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Space-Time Petrov- Discontinuous Galerkin Finite Element Method for Solving Linear Convection-Diffusion Problems

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Abstract. The paper presents the theory of the space-time Petrov-discontinuous Galerkin finite element (PDGFE) method for the discretization of the nonstationary linear convection-diffusion problems. The PDGFE method is modified for the discontinuous Galerkin finite element (DGFE) method in the case of the symmetric interior penalty Galerkin (SIPG) scheme. PDGFE method is applied separately in space using different space gride on different time levels. We prove the properties of the bilinear form $a_{PD,m}(u,v)$ $(V-$ elliptic and continuity) stability and prove the approximate solution converges with the error of order $O(h^2+\tau^3)$. A numerical experiment is carried out to confirm the theoretical conclusions.

Keywords. linear convection-diffusion equation, Petrov-discontinuous Galerkin finite element method, space-time discretization, error estimate.

1. Introduction

The mechanisms of convection-diffusion occur in many fields of science and technology. Fluid mechanics, hydrology for heat and mass transport, for instance, and so on. In this paper, let consider the convection-diffusion problem: Find $u \in Q_T = \Omega \times (0,T) \to \mathbb{R}$, such that [1-2]:

$$u_t - \lambda\Delta u + \boldsymbol{b}\cdot\nabla u = f \quad in\ Q_T\,, \tag{1a}$$
$$u = u^D \quad on \quad \partial\Omega^D \times (0,T)\,, \tag{1b}$$
$$\lambda\frac{\partial u}{\partial n} = u^N \quad on \quad \partial\Omega^N \times (0,T)\,, \tag{1c}$$
$$u(x,0) = u^0(x), \quad x \in \Omega\,, \tag{1d}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polyhedral domain, T > 0.

Assume that $\partial\Omega = \partial\Omega^D \cup \partial\Omega^N$

$$\boldsymbol{b}\cdot n < 0 \quad on \quad \partial\Omega^D\,, \tag{1e}$$
$$\boldsymbol{b}\cdot n \geq 0 \quad on \quad \partial\Omega^N \quad ; \quad \forall\, t \in [0,T], \tag{1f}$$

where n denotes the natural outer unit to the border $\partial\Omega$ of Ω, $\partial\Omega^D$ denotes the inflow boundary, and $\partial\Omega^N$ denotes the outflow boundary.

The two major components of this problem are the terms of convection with the field of convection velocity and the terms of diffusion with the coefficient of diffusion. It is well known that directly solving singly perturbed convection-diffusion problems with the Galerkin finite element (GFE) method can result in spurious oscillation in the approximate solution.i.e. For $h > 0$, $\frac{\lambda}{|\boldsymbol{b}|h} \ll 1$, this situation can arise as a result of any combination of weak diffusion (small), strong convection (big), or a vast domain, with the latter case being the most accurate in geophysical applications.

Several techniques have been thoroughly investigated to overcome such a drawback.

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It is a frequent idea to include stabilizing terms in the problem formulation. This is mostly accomplished by stabilizing procedures such as the upwinding method [3-5], Petrov-Galerkin method [6-10], artificial diffusivity method [11-12], continuous dependence and nonlinear stability [21-23], oscillation criteria [24-25].

In the 1970s, researchers developed a new method called the Discontinues Galerkin finite element (DGFE) method to solve these problems. The DGFE method technique approximates the approximate bounds of the ideal grid solution on finite elements without any consistency constraints. The DGFE method takes a three-class [2], symmetric interior penalty Galerkin (SIPG) method if $\varepsilon = -1$, inconsistent penalty Galerkin (IIPG) method if $\varepsilon = 0$ and nonsymmetric interior penalty Galerkin (NIPG) method if $\varepsilon = 1$. The DGFE method utilizes the same function space as the continuous finite element (FE)method and finite volume (FV) method, but with relaxed continuity at inter-element borders, and may be regarded as an extension of both. When $\lambda < h$, where h is the mesh size, and the standard Galerkin finite element approach produces an oscillating solution that is not close to the precise solution, the convection component dominates over diffusion [13-18]. The PDGFE method is a better and more useful version of the DGFE method. The shape and trail functions are in the same field in the DGFE method, but the test function space in the PDGFE method is distinct from the trial function space.

The SIPG for the linear convection-diffusion problem uses a space-time Petrov-discontinuous Galerkin discretization, which is of importance in this paper. The time interval is divided into subintervals, and a different space mesh may be employed at each time level in general. Furthermore, the spatial discretization triangulations employed may be nonconforming with hanging nodes.

The following is a breakdown of the paper structure. In section 2, we have shown the space-time discretization. Derive the variation formulation in section 3. In section 4, we proved the properties of the bilinear form and stability. The error estimate is presented in section 5. In section 6, We presented numerical evidence to back up our theoretical findings. Finally, the conclusions are presented. in section7.

2. The space-time discretization:

We shall make a partition $0 = t_0 < t_1 < \cdots < t_M = T$, in the time interval $t \in [0, T]$, and denote $I_m = (t_{m-1},\ t_m), \tau_m = t_m - t_{m-1},\ \tau = \max_{m=1,\cdots,M} \tau_m$. We assume a partition $\mathrm{T_{h,m}}$ of the closure $\overline{\Omega}$ of the domain Ω into a limited number of closed triangles with mutually disjoint interiors for each I_m. For different values m, the partitions $\mathrm{T_{h,m}}$ differ. The system of all edges E of all elements $K \in \mathrm{T_{h,m}}$ is represented by $\partial\mathrm{T_{h,m}}$. The set of all inner and boundary edges is further denoted by,[19]:

$$\partial\mathrm{T_{h,m}^{I}} = \{\mathrm{E} \in \partial\mathrm{T_{h,m}},\ \mathrm{E} \subset \Omega\},$$
$$\partial T_{h,m}^{B} = \{E \in \partial T_{h,m},\ E \subset \partial\Omega\},$$
$$\Gamma^{D} = \{E \in \partial\mathrm{T}_{h,m}^{B}, E \subset \partial\Omega^{D}\},$$
$$\Gamma^{\mathrm{N}} = \{E \in \partial\mathrm{T}_{h,m}^{B}, E \subset \partial\Omega^{N}\}.$$

The following notation for $\varphi \in H^1(\Omega, \mathrm{T_{h,m}})$. Obviously
$\mathrm{T}_{h,m} = \partial\mathrm{T_{h,m}^{I}} \cup \partial\, \partial\mathrm{T_{h,m}^{B}},\quad \partial\mathrm{T_{h,m}^{B}} = \Gamma^{D} \cup \Gamma^{\mathrm{N}}$ for each $E \in \partial\mathrm{T_{h,m}}$.

For a function φ defined in $\bigcup_{m=1}^{M} I_m$. Put

$$\varphi_m^{+} = \varphi(t_{m+}) = \lim_{t \to t_{m+}} \varphi(t),\ \varphi_m^{-} = \lim_{t \to t_{m-}} \varphi(t), [\varphi]_m = (\varphi_m^{+} - \varphi_m^{-}),$$
$$\{\varphi\}_E = \frac{1}{2}(\varphi^{+} + \varphi^{-}),$$
$$[\varphi]_E = (\varphi^{+} - \varphi^{-}),$$

further,

$$\partial K_{-}^{i} = \{x \in \partial K^{i}: b \cdot n < 0\},$$
$$\partial K_{+}^{i} = \{x \in \partial K^{i}: b \cdot n \geq 0\},$$

where n denotes the unit outer normal to ∂K^{i}.

