On complete (k,3)-arcs in PG(2,8)

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Salam Abdulqader Alabdullah
University of Basrah

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Salam A. Falih
Department of Petroleum ,College of Engineering
University of Basra
Email: salamkader2006@yahoo.com
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Abstract

In this paper, the classification of the (k,3)-arcs in PG(2,8) with respect to type of their lines has been obtained as well as the group of projectivities of the projectively distinct (k,3)-arcs are found. Furthermore all the complete (k,3)-arcs in PG(2,8) are investigated, also it was shown that PG(2,8) has no maximum arc.

Introduction

Let $GF(q)$ be the Galois field of q elements and $V(3,q)$ be the vector space of dimension three where q is prime power. Let $PG(2,q)$ be the corresponding projective plane. The number of points of $PG(2,q)$ is q^2+q+1. and the number of lines is q^2+q+1, where each line contains exactly $q+1$ points and there are $q+1$ lines throughout every point, and any two distinct points lie exactly on one line , and any two distinct lines have exactly one common point. A (k,n) −arc K in a finite projective plane $PG(2,q)$, is a set of k points, such that there is some n but no $(n+1)$ are collinear where $2 \le n \le q+1$ and a $(k,2)$ −arc generally called a k −arc. A (k,n) −arc is complete if there is no $(k+1,n)$ −arc containing it. The maximum and smallest size of a complete (k,n) −arcs for which a (k,n) −arc K exist in $PG(2,q)$ will be denoted by $m_n(2,q)$ and $t_n(2,q)$ respectively.

In (1938) Singer [24] put down the method to array the points and lines in projective plane $PG(2,q)$. In (1947) Bose[7] proved that $m_2(2,q)=q+1$ for q odd, and $m_2(2,q)=q+2$ for q even. In mid of (1950s), Segre [21,22] proved that for q odd every $q+1$ −arc is a conic, for $q=2, q=4$ and $q=8$ every $q+2$ −arc is a conic plus its nucleus [23], and for $q=16, q=32, q=2^h$ $(h \ge 7)$, there exists a $q+2$ −arc other than the conic plus its nucleus. In (1956) Barlotti [4] proved that the first of many results in the attempt to determine the value of $m_n(2,q)$, and this has proved to be far from simple. Early results by Barlotti bounded $m_n(2,q)$ with $m_n(2,q) \le (n-1)q+n$ and proved for $(n,q)=1$ and $n>2$, $m_n(2,q) \le (n-1)q+n-2$. Hirschfeld [15] and Sadeh [20] had shown the classification and construction of k −arcs over the Galois field $GF(q)$ with $q \le 11$ and gave the example of $(21,3)$ −arc in $PG(2,11)$. Bierbrauer [5] proved that any $(15,3)$ −arc in $PG(2,8)$ is a maximum. The classification and construction of $(k,4)$ −arcs with respect to the type of lines for $q=3$ have been given by Abood [2]. Abdul-Hussain [1] also explained the classification of $(k,4)$ −arcs with respect to the type of lines in $PG(2,5)$. In (2001) Hirschfeld and Storme [17] showed that for q odd this implies immediately that the maximum size of a (k,n) −arc, for $n|q$ is less than $nq-q+n/2$. Ibrahim [18] explained the classification of $(k,4)$ −arcs and $(k,3)$ −arcs with respect to the type of lines in $PG(2,7)$. Ball and Hirschfeld [3] reviewed some of the works of the principal and recently discovered lower and upper bounds on the maximum size of (k,n) −arcs in $PG(2,q)$ for some n,q and put a table for it. The classification of the complete k −arcs in $PG(2,27)$ has been given by Coolsaet and Sticker [8]. The classification and construction of $(k,4)$ −arcs with respect to the type of lines for $q = 8$ have been given by Falih [10].Classification of complete (k,4)-arcs in the projective plane of order eleven have been given by Khalid [19].

The main purpose of this paper is to find the complete $(k, 3)$ −arcs in $PG(2,8)$ through the classification and construction of the projectively distinct $(k, 3)$ −arcs with respect to the type of lines and we found the group of projectivities of each projectively distinct $(k, 3)$ −arcs.

1. Preliminaries :

Definition 1.1 [6]

For p prime, let $GF(p)$ denote a finite field of p elements that consists of the residue classes of integers module p under the natural addition and multiplication. If $f(x)$ is an irreducible polynomial of degree h over $GF(p)$, then :

$GF(p^h) = GF(p)[x] / (f(x)) = \{a_0 + a_1 t + \cdots + a_{h-1} t^{h-1} : a_i \in GF(p), f(t) = 0\}$

$GF(p^h)$ is called a Galois field of order $q = p^h$, where h >1 is an integer number. Notice that, the elements of $GF(q)$ satisfy the equation $x^q = x$ and there exists $y \in GF(q)$ such that: $GF(q) = \{0, 1, y, y^2, \ldots, y^{q-2} : y^{q-1} = 1\}$. The element y is called a primitive element or primitive root of $GF(q)$.

