and diffusion coefficient $(\alpha = 0.00001, 0.0001, 0.001, 0.01,$
$0.1, 1))$.

Case (IIPG) second test: In this test when the diffusion coefficient α is taken from 0.00001 to 1 and the convection filed $\boldsymbol{b} = [1,1]^2$, $h = \frac{1}{32}$, and the Penalty parameter $(\sigma = 2187)$, we see we see an Oscillation disappears in the solution. See (Table 6) and Figure (9).

α	L^2-error	$L^\infty-$ error
0.00001	4.6e-03	10.5e-03
0.0001	1.7e-03	5.6e-03
0.001	8.6102e-04	1.6e-03
0.01	7.2213e-04	9.9546e-04
0.1	6.2389e-04	8.9837e-04
1	5.2108e-04	8.3474e-04

Table 6: The L^2-error and $L^\infty-$error for IIPG method with $\sigma = 2187$.

10 Conclusion

The problem (34)solved by many authors using continuous finite element method, and they saw when the diffusion coefficient $\alpha > \boldsymbol{b}$ or $\alpha > h$ the numerical solution is converge and without any oscillation, but when $\alpha \ll \boldsymbol{b}$ or $\alpha \ll h$, the numerical results got an oscillation and dose not converge to the exact solution (see [17]). In this paper, we solved the problem (34) by DGFE method, we saw that,

1. The effect of parameter σ to the numerical solution

   - When $h > \alpha$, and the Penalty parameter σ is large enough, the oscillation is vanished.
   - When the values of $h = \alpha$, $h > \alpha$ and $h < \alpha$ and the Penalty parameter σ is small, the solution becomes oscillated.

2. The effect of parameterϵ to the numerical

solution

- In the SIPG ($\epsilon = -1$) and the Penalty parameter σ is small, we got an error and the numerical solutions become oscillation see (table 1)and Figures (1 and 2).
- In the NIPG ($\epsilon = 1$) and the Penalty parameter σ is small, we note that the amount $(\{\alpha\nabla u_h{\cdot}n\}[v]-\epsilon[u_h]\{\alpha\nabla v{\cdot}n\})$ will disappear therefore, the error increases and the oscillation increases more than the first case SIPG, see (table 3)and Figures (4 and 5).
- In the IIPG ($\epsilon = 0$) and the Penalty parameter σ is small, we note that the amount $(\epsilon[u_h]\{\alpha\nabla v{\cdot}n\})$ will disappear therefore, the error increases and the oscillation increases more than the first case SIPG, also the error and oscillation becomes less than the second case NIPG, (table 5)and Figures (7 and 8).
- In all cases, when the parameter σ is large enough, the solution convergence and the oscillation will be removed, and in this case, the theoretical side in theorem (7.1) corresponds to the practical side.

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تحليل الأخطاء لطريقة كاركن الغير مستمرة للعناصر المحددة لمسألة الانتشار والحمل الحراري

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الخلاصة:

في هذه البحث، درسنا وحللنا طريقة كاركن الغير مستمرة للعناصر المحددة لمسألة الانتشار والحمل الحراري الخطي. حيث برهنا خصائص الثنائية الخطية $a(u,v)$ (الاهليجية و الاستمرارية). وقمنا بتحليل الخطأ بالتقطيع الشبه التام وأثبتنا الحل التقريبي متقارب بنسبة خطأ $O(h)$. وتم تأكيد النتائج النظرية من خلال التجارب العددية للحالة المستقرة للحالات (SIPG, NIPG, IIPG) ، عندما تكون المعلمة σ كبيره بما فيه الكفاية واظهار تأثير المعلمة σ والمعلمة ε على الحلول العددي.

الكلمات المفتاحية:

مسائلة الانتشار والحمل الحراري، أسلوب DGFE الحالة الثابتة، تحليل الأخطاء