We define the broken Sobolev space:

$$\mathrm{H}^{r}(\Omega, \mathrm{T_{h,m}}) = \{v \in \mathrm{L}^2(\Omega); v|_{\mathrm{K}} \in \mathrm{H^r(K)}\ \forall \mathrm{K} \in \mathrm{T_{h,m}}\},$$

for $v \in \mathrm{H^r}(\Omega, \mathrm{T_{h,m}})$, we define the norm

$$\|u\|_{H^r(\Omega,T_{h,m})} = \left(\sum_{K\in T_h} \|u\|^2_{H^r(\Omega)}\right)^{\frac{1}{2}},$$

and the semi-norm

$$|u|_{H^r(\Omega,T_{h,m})} = \left(\sum_{K\in T_{h,m}} |u|^2_{H^r(\Omega)}\right)^{\frac{1}{2}}.$$

Let $l \geq 1$ be an integer. We denote the space of discontinuous piecewise polynomial functions

$$S_{h,m} = \{v \in L^2(\Omega);\, v|_K \in P_l(K);\ \forall K \in T_{h,m}\},$$

where $P_r(K)$ denotes the space of all polynomials of degree $\leq l$ on K. We call the number l the degree of polynomial approximation. Obviously, $S_{h,m} \subset H^r(\Omega, T_{h,m})$.

Let ϑ be trial space and $\emptyset$ be a test space,

$$\vartheta = H^r(\Omega, T_{h,m}) = \{v : v \in L^2(\Omega);\, v|_K \in H^r(K);\ \forall K \in T_{h,m}\},$$
$$\emptyset = \{w : w = v + \delta \boldsymbol{b}\cdot\nabla v;\, v \in \vartheta\}.$$

We defined PDGFE space

$$\vartheta_{h,m} = \{v : v \in L^2(\Omega);\, v|_K \in P_l(K);\ \forall K \in T_{h,m}\},$$
$$\emptyset_{h,m} = \{w : w = v + \delta \boldsymbol{b}\cdot\nabla v;\, v \in \vartheta_{h,m}\},$$

where δ denotes a constant stability parameter in Q_T.

It can be selected as [7],

$$\delta \equiv \begin{cases} \eta h & \text{if } \lambda < h \\ 0 & \text{if } \lambda \geq h \end{cases};\ 0 < \eta < \frac{1}{4},$$

and dim ϑ = dim $\emptyset$.

Assumption:

a) $f \in C([0,T]; L^2(\Omega)),\, u^0, u,\, u_t \in L^2(\Omega)$.

b) u^D is the trace of some $u \in C([0,\ T];\ H^1(\Omega)) \cap L^\infty(Q_T)$ on $\partial\Omega^D \times (0, T)$.

c) $u^N \in C([0,T]; L^2(\partial\Omega_N))$.

d) $|K|$ = the area of $K \in T_{h,m}$, and $\sigma = \frac{\sigma^0}{|E|^{\beta_0}}$, $\beta_0 \geq (d-1)^{-1}, \sigma^0 > 0$.

e) Define h_K = the length of the longest side of the triangle $K \in T_{h,m}$ and h_K= diameter of K. $h = \max_{K\in T_{h,m}} h_K$.

3. The variation formulation of the Problem.

We present the SIPG variant of the PDG approximation by multiplying equation (1a) by test function w to find $u \in \vartheta$ such that:

$$\int_{I_m} (u_t, w)\, dt + (\{u\}_{m-1}, w^+_{m-1})$$
$$+ \int_{I_m} \Bigg(\sum_{K\in T_{h,m}} \lambda(\nabla u, \nabla w)_K - \sum_{E\in\partial T_{h,m}} \int (\{\lambda\nabla u\cdot n\}[w] - \varepsilon[u]\{\lambda\nabla w\cdot n\})\, ds$$
$$+ \sum_{E\in\partial T_{h,m}} \int (\{|\boldsymbol{b}\cdot n|u\}[w])\, ds - \sum_{K\in T_{h,m}} (\boldsymbol{b}\cdot\nabla u, w)_K + \sigma \sum_{E\in\partial T_{h,m}} \int [u][w] ds \Bigg) dt$$
$$= \int_{I_m} \Bigg((f, w) + \sum_{E\in\Gamma^N} \int u^N w ds + \sum_{E\in\Gamma^D} \int \lambda\nabla w\cdot n u^D\, ds - \sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n| u^D w\, ds$$
$$- \sigma \sum_{E\in\Gamma^D} \int u^D w\, ds \Bigg) dt.$$

Since $\varepsilon = -1$(SIPG) then

$$\int_{I_m} (u_t, w)\, dt + (\{u\}_{m-1}, w_{m-1}^+) + \int_{I_m} \Bigg(\sum_{K\in\mathrm{T}_{h,m}} \lambda(\nabla u, \nabla w)_K - \sum_{E\in\partial\mathrm{T}_{h,m}} \int (\{\lambda\nabla u\cdot n\}[w] + [u]\{\lambda\nabla w\cdot n\})\, ds + \sum_{E\in\partial\mathrm{T}_{h,m}} \int (\{|\boldsymbol{b}\cdot n|u\}[w])\, ds - \sum_{K\in\mathrm{T}_{h,m}} (\boldsymbol{b}\cdot\nabla u, w)_K + \sigma \sum_{E\in\partial\mathrm{T}_{h,m}} \int [u][w] ds \Bigg) dt = \int_{I_m} \Bigg((f, w) + \sum_{E\in\Gamma^N} \int u^N w ds + \sum_{E\in\Gamma^D} \int \lambda\nabla w\cdot n u^D\, ds - \sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n| u^D w\, ds - \sigma \sum_{E\in\Gamma^D} \int u^D w\, ds \Bigg) dt\ ; \quad \forall w \in \emptyset\,. \quad (2)$$

$$\int_{I_m} (u_t, v + \delta\cdot\boldsymbol{b}\nabla v)\, dt + (\{u\}_{m-1}, v_{m-1}^+ + \delta\cdot\boldsymbol{b}\nabla v_{m-1}^+) + \int_{I_m} \Bigg(\sum_{K\in\mathrm{T}_{h,m}} \lambda(\nabla u, \nabla(v + \delta\boldsymbol{b}\cdot\nabla v))_K - \sum_{E\in\partial\mathrm{T}_{h,m}} \int (\{\lambda\nabla u\cdot n\}[v + \delta\boldsymbol{b}\cdot\nabla v] + [u]\{\lambda\nabla(v + \delta\boldsymbol{b}\cdot\nabla v)\cdot n\}) ds + \sum_{E\in\partial\mathrm{T}_{h,m}} \int (\{|\boldsymbol{b}\cdot n|u\}[v + \delta\boldsymbol{b}\cdot\nabla v])\, ds - \sum_{K\in\mathrm{T}_{h,m}} (\boldsymbol{b}\cdot\nabla u, v + \delta\boldsymbol{b}\cdot\nabla v)_K + \sigma \sum_{E\in\partial\mathrm{T}_{h,m}} \int [u][v + \delta\boldsymbol{b}\cdot\nabla v] ds = (f, v + \delta\boldsymbol{b}\cdot\nabla v) + \sum_{E\in\Gamma^N} \int u^N (v + \delta\boldsymbol{b}\cdot\nabla v) ds + \sum_{E\in\Gamma^D} \int \lambda\nabla(v + \delta\boldsymbol{b}\cdot\nabla v)\cdot n u^D\, ds - \sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n| u^D (v + \delta\boldsymbol{b}\cdot\nabla v) ds - \sigma \sum_{E\in\Gamma^D} \int u^D (v + \delta\boldsymbol{b}\cdot\nabla v)\, ds \Bigg) dt\,; \quad \forall v \in \vartheta. \quad (3)$$