Definition 1.2 [15]

Let $V = V(n+1, F)$ be a $(n+1)$ −dimensional vector space over a field F with zero vector 0. Define an equivalence relation $\sim$ on the vectors of $V^* = V \backslash \{0\}$ as follows:

If $X = (x_1, x_2, \ldots, x_{n+1})$, $Y = (y_1, y_2, \ldots, y_{n+1}) \in V \backslash \{0\}$, we say that X is equivalent to Y if, $Y = \lambda X$, for some $\lambda \in F \backslash \{0\}$. Then the space $V(n+1, F)/\sim$ is said to be the n-dimensional projective space over F and is denoted by $PG(n, F)$ or, when $F = GF(q)$, by $PG(n, q)$. The equivalence classes are called points of $PG(n, F)$.

For any $m = 0, 1, 2, \ldots, n$, a subspace of dimension m (or m −space) of $PG(n, q)$ is the set of points all of whose representing vectors form, (together with the zero), a subspace of dimension $m + 1$ of V. A subspace of the dimensions zero, one, two, and three are respectively called a point, a line, a plane, and a solid. Subspaces of dimension $n - 1$ and $n - 2$ are respectively called a prime (hyperplane) and secundum. A subspace of dimension $n - r$ is also referred to as a subspace of codimension r. The set of m −spaces is denoted by $PG^{(m)}(n, q)$.

Theorem 1.1 [15]

The number of points in $PG(n, q)$ is $\theta(n) = \frac{q^{n+1} - 1}{q - 1}$.

In particular, $\theta(0) = 1$, $\theta(1) = q + 1$ and $\theta(2) = q^2 + q + 1$.

Definition 1.3 [15]

A projective plane over $GF(q)$ is 2-dimensional projective space denoted by $PG(2, q)$ and it has the following properties:

1. The number of points is $q^2 + q + 1$.
2. The number of lines is $q^2 + q + 1$.
3. Each line contains exactly $q + 1$ points.
4. Each point lies on $q + 1$ lines.

The fundamental theorem in projective geometry 1.2 [15]

If $\{P_1, P_2, \dots, P_{n+2}\}$ and $\{Q_1, Q_2, \dots, Q_{n+2}\}$ are two sets of points of $PG(n,q)$ such that no $n+1$ points chosen from the same set lie in a prime, then there exists a unique projectivity T, such that $Q_i = P_i T$, for all $i = 1,2,\dots,n+2$.

For n=1, there exists a unique projectivity transforming any three distinct points on a line to any other three.

For n=2, there exists a unique projectivity transforming the four points P_1, P_2, P_3, P_4 (no three are collinear) to the four points Q_1, Q_2, Q_3, Q_4 (no three are collinear) respectively

Primitive and subprimitive roots of polynomials 1.4 [15]

Let $N(m,q)$ be the set of monic irreducible polynomials over $GF(q)$ of degree m then:

1- If $f \in N(m,q)$, then f has exponent e, if e is the smallest positive integer such that $f(x)$ divided $x^e - 1$. The exponent e always divides $q^m - 1$. If $e = q^m - 1$, then f is called a primitive and has a primitive root in $GF(q^m)$. So, if α is a root in $GF(q^m)$ of a primitive f, then α has order $q^m - 1$.

2- If $f(x) \in N(m,q)$, then $f(x)$ has a subexponent e, if e is the smallest positive integer number such that $f(x)$ divided $x^e - c$ for some $c \in GF(q)$.The subexponent e always divides $\theta(m-1) = \frac{q^m - 1}{q-1}$ .If $e = \frac{q^m-1}{q-1}$, then $f(x)$ is subprimitive polynomial and has a subprimitive root.

Definition 1.5 [15]

Let $f(x) = x^{r+1} - a_r x^r - \cdots - a_0$ be any monic polynomial, then its companion matrix, $C(f)$ is given by the $(r+1) \times (r+1)$ matrix;

$$C(f) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_r \end{bmatrix}$$. In particular, when $r = 2$ therefore;

$f(x) = x^3 - a_2 x^2 - a_1 x - a_0$, and $C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix}$.

Definition 1.6 [15]

A projectivity T which permutes the $\theta(n)$ points of $PG(n,q)$ in a single cycle is called a cyclic projectivity.

Theorem 1.2 [14]

A projectivity T of $PG(n,q)$ is cyclic if and only if the characteristic polynomial of an associated matrix is subprimitive .

If $f(x) \in N(m,q)$ and $f(x)$ is a subprimitive, then the companion matrix $C(f)$ is the cyclic projectivity of $PG(n,q)$.