The variation formulation of the PDGFE method is found $u \in \vartheta \ni$

$$\int_{I_m} \Big((u_t, v) + (u_t, \delta\boldsymbol{b}\cdot\nabla v) + a_{PD,m}(u, v) \Big) dt + (\{u\}_{m-1}, v_{m-1}^+) + (\{u\}_{m-1}, \delta\cdot\boldsymbol{b}\nabla v_{m-1}^+) = \int_{I_m} \Bigg(\sum_{E\in\Gamma^N} \int u^N (v + \delta\boldsymbol{b}\cdot\nabla v) ds + (f, v) + (f, \delta\boldsymbol{b}\cdot\nabla v) + \sum_{E\in\Gamma^D} \int \lambda\nabla(v + \delta\boldsymbol{b}\cdot\nabla v)\cdot n u^D ds - \sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n| u^D (v + \delta\boldsymbol{b}\cdot\nabla v) ds - \sigma \sum_{E\in\Gamma^D} \int u^D (v + \delta\boldsymbol{b}\cdot\nabla v)\, ds \Bigg) dt\,; \quad \forall v \in \vartheta\,, \quad (4)$$

where

$$a_{PD,m}(u, v) = \sum_{K\in\mathrm{T}_{h,m}} \lambda(\nabla u, \nabla v)_K - \sum_{E\in\partial\mathrm{T}_{h,m}} \int (\{\lambda\nabla u\cdot n\}[v] + [u]\{\lambda\nabla v\cdot n\}) ds$$

$$+\sum_{E\in\partial \mathrm{T}_{h,m}}\int(\{|\boldsymbol{b}\cdot n|u\}[v])\,ds-\sum_{K\in\mathrm{T}_{h,m}}(\boldsymbol{b}\cdot\nabla u,v+\delta\boldsymbol{b}\cdot\nabla v)_K+\sigma\sum_{E\in\partial\mathrm{T}_{h,m}}\int[u][\,v]ds. \quad (5)$$

The PDGFE method is to find $u_h\in\vartheta_{h,m},\forall v\in\vartheta_{h,m}$, such that:

$$\int_{I_m}\Big((u_{h,t},v)+(u_{h,t},\delta\boldsymbol{b}\cdot\nabla v)+a_{PD,m}(u_h,v)\Big)dt+(\{u_h\}_{m-1},v_{m-1}^{+})$$
$$+(\{u_h\}_{m-1},\delta\cdot\boldsymbol{b}\nabla v_{m-1}^{+})=\int_{I_m}\Bigg(\sum_{E\in\Gamma^N}\int u^N(v+\delta\boldsymbol{b}\cdot\nabla v)ds+(f,v)+(f,\delta\boldsymbol{b}\cdot\nabla v)$$
$$+\sum_{E\in\Gamma^D}\int\lambda\nabla(v+\delta\boldsymbol{b}\cdot\nabla v)\cdot nu^D ds-\sum_{E\in\Gamma^D}\int|\boldsymbol{b}\cdot n|u^D(v+\delta\boldsymbol{b}\cdot\nabla v)ds$$
$$-\sigma\sum_{E\in\Gamma^D}\int u^D(v+\delta\boldsymbol{b}\cdot\nabla v)\,ds\Bigg)dt, \qquad (6)$$

where

$$a_{PD,m}(u_h,v)=\sum_{K\in\mathrm{T}_{h,m}}\lambda(\nabla u_h,\nabla v)_K-\sum_{E\in\partial\mathrm{T}_{h,m}}\int(\{\lambda\nabla u_h\cdot n\}[v]+[u_h]\{\lambda\nabla v\cdot n\})\,ds$$
$$+\sum_{E\in\partial\mathrm{T}_{h,m}}\int(\{|\boldsymbol{b}\cdot n|u_h\}[v])\,ds-\sum_{K\in\mathrm{T}_{h,m}}(\boldsymbol{b}\cdot\nabla u_h,v+\delta\boldsymbol{b}\cdot\nabla v)_K+\sigma\sum_{E\in\partial\mathrm{T}_{h,h}}\int[u_h][\,v]ds\,.$$

4. The properties of $a_{PD,m}(u,v)$ and stability.

In this section, we prove the bilinear form (V −elliptic, continuous) and stability.

Lemma 1. ($V-$ elliptic). Suppose that the penalty value σ is sufficiently large and that, $\beta_0\geq(d-1)^{-1}$, there exists a positive constant $\aleph$ independent of h and τ such that,

$$a_{PD,m}(u,u)\geq\ \aleph\,\|u\|^2_{H^1(\mathrm{T}_{h,m})},\quad\forall u\in\vartheta\,,K\in\mathrm{T}_{h,m}. \qquad (7)$$

Proof: Let $v=u$ in the equation (5) we get

$$a_{PD,m}(u,u)=\sum_{K\in\mathrm{T}_{h,m}}\lambda(\nabla u,\nabla u)_K-\sum_{E\in\partial\mathrm{T}_{h,m}}\int(\{\lambda\nabla u\cdot n\}[u]+[u\,]\{\lambda\nabla u\cdot n\})ds$$
$$+\sum_{E\in\partial\mathrm{T}_{h,m}}\int(\{|\boldsymbol{b}\cdot n|u\}[u])ds-\sum_{K\in\mathrm{T}_{h,m}}(\boldsymbol{b}\cdot\nabla u,u+\delta\boldsymbol{b}\cdot\nabla u)_K$$
$$+\sigma\sum_{E\in\partial\mathrm{T}_{h,m}}\int[u]^2ds\ . \qquad (8)$$

We define the energy norm

$$||u||_{H^1(\mathrm{T}_{h,m})}=\left(\sum_{K\in\mathrm{T}_{h,m}}\left\|\lambda^{\frac{1}{2}}\nabla u\right\|^2_{L^2(K)}+\sum_{K\in\mathrm{T}_{h,m}}\|\boldsymbol{b}.\nabla u\|^2_{L^2(K)}+\sigma\sum_{E\in\partial\mathrm{T}_{h,m}}||[u]||^2_{L^2(E)}\right)^{\frac{1}{2}}.$$

By using Schwartz and young inequalities [20] we get,

$$a_{PD,m}(u,u)=\sum_{K\in\mathrm{T}_{h,m}}\left\|\lambda^{\frac{1}{2}}\nabla u\right\|^2_{L^2(K)}+\varrho\|u\|^2_{H^1(\mathrm{T}_{h,m})}+\varrho\|u\|^2_{H^1(\mathrm{T}_{h,m})}$$
$$+\frac{\beta}{2}\sum_{E\in\partial\mathrm{T}_{h,m}}\sigma\ \ \|[u]\|^2{}_{L^2(E)}+\frac{\omega^2}{2\beta}\|u\|^2_{H^1(\mathrm{T}_{h,m})}\quad+\sigma\sum_{E\in\partial\mathrm{T}_{h,m}}||[u]||^2_{L^2(E)}$$
$$+\sum_{K\in\mathrm{T}_{h,m}}\delta\ \|\ \boldsymbol{b}\cdot\nabla u\ \|^2_{L^2(\mathrm{K})}.$$

$$a_{PD,m}(u,u) \geq g\left(\sum_{K\in \mathrm{T}_{h,m}} \left\|\lambda^{\frac{1}{2}}\nabla u\right\|_{L^2(K)}^2 + \sum_{K\in \mathrm{T}_{h,m}} \|\boldsymbol{b}.\nabla u\|_{L^2(K)}^2 + \sigma \sum_{E\in\partial \mathrm{T}_{h,m}} \left|\left|[u]\right|\right|_{L^2(E)}^2\right)^{\frac{1}{2}} + q\|u\|_{H^{1`}(\mathrm{T}_{h,m})}^2,$$

where, $g = \min\left(1, \left(1+\frac{\beta}{2}\right), \delta\right)$.

$$a_{PD,m}(u\,,u) \geq g\|u\|_{H^{1`}(\mathrm{T}_{h,m})}^2 + q\|u\|_{H^{1`}(\mathrm{T}_{h,m})}^2,$$

then,

$$a_{PD,m}(u,u) \geq \aleph\,\|u\|_{H^1(\mathrm{T}_{h,m})}^2,$$

where $q \leq \left(2\varrho + \frac{\omega^2}{2\beta}\right)$, and $\aleph \leq (g+q)$.

Lemma 2. ($countinuity$): Let $v \in \vartheta$ be the test function and u the solution of equation (4), then $a_{PD}(u\,,v)$ is continuous if there is a positive constant κ such that:

$$\left|a_{PD,m}(u\,,v)\right| \leq \kappa\|u\|_{H^1(\mathrm{T}_{h,m})}\,\|v\|_{H^1(\mathrm{T}_{h,m})}, \quad \forall\, u,v \in \vartheta.$$

Proof: From the equation (5) we have,

$$|a_{PD,m}(u,v)| = |\sum_{K\in \mathrm{T}_{h,m}} \lambda(\nabla u,\nabla v)_K - \sum_{E\in\partial \mathrm{T}_{h,m}} \int (\{\lambda\nabla u\cdot n\,\}[v] + [u\,]\{\lambda\nabla v\cdot n\})\,ds + \sum_{E\in\partial \mathrm{T}_{h,m}} \int (\{|\boldsymbol{b}\cdot n|u\}[v])\,ds - \sum_{K\in \mathrm{T}_{h,m}} (\boldsymbol{b}\cdot\nabla u, v+\delta\boldsymbol{b}\cdot\nabla v)_K + \sigma \sum_{E\in\partial \mathrm{T}_{h,m}} \int [u][\,v]ds\,|.$$

$$|a_{PD,m}(u,v)| \leq \sum_{K\in \mathrm{T}_{h,m}} |\lambda(\nabla u,\nabla v)_K| - \sum_{E\in\partial \mathrm{T}_{h,m}} \int |\{\lambda\nabla u\cdot n\,\}[v] + [u\,]\{\lambda\nabla v\cdot n\}|\,ds + \sum_{E\in\partial \mathrm{T}_{h,m}} \int |(\{|\boldsymbol{b}\cdot n|u\}[v])|ds - \sum_{K\in \mathrm{T}_{h,m}} |(\boldsymbol{b}\cdot\nabla u, v+\delta\boldsymbol{b}\cdot\nabla v)_K| + \sigma \sum_{E\in\partial \mathrm{T}_{h.m}} \int |[u][\,v]|ds\,. \quad (9)$$

By using Cauchy –Schwarz inequality

$$|a_{PD,m}(u,v)| \leq \sum_{K\in \mathrm{T}_{h,m}} (|\lambda|_{L^\infty}\|\nabla u\|_{L^2(K)}\|\nabla v\|_{L^2(K)} + |b|_{L^\infty}\|\nabla u\|_{L^2(K)}\|v\|_{L^2(K)}) + \sum_{E\in\partial \mathrm{T}_{h,m}} (\|\{\lambda\,\nabla u\cdot n\}\|_{L^2(E)}\,\|[v]\|_{L^2(E)} + \|\{\lambda\,\nabla v\cdot n\}\|_{L^2(E)}\,\|[u]\|_{L^2(E)}) + \sum_{E\in\partial \mathrm{T}_{h,m}} \|\{|\boldsymbol{b}\cdot n|u\}\|_{L^2(E)}\,\|[v]\|_{L^2(E)} + \sigma \sum_{E\in\partial \mathrm{T}_{h,m}} \|[u]\|_{L^2(E)}\|[v]\|_{L^2(E)} + \sum_{K\in \mathrm{T}_{h,m}} |\delta|_{L^\infty}|\boldsymbol{b}^2|_{L^\infty}|\nabla u|_{L^2(K)}|\nabla v|_{L^2(K)}.$$

$$\begin{aligned}\left|a_{PD,m}(u,v)\right| &\leq \varsigma\|u\|_{H^1(T_{h,m})}\|v\|_{H^1(T_{h,m})} + 2|\lambda|\sigma {G_t}^2\|u\|_{H^1(T_{h,m})}\|v\|_{H^1(T_{h,m})} \\ &\quad +\sigma {G_t}^2\|u\|_{H^1(T_{h,m})}\|v\|_{H^1(T_{h,m})} + \sigma^2 {G_t}^2\|u\|_{H^1(T_{h,m})}\|v\|_{H^1(T_{h,m})} \\ &\quad + \Lambda\|u\|_{H^1(T_{h,m})}\|v\|_{H^1(T_{h,m})} \\ &= \left(\varsigma + 2|\lambda|\sigma {G_t}^2 + \sigma {G_t}^2 + \sigma^2 {G_t}^2 + \Lambda\right)\|u\|_{H^1(T_{h,m})}\|v\|_{H^1(T_{h,m})} \\ &\leq \kappa\|u\|_{H^1(\mathrm{T}_{h,m})}\,\|v\|_{H^1(\mathrm{T}_{h,m})},\end{aligned}$$

where, $\kappa \geq \left(\varsigma + 2|\lambda|\sigma {G_t}^2 + \sigma {G_t}^2 + \sigma^2 {G_t}^2 + \Lambda\right)$.