Theorem 1.4 [14]

The number of cyclic projectivities in $PG(n,q)$ is given by;

$\sigma(n,q) = q^{n(n+1)/2} \prod_{i=1}^{n} (q^i - 1)\, \square(\theta(n)) / (n+1)$,where $\square$ is the Euler function .

Definition 1.7 [15]

1. A (k,n) −arc K is a set of k points, such that there is some n but no $(n+1)$ are collinear where $n \geq 2$. When n=2 a $(k,2)$ −arc is called a k −arc.

2. A (k,n) −arc is complete if, there is no $(k+1,n)$ −arc containing it.

3. A line ℓ of $PG(2,q)$ is an i −secant of a (k,n) −arc K if, $|\ell \cap K| = i$. A 0 −secant is called an external line of k −arc, a 1 −secant is called unisecant and a 2 −secant is called a bisecant.

4. A (k,n) −arc K is maximal arc if it satisfies $k = (n-1)q + n$.

5. The maximum and smallest size of a complete (k, n) −arc for which a (k, n) −arc K exists in $PG(2, q)$ will be denoted by $m_n(2, q)$ and $t_n(2, q)$ respectively.

Notation : Let r_i denotes the total number of i −secants of (k, n) −arc K in $PG(2, q)$, R_i=$R_i(P)$ the number of i −secants through a point P of K and $S_i = S_i(Q)$ the number of i −secants through a point Q of $PG(2, q) \setminus K$.

Lemma 1.1 [15]

For a (k, n) −arc K, the following equations hold:

$\sum_{i=0}^{n} r_i = q^2 + q + 1$ ……………(1)

$\sum_{i=1}^{n} ir_i = k(q + 1)$ ……………(2)

$\sum_{i=2}^{n} \frac{i(i-1)r_i}{2} = \frac{k(k-1)}{2}$ ……………(3)

$\sum_{i=1}^{n} R_i = q + 1$ ……………(4)

$\sum_{i=2}^{n} (i - 1)R_i = k - 1$ ……………(5)

$\sum_{i=0}^{n} S_i = q + 1$ ……………(6)

$\sum_{i=1}^{n} iS_i = k$ ……………(7)

$\sum_{P} R_i = ir_i$ ……………(8)

$\sum_{Q} S_i = (q + 1 - i)r_i$ ……………(9)

Where the summation in the equation (8) taken over all $P \in K$, and taken over all $Q \in PG(2, q) \setminus K$ in the equation (9).

Notation : Assume the equations (4) and (5) in the above lemma have v distinct solutions $B_j = (R_{1j}, \dots, R_{nj})$; $j = 1, \dots, v$ and the equations (6), (7) have g distinct solutions $M_j = (S_{0j}, \dots, S_{nj})$; $j = 1, \dots, g$.

Suppose there are b_j points on the (k, n) −arc K with solution B_j, and m_j points on $PG(2, q) \setminus K$ with solution M_j.

Lemma 1.2 [12]

For a (k, n) −arc K in $PG(2, q)$, the following equations hold:

$\sum_{j=1}^{v} b_j R_{ij} = ir_i$ ……………………. (1)

$\sum_{j=1}^{v} b_j = k$ ………………………. (2)

$\sum_{j=1}^{g} m_j S_{ij} = (q + 1 - i)r_i$ ………….(3)

$\sum_{j=1}^{g} m_j = q^2 + q + 1 - k$ ………. (4)

Lemma 1.3 [12]

Let $t(P)$ be the number of unisecants through P, where P is a point of the k −arc K. Let r_i be the total number of i −secants of K in the plane, then :

1. $t(p) = q + 2 - k = t$
2. $r_2 = k(k - 1)/2$, $r_1 = kt$ and $r_0 = q(q - 1)/2 + t(t - 1)/2$

Definition 1.8 [1]

If P is a point of $PG(2, q)$ not on the (k, n) −arc K and not on any $n -$ secants of the (k, n) −arc K, then P is called a point of index zero.

Theorem 1.5 [7]

$$m_2(2, q) = \begin{cases} q + 2 & ,\text{for } q \text{ even} \\ q + 1 & ,\text{for } q \text{ odd} \end{cases}$$

Theorem 1.6 [13]

For $2 \leq n \leq q+1$,

1- The maximum size $m_n(2,q) \leq (n-1)q+n$.

2- If $n \leq q$ and equality occur in (1), then n is a divisor of q.

Corollary 1.1 [16]

$$m_n(2,q)\begin{cases} = (n-1)q+n & ,\text{for } q \text{ even and } n|q \\ < (n-1)q+n & ,\text{for } q \text{ odd}\end{cases}$$

Theorem 1.7 [15]

If K is a maximal (k,n) −arc in $PG(2,q)$, then:

(i) $K = PG(2,q)$ if $n = q+1$ and;

(ii) $K = PG(2,q)\backslash\ell$, if $n = q$, where ℓ is a line.