Lemma 3. (stability): There exists a constant $\varpi > 0$ independent of h and τsuch that:

$$\|(u_h)_m^-\|_{L^2(\mathrm{T}_{h,m})}^2 + \|\{u_h\}_{m-1}\|_{L^2(\mathrm{T}_{h,m})}^2 + \xi\|u_h\|_{L^2(I_m;H^1(\mathrm{T}_{h,m}))}^2$$

$$\leq \varpi \int_{I_m} \left(\| f \|^2_{L^2(\mathrm{T}_{h,m})} + \sum_{E \in \partial \mathrm{T}_{h,m}} \left(\|u_N\|^2_{L^2(\Gamma^N)} + \|u_D\|^2_{L^2(\Gamma^D)} \right) \right) dt.$$

Proof:

$$\int_{I_m} \left((u_{h,t}, w) + a_{PD,m}(u_h, w) \right) dt + (\{u_h\}_{m-1}, w^+_{m-1})$$
$$= \int_{I_m} \left((f, w) + \sum_{E \in \Gamma^N} \int u^N w ds + \sum_{E \in \Gamma^D} \int \lambda \nabla w \cdot n u^D \, ds \right. \quad (10)$$
$$\left. - \sum_{E \in \Gamma^D} \int |\boldsymbol{b} \cdot n| u^D w \, ds - \sigma \sum_{E \in \Gamma^D} \int u^D w \, ds \right) dt \ ; \quad \forall w \in \emptyset_{h,m}.$$

Let $w = u_h$ we get

$$\int_{I_m} \left((u_{h,t}, u_h) + a_{PD,m}(u_h, u_h) \right) dt + (\{u_h\}_{m-1}, (u_h)^+_{m-1})$$
$$= \int_{I_m} \left((f, u_h) + \sum_{E \in \Gamma^N} \int u^N u_h ds + \sum_{E \in \Gamma^D} \int \lambda \nabla u_h \cdot n u^D \, ds \right.$$
$$\left. - \sum_{E \in \Gamma^D} \int |\boldsymbol{b} \cdot n| u^D u_h \, ds - \sigma \sum_{E \in \Gamma^D} \int u^D u_h \, ds \right) dt. \quad (11)$$

To estimate the left-hand side, by using Lemma (1), we have

$$\int_{I_m} \left((u_{h,t}, u_h) + a_{PD,m}(u_h, u_h) \right) dt + (\{u_h\}_{m-1}, (u_h)^+_{m-1}) \geq \frac{1}{2} \int_{I_m} \frac{d}{dt} \|u_h\|^2_{L^2(\mathrm{T}_{h,m})}$$
$$+ \aleph \|u_h\|^2_{L^2\left(I_m, H^1(\mathrm{T}_{h,m})\right)} + \|\{u_h\}_{m-1}\|^2_{L^2(\mathrm{T}_{h,m})} \|(u_h)^+_{m-1}\|^2_{L^2(\mathrm{T}_{h,m})}$$
$$= \|(u_h)^-_m\|^2_{L^2(\mathrm{T}_{h,m})} - \|(u_h)^+_{m-1}\|^2_{L^2(\mathrm{T}_{h,m})} + \aleph \|u_h\|^2_{L^2\left(I_m, H^1(\mathrm{T}_{h,m})\right)}$$
$$+ \left(\left|\left|\{u_h\}_{m-1}\right|\right|_{L^2(\mathrm{T}_{h,m})} + \left|\left|(u_h)^+_{m-1}\right|\right|_{L^2(\mathrm{T}_{h,m})} \right)$$
$$= \|(u_h)^-_m\|^2_{L^2(\mathrm{T}_{h,m})} + \aleph \|u_h\|^2_{L^2\left(I_m, H^1(\mathrm{T}_{h,m})\right)} + \left|\left|\{u_h\}_{m-1}\right|\right|_{L^2(\mathrm{T}_{h,m})}. \quad (12)$$

To estimate the right-hand side, by using the Cauchy inequality and Young inequality [20], we have

$$\int_{I_m} \left(\sum_{E \in \Gamma^N} \int u^N u_h \, ds + (f, u_h) + \sum_{E \in \Gamma^D} \int \lambda \nabla u_h \cdot n u^D \, ds \right) dt$$

$$\leq \int_{I_m} \left(C \left(\| f \|^2_{L^2(\mathrm{T}_{h,m})} + \|u_h\|^2_{L^2(\mathrm{T}_{h,m})} \right) + C \sum_{E \in \partial \mathrm{T}_{h,m}} \left(\|u_h\|^2_{H^1(K)} + \|u_D\|^2_{L^2(\Gamma^D)} \right) \right.$$
$$+ C \sum_{E \in \partial \mathrm{T}_{h,m}} \left(\|u_h\|^2_{H^1(K)} + \|u_N\|^2_{L^2(\Gamma^N)} \right) + C \sum_{E \in \partial \mathrm{T}_{h,m}} \left(\|u_h\|^2_{H^1(K)} + \|u_D\|^2_{L^2(\Gamma^D)} \right)$$
$$\left. + C \sum_{E \in \partial \mathrm{T}_h} \left(\|u_h\|^2_{H^1(K)} + \|u_D\|^2_{L^2(\Gamma^D)} \right) \right) dt,$$

rearrangement above inequality, we have,

$$\int_{I_m} \left(\sum_{E \in \Gamma^N} \int u^N u_h \, ds + (f, u_h) + \sum_{E \in \Gamma^D} \int \lambda \nabla u_h \cdot n u^D \, ds \right) dt$$

$$\leq \int_{I_m} \left(C \left(\| f \|^2_{L^2(\mathrm{T}_{h,m})} + \|u_h\|^2_{H^1(K)} \right) + 4C\|u_h\|^2_{H^1(K)} \right.$$
$$\left. + C \sum_{E\in\partial \mathrm{T}_{h,m}} \left(\|u_N\|^2_{L^2(\Gamma^N)} + 3\|u_D\|^2_{L^2(\Gamma^D)} \right) \right) dt. \quad (13)$$

Substituting (12) and (13) in (11), we have

$$\|(u_h)^-_m\|^2_{L^2(\mathrm{T}_{h,m})} + \aleph\|u_h\|^2_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)} + \left\|\{u_h\}_{m-1}\right\|_{L^2(\mathrm{T}_{h,m})}$$
$$\leq \int_{I_m} \left(C \| f \|^2_{L^2(\mathrm{T}_{h,m})} + 5C\|u_h\|^2_{H^1(K)} + C \sum_{E\in\partial \mathrm{T}_{h,m}} \left(\|u_N\|^2_{L^2(\Gamma^N)} + 3\|u_D\|^2_{L^2(\Gamma^D)} \right) \right) dt.$$

$$\|(u_h)^-_m\|^2_{L^2(\mathrm{T}_{h,m})} + \left\|\{u_h\}_{m-1}\right\|_{L^2(\mathrm{T}_{h,m})} + (\aleph - 5C)\|u_h\|^2_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)}$$
$$\leq \varpi \int_{I_m} \left(\| f \|^2_{L^2(\mathrm{T}_{h,m})} + \sum_{E\in\partial \mathrm{T}_{h,m}} \left(\|u_N\|^2_{L^2(\Gamma^N)} + \|u_D\|^2_{L^2(\Gamma^D)} \right) \right) dt.$$

$$\|(u_h)^-_m\|^2_{L^2(\mathrm{T}_{h,m})} + \|\{u_h\}_{m-1}\|^2_{L^2(\mathrm{T}_{h,m})} + \xi\|u_h\|^2_{L^2(I_m;H^1(\mathrm{T}_{h,m}))}$$
$$\leq \varpi \int_{I_m} \left(\| f \|^2_{L^2(\mathrm{T}_{h,m})} + \sum_{E\in\partial \mathrm{T}_{h,m}} \left(\|u_N\|^2_{L^2(\Gamma^N)} + \|u_D\|^2_{L^2(\Gamma^D)} \right) \right) dt,$$

where $C = \left\{\frac{\beta}{2}, \frac{w^2}{2\beta}\right\}, \varpi \geq 3C$ and $\xi \leq (\aleph - 5C)$. (14)

5. The error estimate

Theorem 5.1. Let u be the solution of equation (4), $u_h \in \vartheta_{h,m}$ be the approximate solution of equation (6) and $u \in L^2\left(H^1(\Omega)\right)$,$u_t \in L^2\left(I_m;\ H^1(\Omega)\right)$ and σ sufficiently large, then there exists a constant C such that:

$$\|u - u_h\|_{L^2\left(I_m,L^2(T_{h,m})\right)} \leq C(h^2 + \tau^3).$$

Proof: Let Pu be the L^2 − projection of u, and $e = u - u_h = (u - \mathrm{P}u) + (\mathrm{P}u - u_h) = \Sigma - \Xi$, then

$$\|u - u_h\|_{L^2\left(I_m,L^2(T_{h,m})\right)} \leq \|\Sigma\|_{L^2\left(I_m,L^2(T_{h,m})\right)} + \|\Xi\|_{L^2\left(I_m,L^2(T_{h,m})\right)}. \quad (15)$$

From [19],

$$\|\Sigma\|_{L^2\left(I_m,L^2(T_{h,m})\right)} \leq \sqrt{c}(h^2 + \tau^3). \quad (16)$$

Now,

$$\int_{I_m} \left((u_t, w) + a_{PD,m}(u, w) \right) dt + (\{u\}_{m-1}, w^+_{m-1})$$
$$= \int_{I_m} \left((f, w) + \sum_{E\in\Gamma^N} \int u^N w ds + \sum_{E\in\Gamma^D} \int \lambda\nabla w \cdot n u^D\, ds \right.$$
$$\left. - \sum_{E\in\Gamma^D} \int |\boldsymbol{b} \cdot n| u^D w\, ds - \sigma \sum_{E\in\Gamma^D} \int u^D w\, ds \right) dt \;;\quad \forall w \in \emptyset\,. \quad (17)$$

$$\int_{I_m}\left((u_{h,t},w)+a_{PD,m}(u_h,w)\right)dt+(\{u_h\}_{m-1},w_{m-1}^{+})$$
$$=\int_{I_m}\left((f,w)+\sum_{E\in\Gamma^N}\int u^N w ds+\sum_{E\in\Gamma^D}\int \lambda\nabla w\cdot n u^D\, ds\right.$$
$$\left.-\sum_{E\in\Gamma^D}\int|\boldsymbol{b}\cdot n|u^D w\, ds-\sigma\sum_{E\in\Gamma^D}\int u^D w\, ds\right)dt\ ;\quad \forall w\in\emptyset_{h,m}. \tag{18}$$

Subtracting (17) from (18), we obtain,

$$\int_{I_m}\left(((u-u_h)_t\,,w)+a_{PD,m}(u-u_h,w)\right)dt+(\{u-u_h\}_{m-1},w_{m-1}^{+})$$
$$=\int_{I_m}\left(((\Sigma-\Xi)_t\,,w)+a_{PD,m}(\Sigma-\Xi,w)\right)dt+(\{\Sigma-\Xi\}_{m-1},w_{m-1}^{+})=0;\ \forall w\in\emptyset_{h,m}.$$

Then,

$$\int_{I_m}\left((\Sigma_t,w)+a_{PD,m}(\Sigma,w)\right)dt+(\{\Sigma\}_{m-1},w_{m-1}^{+})$$
$$=\int_{I_m}\left((\Xi_t,w)+a_{PD,m}(\Xi,w)\right)dt+(\{\Xi\}_{m-1},w_{m-1}^{+}). \tag{19}$$

Let $w=\Xi$, we have,

$$\int_{I_m}\left((\Sigma_t,\Xi)+a_{PD,m}(\Sigma,\Xi)\right)dt+(\{\Sigma\}_{m-1},\Xi_{m-1}^{+})$$
$$=\int_{I_m}\left((\Xi_t,\Xi)+a_{PD,m}(\Xi,\Xi)\right)dt+(\{\Xi\}_{m-1},\Xi_{m-1}^{+}). \tag{20}$$

From Lemma 1, we have,

$$\int_{I_m}\left((\Xi_t,\Xi)+a_{PD,m}(\Xi,\Xi)\right)dt\geq\frac{1}{2}\int_{I_m}\frac{d}{dt}\parallel\Xi\parallel_{L^2(\mathrm{T}_h,m)}^2\,dt+\aleph\int_{I_m}\|\Xi\|_{L^2(\mathrm{T}_{h,m})}^2\,dt$$
$$=\left(||\Xi_m^-||_{L^2(\mathrm{T}_{h,m})}^2-||\Xi_{m-1}^+||_{L^2(\mathrm{T}_{h,m})}^2\right)+\aleph\|\Xi\|_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)}^2. \tag{21}$$

$$(\{\Xi\}_{m-1},\Xi_{m-1}^{+})\leq||\{\Xi\}_{m-1}||_{L^2(\mathrm{T}_{h,m})}||\Xi_{m-1}^{+}||_{L^2(\mathrm{T}_{h,m})}\leq\frac{1}{2}\left(\left||\{\Xi\}_{m-1}|\right|_{L^2(\mathrm{T}_{h,m})}+\left||\Xi_{m-1}^{+}|\right|_{L^2(\mathrm{T}_{h,m})}\right). \tag{22}$$

From equations (21) and (22), we have

$$\int_{I_m}\left((\Xi_t,\Xi)+a_{PD,m}(\Xi,\Xi)\right)dt+(\{\Xi\}_{m-1},\Xi_{m-1}^{+})=\left||\Xi_m^-|\right|_{L^2(\mathrm{T}_{h,m})}^2$$
$$+\aleph\|\Xi\|_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)}^2+\frac{1}{2}\left(\left||\{\Xi\}_{m-1}|\right|_{L^2(\mathrm{T}_{h,m})}-\left||\Xi_{m-1}^{+}|\right|_{L^2(\mathrm{T}_{h,m})}\right). \tag{23}$$

Clearly

$$\int_{I_m}(\Sigma_t,\Xi)dt=(\Sigma_m^-,\Xi_m^-)-(\Sigma_{m-1}^+,\Xi_{m-1}^+)-\int_{I_m}(\Sigma,\ \Xi_t)\,dt.$$

Since

$$\int_{I_m}(\Sigma,\ \Xi_t)\,dt=0\text{ and }(\Sigma_m^-,\Xi_m^-)=0.$$

From Lemma 2, by Young inequality and Schwartz, we have

$$\int_{I_m}\left((\Sigma_t,\Xi)+a_{PD,m}(\Sigma,\Xi)\right)dt+(\{\Sigma\}_{m-1},\Xi_{m-1}^{+})\leq-(\Sigma_{m-1}^{+},\Xi_{m-1}^{+})$$

$$\int_{I_m}\left(\kappa\|\Sigma\|_{H^1(\mathrm{T}_{h,m})}\,\|\Xi\|_{H^1(\mathrm{T}_{h,m})}\right)dt+\frac{1}{2}\left((\Sigma_{m-1}^{+},\Xi_{m-1}^{+})+(\Sigma_{m-1}^{-},\Xi_{m-1}^{+})\right)$$
$$\le\frac{\beta\kappa}{2}\|\Sigma\|^2_{L^2(I_m,H^1(\mathrm{T}_{h,m}))}+\frac{\kappa}{2\beta}\|\Xi\|^2_{L^2(I_m,H^1(\mathrm{T}_{h,m}))}+\frac{1}{2}\left((\Sigma_{m-1}^{-},\Xi_{m-1}^{+})-(\Sigma_{m-1}^{+},\Xi_{m-1}^{+})\right).$$

For the first term,

$$\frac{\beta\kappa}{2}\|\Sigma\|^2_{L^2(I_m,H^1(\mathrm{T}_{h,m}))}\le c(h^4+\tau^6). \tag{24}$$

Then,

$$\|\Sigma\|_{L^2(I_m,H^1(\mathrm{T}_{h,m}))}\le c_1(h^2+\tau^3),$$

where $c_1=\sqrt{\frac{2c}{\beta\kappa}}$.