Corollary 1.2 [15]

A (k,n) −arc K is maximal if and only if every line in $PG(2,q)$ is either an n −secant or an external line.

Lemma 1.4 [15]

If K is a complete (k,n) −arc, then: $(q+1-n)r_n \geq q^2+q+1-k$, with equality if and only if $S_n = 1$ for all Q in $PG(2,q)\backslash K$.

Definition 1.9 [1]

Let r_i be the total number of i −secants of the (k,n) −arc K in $PG(2,q)$. Then the type of K with respect to its lines is denoted by $(r_n, \ldots, r_0)$. Let K_1 be of type $(r_n, \ldots, r_0)$ and K_2 be of type $(t_n, \ldots, t_0)$, then K_1 and K_2 have the same type of lines iff $r_i = t_i$ for all $i = 0,1,\ldots,n$.

Definition 1.10 [1]

Two arcs K_1 and K_2 in $PG(2,q)$ are called projectively equivalents with respect to the types of lines if and only if they have the same type.

Definition 1.11 [1]

Let Q_1 and Q_2 be two points of index zero not on the (k,n) −arc K, and let $K_1 = K \cup \{Q_1\}$, $K_2 = K \cup \{Q_2\}$ be two arcs, then Q_1 and Q_2 have the same type if and only if K_1 and K_2 are projectively equivalents with respect to the types of lines.

Lemma 1.5 [1]

Let Q_1 and Q_2 be two points of index zero not on the (k,n) −arc, then:

(1) Q_1 and Q_2 are in the same set if they have the same type.

(2) Q_1 and Q_2 are in different sets if they have different types.

2.The cyclic projectivity of $PG(2,8)$

The plane $PG(2,8)$ contains 73 points and 73 lines, every line contains 9 points and every point passes through it 9 lines. It is convenience to use the numbers 0,1,2,3,4,5,6,7 will be the elements of $GF(8)$. Let $f(x) = x^3+x+\lambda^4$ be an irreducible polynomial over GF(8), then the matrix $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & 0 \end{bmatrix}$ is cyclic projecivity which is given by right multiplication on the points of PG(2,8).

2.1.The points of PG(2,8)

Let the point P_1 be represented by the vector (1,0,0). Then $P_1T^i = P_i$, i=1,…,73 are the 73 points of PG(2,8) .Writing i for P_i, the vectors of the 73 points of PG(2,8) are given in the table (2,1).

Table (2.1)

P_1=(1 0 0)	P_{16}=(1 3 0)	P_{31}=(1 0 5)	P_{46}=(1 7 4)	P_{61}=(1 6 2)
P_2=(0 1 0)	P_{17}=(0 1 3)	P_{32}=(1 5 0)	P_{47}=(1 2 7)	P_{62}=(1 6 1)
P_3=(0 0 1)	P_{18}=(1 4 2)	P_{33}=(0 1 5)	P_{48}=(1 7 6)	P_{63}=(1 0 2)
P_4=(1 4 0)	P_{19}=(1 6 6)	P_{34}=(1 4 7)	P_{49}=(1 3 5)	P_{64}=(1 6 0)
P_5=(0 1 4)	P_{20}=(1 3 4)	P_{35}=(1 7 1)	P_{50}=(1 5 2)	P_{65}=(0 1 6)
P_6=(1 4 1)	P_{21}=(1 2 3)	P_{36}=(1 0 3)	P_{51}=(1 6 7)	P_{66}=(1 4 6)
P_7=(1 0 7)	P_{22}=(1 1 3)	P_{37}=(1 10)	P_{52}=(1 7 3)	P_{67}=(1 3 2)
P_8=(1 7 0)	P_{23}=(1 1 2)	P_{38}=(0 1 1)	P_{53}=(1 1 1)	P_{68}=(1 6 5)
P_9=(0 1 7)	P_{24}=(1 6 3)	P_{39}=(1 4 4)	P_{54}=(1 0 4)	P_{69}=(1 5 5)
P_{10}=(1 4 5)	P_{25}=(1 1 7)	P_{40}=(1 2 4)	P_{55}=(1 2 0)	P_{70}=(1 5 4)
P_{11}=(1 5 3)	P_{26}=(1 7 5)	P_{41}=(1 2 2)	P_{56}=(0 1 2)	P_{71}=(1 2 5)
P_{12}=(1 1 6)	P_{27}=(1 5 6)	P_{42}=(1 6 4)	P_{57}=(1 4 3)	P_{72}=(1 5 1)
P_{13}=(1 3 6)	P_{28}=(1 3 3)	P_{43}=(1 2 6)	P_{58}=(1 1 5)	P_{73}=(1 0 1)
P_{14}=(1 3 1)	P_{29}=(1 1 4)	P_{44}=(1 3 7)	P_{59}=(1 5 7)	
P_{15}=(1 0 6)	P_{30}=(1 2 1)	P_{45}=(1 7 7)	P_{60}=(1 7 2)	