For the third term

$$\frac{1}{2}\left((\Sigma_{m-1}^{-},\Xi_{m-1}^{+})-(\Sigma_{m-1}^{+},\Xi_{m-1}^{+})\right)$$
$$\le\frac{1}{2}\left(\frac{1}{2}\left(\left\|\Sigma_{m-1}^{-}\right\|^2_{L^2(\mathrm{T}_{h,m})}+\left\|\Xi_{m-1}^{+}\right\|^2_{L^2(\mathrm{T}_{h,m})}\right)-\frac{1}{2}\left(\left\|\Sigma_{m-1}^{+}\right\|^2_{L^2(\mathrm{T}_{h,m})}+\left\|\Xi_{m-1}^{+}\right\|^2_{L^2(\mathrm{T}_{h,m})}\right)\right)$$
$$=\frac{1}{4}\left(\left\|\Sigma_{m-1}^{-}\right\|^2_{L^2(\mathrm{T}_{h,m})}-\left\|\Sigma_{m-1}^{+}\right\|^2_{L^2(\mathrm{T}_{h,m})}\right). \tag{25}$$

Then

$$\int_{I_m}\left((\Sigma_t,\Xi)+a_{PD,m}(\Sigma,\Xi)\right)dt+(\{\Sigma\}_{m-1},\Xi_{m-1}^{+})$$
$$\le c(h^4+\tau^6)\ +\frac{\kappa}{2\beta}\|\Xi\|^2_{L^2(I_m,H^1(\mathrm{T}_{h,m}))}\ \frac{1}{4}\left(\left\|\Sigma_{m-1}^{-}\right\|^2_{L^2(\mathrm{T}_{h,m})}-\left\|\Sigma_{m-1}^{+}\right\|^2_{L^2(\mathrm{T}_{h,m})}\right). \tag{26}$$

By substituting (23) and (26) in (20), we have,

$$\left\|\Xi_m^{-}\right\|^2_{L^2(\mathrm{T}_{h,m})}+\aleph\|\Xi\|^2_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)}+\frac{1}{2}\left(\left\|\{\Xi\}_{m-1}\right\|_{L^2(\mathrm{T}_{h,m})}-\left\|\Xi_{m-1}^{+}\right\|_{L^2(\mathrm{T}_{h,m})}\right)$$
$$\le c(h^4+\tau^6)\ +\frac{\kappa}{2\beta}\|\Xi\|^2_{L^2(I_m,H^1(\mathrm{T}_{h,m}))}\ \frac{1}{4}\left(\left\|\Sigma_{m-1}^{-}\right\|^2_{L^2(\mathrm{T}_{h,m})}-\left\|\Sigma_{m-1}^{+}\right\|^2_{L^2(\mathrm{T}_{h,m})}\right).$$
$$\aleph\|\Xi\|^2_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)}\ \le c(h^4+\tau^6)\ +\frac{\kappa}{2\beta}\|\Xi\|^2_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)}.$$

Rearrangement above inequality, we have

$$\left(\aleph-\frac{\kappa}{2\beta}\right)\|\Xi\|^2_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)}\le c(h^4+\tau^6).$$
$$\|\Xi\|^2_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)}\le Q(h^4+\tau^6).$$

Hence,

$$\|\Xi\|_{L^2\left(I_m,H^1(\mathrm{T}_{h,m})\right)}\le\sqrt{Q}(h^2+\tau^3). \tag{27}$$

Substituting (16) and (27) in (15), we have

$$\|u-u_h\|_{L^2(I_m,L^2(T_{h,m}))}\le C(h^2+\tau^3),$$

where

$$Q\ge\frac{c}{\left(\aleph-\frac{\kappa}{2\beta}\right)},\ C=max\{\sqrt{c},\sqrt{Q}\}\ \text{and}\ \aleph\ne\frac{\kappa}{2\beta}.$$

6. Numerical experiment

In this section, we calculate the error $u-u_h$ of L^2-error and H^1-error of the SIPG space-time PDGFE method and space-time DGFE method by using Matlab software. We consider the following convection-diffusion problem

$$u_t-\lambda\Delta u+\boldsymbol{b}\cdot\nabla u=f,\qquad in\ \Omega\times \mathrm{J}, \tag{28}$$

Suppose we have a homogeneous Dirichlet boundary condition and a homogeneous initial condition.

This equation has an analytical solution:

$$u(x, y, t) = \sin(\pi x) \sin(\pi y)\, e^{-t}.$$

Let $\Omega = [0,1] \times [0,1]$, time interval be $J = (0,1)$, $\boldsymbol{b} = [0,1], \sigma = 2782$, and f it is worked out by plugging the true solution into the left side of the equation (28).

The square domain is divided into sections $\Omega = (0,1) \times (0,1)$ into $\mathrm{N} \times \mathrm{N}$ uniformly in the subsquare. The PDGFE method with $h = 1/\mathrm{N}(\mathrm{N} = 4,8,16,32,64), \tau = 0.015625$ and $\delta = \mathrm{h}/6$ found the numerical solution at $t = 1$.

When $\delta = 0$, the numerical error results and degree of convergence for the DGFE technique are in Table 1 and convergence rate is in Figure 1, and when $\delta = \mathrm{h}/6$, the numerical error results and degree of convergence for the PDGF method are in Table 2 and convergence rate is in Figure 2.

The precise solution and numerical solution in the DGFE method are incompatible (see Figure 3), whereas the exact solution and numerical solution in the PDGFE method are identical (see Figure 4).