2.1.The lines of PG(2,8)

Let L_1 be the line which contains the points $\{P_1,P_2,P_4,P_8,P_{16},P_{32},P_{37},P_{55},P_{64}\}$

Let $L_1T^i = L_i$, i=1,2,…,73 are the lines of PG(2,8). The 73 lines, L_i are given by the rows in the table (2,2).

Table (2.2)

L_1	P_1	P_2	P_4	P_8	P_{16}	P_{32}	P_{37}	P_{55}	P_{64}
L_2	P_2	P_3	P_5	P_9	P_{17}	P_{33}	P_{38}	P_{56}	P_{65}
L_3	P_3	P_4	P_6	P_{10}	P_{18}	P_{34}	P_{39}	P_{57}	P_{66}
L_4	P_4	P_5	P_7	P_{11}	P_{19}	P_{35}	P_{40}	P_{58}	P_{67}
L_5	P_5	P_6	P_8	P_{12}	P_{20}	P_{36}	P_{41}	P_{59}	P_{68}
L_6	P_6	P_7	P_9	P_{13}	P_{21}	P_{37}	P_{42}	P_{60}	P_{69}
L_7	P_7	P_8	P_{10}	P_{14}	P_{22}	P_{38}	P_{43}	P_{61}	P_{70}
L_8	P_8	P_9	P_{11}	P_{15}	P_{23}	P_{39}	P_{44}	P_{62}	P_{71}
L_9	P_9	P_{10}	P_{12}	P_{16}	P_{24}	P_{40}	P_{45}	P_{63}	P_{72}
L_{10}	P_{10}	P_{11}	P_{13}	P_{17}	P_{25}	P_{41}	P_{46}	P_{64}	P_{73}
L_{11}	P_{11}	P_{12}	P_{14}	P_{18}	P_{26}	P_{42}	P_{47}	P_{65}	P_1
L_{12}	P_{12}	P_{13}	P_{15}	P_{19}	P_{27}	P_{43}	P_{48}	P_{66}	P_2
L_{13}	P_{13}	P_{14}	P_{16}	P_{20}	P_{28}	P_{44}	P_{49}	P_{67}	P_3
L_{14}	P_{14}	P_{15}	P_{17}	P_{21}	P_{29}	P_{45}	P_{50}	P_{68}	P_4
L_{15}	P_{15}	P_{16}	P_{18}	P_{22}	P_{30}	P_{46}	P_{51}	P_{69}	P_5
L_{16}	P_{16}	P_{17}	P_{19}	P_{23}	P_{31}	P_{47}	P_{52}	P_{70}	P_6
L_{17}	P_{17}	P_{18}	P_{20}	P_{24}	P_{32}	P_{48}	P_{53}	P_{71}	P_7
L_{18}	P_{18}	P_{19}	P_{21}	P_{25}	P_{33}	P_{49}	P_{54}	P_{72}	P_8
L_{19}	P_{19}	P_{20}	P_{22}	P_{26}	P_{34}	P_{50}	P_{55}	P_{73}	P_9
L_{20}	P_{20}	P_{21}	P_{23}	P_{27}	P_{35}	P_{51}	P_{56}	P_1	P_{10}
L_{21}	P_{21}	P_{22}	P_{24}	P_{28}	P_{36}	P_{52}	P_{57}	P_2	P_{11}
L_{22}	P_{22}	P_{23}	P_{25}	P_{29}	P_{37}	P_{53}	P_{58}	P_3	P_{12}
L_{23}	P_{23}	P_{24}	P_{26}	P_{30}	P_{38}	P_{54}	P_{59}	P_4	P_{13}
L_{24}	P_{24}	P_{25}	P_{27}	P_{31}	P_{39}	P_{55}	P_{60}	P_5	P_{14}
L_{25}	P_{25}	P_{26}	P_{28}	P_{32}	P_{40}	P_{56}	P_{61}	P_6	P_{15}
L_{26}	P_{26}	P_{27}	P_{29}	P_{33}	P_{41}	P_{57}	P_{62}	P_7	P_{16}
L_{27}	P_{27}	P_{28}	P_{30}	P_{34}	P_{42}	P_{58}	P_{63}	P_8	P_{17}
L_{28}	P_{28}	P_{29}	P_{31}	P_{35}	P_{43}	P_{59}	P_{64}	P_9	P_{18}

L_{29}	P_{29}	P_{30}	P_{32}	P_{36}	P_{44}	P_{60}	P_{65}	P_{10}	P_{19}
L_{30}	P_{30}	P_{31}	P_{33}	P_{37}	P_{45}	P_{61}	P_{66}	P_{11}	P_{20}
L_{31}	P_{31}	P_{32}	P_{34}	P_{38}	P_{46}	P_{62}	P_{67}	P_{12}	P_{21}
L_{32}	P_{32}	P_{33}	P_{35}	P_{39}	P_{47}	P_{63}	P_{68}	P_{13}	P_{22}
L_{33}	P_{33}	P_{34}	P_{36}	P_{40}	P_{48}	P_{64}	P_{69}	P_{14}	P_{23}
L_{34}	P_{34}	P_{35}	P_{37}	P_{41}	P_{49}	P_{65}	P_{70}	P_{15}	P_{24}
L_{35}	P_{35}	P_{36}	P_{38}	P_{42}	P_{50}	P_{66}	P_{71}	P_{16}	P_{25}
L_{36}	P_{36}	P_{37}	P_{39}	P_{43}	P_{51}	P_{67}	P_{72}	P_{17}	P_{26}
L_{37}	P_{37}	P_{38}	P_{40}	P_{44}	P_{52}	P_{68}	P_{73}	P_{18}	P_{27}
L_{38}	P_{38}	P_{39}	P_{41}	P_{45}	P_{53}	P_{69}	P_{1}	P_{19}	P_{28}
L_{39}	P_{39}	P_{40}	P_{42}	P_{46}	P_{54}	P_{70}	P_{2}	P_{20}	P_{29}
L_{40}	P_{40}	P_{41}	P_{43}	P_{47}	P_{55}	P_{71}	P_{3}	P_{21}	P_{30}
L_{41}	P_{41}	P_{42}	P_{44}	P_{48}	P_{56}	P_{72}	P_{4}	P_{22}	P_{31}
L_{42}	P_{42}	P_{43}	P_{45}	P_{49}	P_{57}	P_{73}	P_{5}	P_{23}	P_{32}
L_{43}	P_{43}	P_{44}	P_{46}	P_{50}	P_{58}	P_{1}	P_{6}	P_{24}	P_{33}
L_{44}	P_{44}	P_{45}	P_{47}	P_{51}	P_{59}	P_{2}	P_{7}	P_{25}	P_{34}
L_{45}	P_{45}	P_{46}	P_{48}	P_{52}	P_{60}	P_{3}	P_{8}	P_{26}	P_{35}
L_{46}	P_{46}	P_{47}	P_{49}	P_{53}	P_{61}	P_{4}	P_{9}	P_{27}	P_{36}
L_{47}	P_{47}	P_{48}	P_{50}	P_{54}	P_{62}	P_{5}	P_{10}	P_{28}	P_{37}
L_{48}	P_{48}	P_{49}	P_{51}	P_{55}	P_{63}	P_{6}	P_{11}	P_{29}	P_{38}
L_{49}	P_{49}	P_{50}	P_{52}	P_{56}	P_{64}	P_{7}	P_{12}	P_{30}	P_{39}
L_{50}	P_{50}	P_{51}	P_{53}	P_{57}	P_{65}	P_{8}	P_{13}	P_{31}	P_{40}
L_{51}	P_{51}	P_{52}	P_{54}	P_{58}	P_{66}	P_{9}	P_{14}	P_{32}	P_{41}
L_{52}	P_{52}	P_{53}	P_{55}	P_{59}	P_{67}	P_{10}	P_{15}	P_{33}	P_{42}
L_{53}	P_{53}	P_{54}	P_{56}	P_{60}	P_{68}	P_{11}	P_{16}	P_{34}	P_{43}
L_{54}	P_{54}	P_{55}	P_{57}	P_{61}	P_{69}	P_{12}	P_{17}	P_{35}	P_{44}
L_{55}	P_{55}	P_{56}	P_{58}	P_{62}	P_{70}	P_{13}	P_{18}	P_{36}	P_{45}
L_{56}	P_{56}	P_{57}	P_{59}	P_{63}	P_{71}	P_{14}	P_{19}	P_{37}	P_{46}
L_{57}	P_{57}	P_{58}	P_{60}	P_{64}	P_{72}	P_{15}	P_{20}	P_{38}	P_{47}
L_{58}	P_{58}	P_{59}	P_{61}	P_{65}	P_{73}	P_{16}	P_{21}	P_{39}	P_{48}
L_{59}	P_{59}	P_{60}	P_{62}	P_{66}	P_{1}	P_{17}	P_{22}	P_{40}	P_{49}
L_{60}	P_{60}	P_{61}	P_{63}	P_{67}	P_{2}	P_{18}	P_{23}	P_{41}	P_{50}
L_{61}	P_{61}	P_{62}	P_{64}	P_{68}	P_{3}	P_{19}	P_{24}	P_{42}	P_{51}
L_{62}	P_{62}	P_{63}	P_{65}	P_{69}	P_{4}	P_{20}	P_{25}	P_{43}	P_{52}
L_{63}	P_{63}	P_{64}	P_{66}	P_{70}	P_{5}	P_{21}	P_{26}	P_{44}	P_{53}
L_{64}	P_{64}	P_{65}	P_{67}	P_{71}	P_{6}	P_{22}	P_{27}	P_{45}	P_{54}
L_{65}	P_{65}	P_{66}	P_{68}	P_{72}	P_{7}	P_{23}	P_{28}	P_{46}	P_{55}
L_{66}	P_{66}	P_{67}	P_{69}	P_{73}	P_{8}	P_{24}	P_{29}	P_{47}	P_{56}
L_{67}	P_{67}	P_{68}	P_{70}	P_{1}	P_{9}	P_{25}	P_{30}	P_{48}	P_{57}
L_{68}	P_{68}	P_{69}	P_{71}	P_{2}	P_{10}	P_{26}	P_{31}	P_{49}	P_{58}
L_{69}	P_{69}	P_{70}	P_{72}	P_{3}	P_{11}	P_{27}	P_{32}	P_{50}	P_{59}
L_{70}	P_{70}	P_{71}	P_{73}	P_{4}	P_{12}	P_{28}	P_{33}	P_{51}	P_{60}
L_{71}	P_{71}	P_{72}	P_{1}	P_{5}	P_{13}	P_{29}	P_{34}	P_{52}	P_{61}
L_{72}	P_{72}	P_{73}	P_{2}	P_{6}	P_{14}	P_{30}	P_{35}	P_{53}	P_{62}
L_{73}	P_{73}	P_{1}	P_{3}	P_{7}	P_{15}	P_{31}	P_{36}	P_{54}	P_{63}