Tabe 1: Numerical results for $\lambda = 0.001$ in the DGFE method

h	H^1 −error	H^1 −order	L^2 −error	L^2 −order
$1/4$	1.2605	0	0.1353	0
$1/8$	0.9930	0.3442	0.0451	1.5846
$1/16$	0.7567	0.3920	0.0162	1.4746
$1/32$	0.5481	0.4654	0.0055	1.5537
$1/64$	0.3902	0.4902	0.0019	1.5065

Tabe 2: Numerical results for $\lambda = 0.001$ in the PDGFE method

h	H^1 −error	H^1 −order	L^2 −error	L^2 −order
$1/4$	0.2011	0	0.0422	0
$1/8$	0.1013	0.9889	0.0125	1.7560
$1/16$	0.0514	0.9786	0.0036	1.7905
$1/32$	0.0273	0.9140	0.0009	1.9680
$1/64$	0.0163	0.7425	0.0002	1.9648

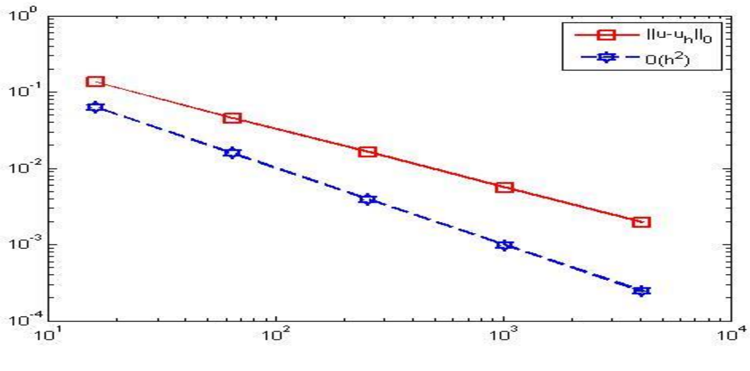


Figure 1: Convergence rate in the DGFE method for $\lambda = 0.001$ in L^2 −norm

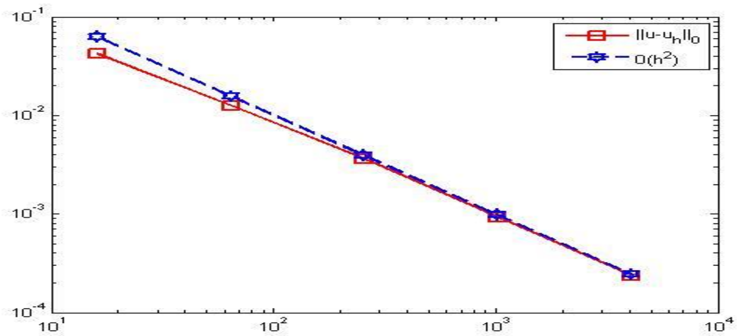
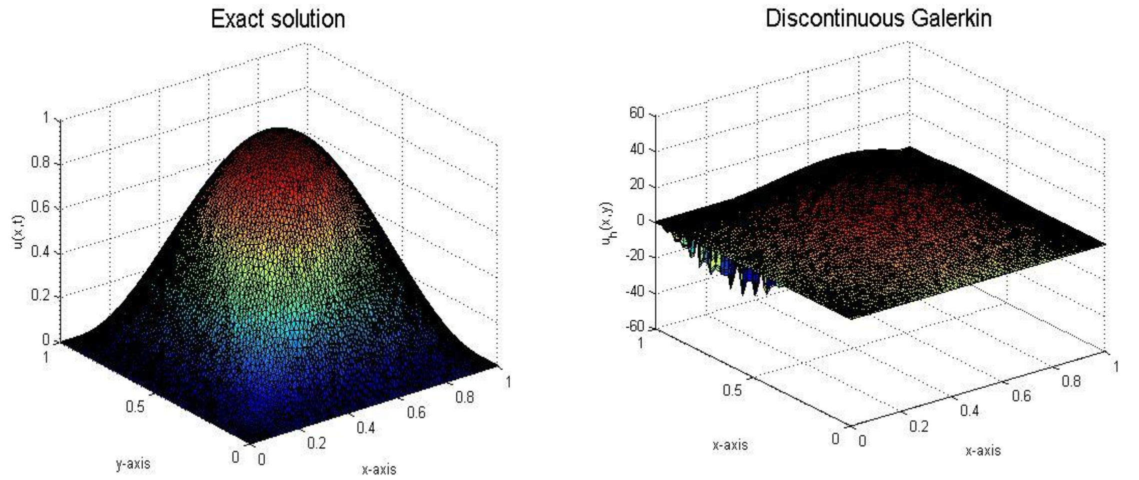
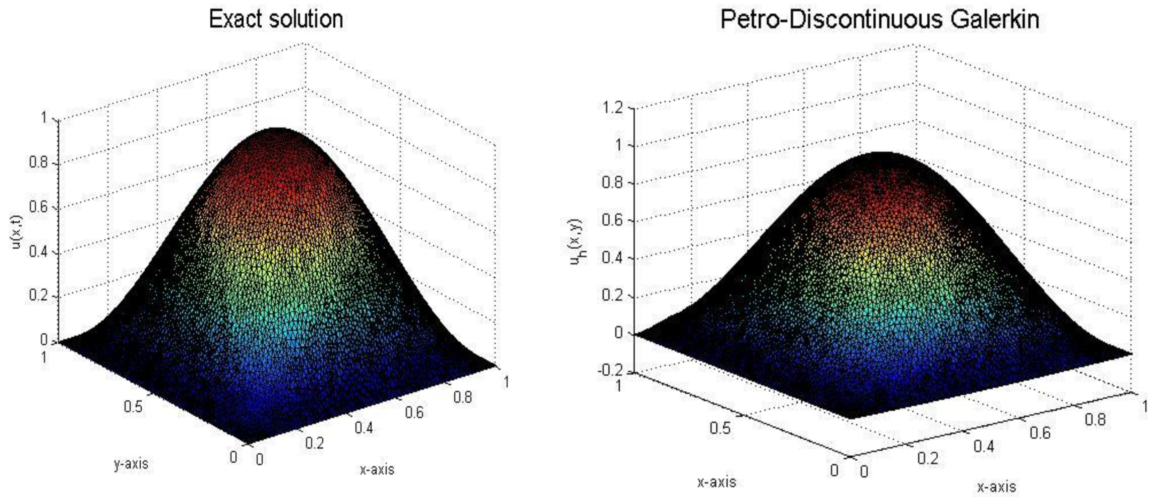


Figure 2: Convergence rate in the PDGFE method for $\lambda = 0.001$ in L^2 −norm



(a) (b)

Figure 3: (a) The exact solution with$\lambda = 0.001, h = 1/64$, and $\tau = 0.015625$. (b) The numerical solution of the DGFE method with $\lambda = 0.001, h = 1/64$, and $\tau = 0.015625$.



(a) (b)

Figure 4: (a) The exact solution with $\lambda = 0.001, h = 1/64$, and $\tau = 0.015625$. (b) The numerical solution of the PDGFE method with $\lambda = 0.001, h = 1/64$, and $\tau = 0.015625$.

7. Conclusion

The theoretical study and numerical findings in the space-time PDGFE technique lead to the following conclusions:

1- The characteristics of the bilinear form $a_{PD,m}(u, v)$ ($V -$ elliptic and continuous) of the PDGFE technique were established.
2- The PDGFE method stability was demonstrated.
3- We established that the approximate solution converges with an order of $O(h^2 + \tau^3)$ magnitude inaccuracy.
4- When the numerical results of the PDGFE technique (see Table 2, Figure 2, and Figure 4), were compared to the numerical results of the DGFE method (see Table 1, Figure 1, and Figure 3), the numerical results of the PDGFE approach demonstrated improvement and regularity.
5- We found that when we smoothed the network with n = 64, the numerical results in the DGFE method oscillated as seen in Figure 3, but that the numerical results in the PDGFE method were close and devoid of oscillation as shown in Figure 4.

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