2.3 Algorithm [10]:

For any (k,3)-arc K , $k \geq 6$, there are at least four points in K no three of which are collinear . Let $C_1 = \{P_i : i= 1,\dots,k\}$ and $C_2 = \{W_i : i= 1,\dots,k\}$ be two (k,3)-arcs in PG(2,q) , where the coordinates of the points P_i and W_i are :

$P_i=p\,(x_i\,(1), x_i\,(2), x_i\,(3))$ and $W_i= w\,(x_i\,(1), x_i\,(2), x_i\,(3))$.

By the fundamental theorem, there exists a unique projectivity which takes any set of four points of the (k,4)-arc C_1 no three are collinear to any set of four points of C_2 no three are collinear . Let the (3x3) matrix $Z=(\square_{i,j})$, i,j = 1,2,3 take a fixed set of four points of C_1 no three are collinear , say { P_1 , P_2 , P_3, P_4 } , to any set of four points of C_2 no three are collinear , say {W_1, W_2, W_3, W_4 }, to determine Z we fixed a set of four points C_1 no three are collinear .

Then we work out the projectivity matrix Z that takes the fixed set of four points of C_1 to one of the J sets of four points of C_2 no three collinear , where J is the number of the sets of four points in C_2 no three of which are collinear. Therefore, there are J matrices Z to be checked. Now Z is the projectivity matrix takes the points of C_1 to the points of C_2 if Z takes the remaining points of C_1 to the remaining points of C_2.

The following is the matrix arithmetic to determine the matrix Z.

Let

$$X = \begin{bmatrix} x_1(0) & x_2(0) & x_3(0) \\ x_1(1) & x_2(1) & x_3(1) \\ x_1(2) & x_2(2) & x_3(2) \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1(1) & y_2(1) & y_3(1) \\ y_1(2) & y_2(2) & y_3(2) \end{bmatrix}$$

Let $Z=(\square_{i,j})$, i,j = 1,2,3 be a (3x3) matrix that takes three points P_i of C_1 to three points W_i of C_2 , when i = 1,2,3 . So Let

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Thus $Z=Y\,B\,X^{-1}$ …………………… (1)

The matrix Z also has to take the fourth point P_4 of C_1 to the point W_4 of C_2 . So $Z\,X_4 = Y_4$ where X_4 and Y_4 are the column vectors represent the points P_4 and W_4 respectively. Therefore, (1) gives

$Y\,B\,X^{-1}\,X_4 = Y_4$ …………………… (2)

Let

$$X^{-1}X_4 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \text{ and let } \quad D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Thus (2) can be written as

$$YD\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = Y_4, \quad \text{So} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = Y^{-1}Y_4$$

Substituting the values of λ_1, λ_2 and λ_3 in matrix B, we have the projectivity matrix given in (1)

The set of projectivities fixing a (k,4)-arc K in the group G(K) . To determine this group, we used a computer program . In this case the program is set to compare K with itself, that is the projectivity matrix Z is an element of the group G(K) if $ZX_i=bX_j$ i,j=5,…,k, where X_i is the column vector represents the point P_i. When we choose the points of triangle of reference and the unit points are (1,0,0) , (0,1,0) , (0,0,1) and (1,1,1) to be fixed four points , then (1) becomes : Z=YB.

3. classification of (k,3)-arcs in PG(2,8) ; (k=3,4,…,15)

3.1 The construction of the projectively distinct (3,3)-arcs

Let A = {1, 2, 37} be a (3,3)-arc in PG(2,8) . Then all (3,3)-arcs are projectively equivalent with respect to the type of their lines to A, therefore there is only (up to projectively equivalent) one (3,3)-arc in PG(2,8) with the type can be calculated as follows:

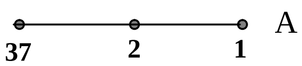
By using equations 1,2 and 3 of lemma (1.1), we have the following equations:

$$r_0 + r_1 + r_2 + r_3 = 73$$
$$r_1 + 2r_2 + 3r_3 = 27$$
$$2r_2 + 6r_3 = 6$$

The only type of (3,3)-arc which satisfies the above equations is:

$r_3 = 1$ $r_2 = 0$ $r_1 = 24$ $r_0 = 48$

So a (3,3)-arc is of type (1, 0, 24,48)



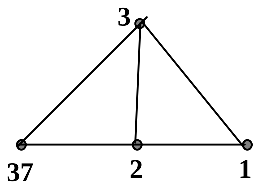
3.2 The construction of the projectively distinct (4,3)-arcs

From (2.1) there is only one (3,3)-arc A. There are 64 points of index zero for A . So by adding one point of them to (3,3)-arc A , we have all these points lie in the same set.

Therefore there is only one (4,3)-arc can be constructed by adding one point from this set to A. So there is only one type of (4,3)-arc denoted it by B, can be calculated as follows :

By using equations 1,2 and 3 of lemma (1.1), we have the following equations:

$$r_0 + r_1 + r_2 + r_3 = 73$$
$$r_1 + 2r_2 + 3r_3 = 36$$
$$2r_2 + 6r_3 = 12$$



B

The only type of (4,3)-arc which satisfies the above equations is:

$r_3 = 1$, $r_2 = 3$, $r_1 = 27$, and $r_0 = 42$

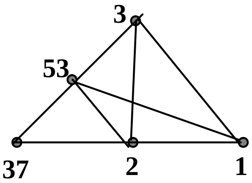
So a (4,3)-arc is of type (1, 3, 27,42)

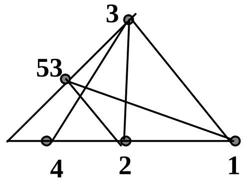
3.3 The construction of the projectively distinct (5,3)-arcs

From (2.2) there is only one (4,3)-arc B.There are 63 points of index zero for B. So by adding one point of index zero from PG(2,8)\B, we get only two projectively distinct (5,3)-arcs, we denoted it by C_1, C_2 . which are shown in following :

C_1 = {1,2,3,53,37} and C_2 = {1,2,3,53,4}

So a (5,3)-arc is of types (2,4,31,36) and (1,7,28,37) respectively

C_1




C_2

3.4 The construction of the projectively distinct (6,3)-arcs

From (2.3), we have get two sets C_1 and C_2 , Now we have 62 points of index zero for C_1 and C_2 . So by adding one point of index zero from PG(2,8)\$C_1$ or by adding one point of index zero from PG(2,8)\$C_2$, we get :

$D_1 = C_1 \cup \{5\}$, $D_2 = C_1 \cup \{10\}$, $D_3 = C_1 \cup \{38\}$, $D_4 = C_2 \cup \{11\}$

By using a computer program the group G(D_1) consists of three elements which are :

$$T_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 4 & 4 & 0 \end{bmatrix}, T_3 = \begin{bmatrix} 4 & 4 & 0 \\ 4 & 4 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

The order of the projectivity T_2 and T_3 are 2, So G (D_1) is isomorphic to Z_3. The groups G(D_2) is consist of I.Thus the group G(D_2) is isomorphic to the trivial group . The group G(D_3) consists of twenty four elements, So G (D_3) is isomorphic to S_4. The groups G(D_4) is consist of I. Thus the group G(D_4) is isomorphic to the trivial group.

All the above results are written in the following table :

Table (3.1)

Sy.	Distinct (6,3)-arc						G	|G|	r_3	r_2	r_1	r_0
D_1	1	2	3	53	37	5	Z_3	3	3	6	33	31
D_2	1	2	3	53	37	10	I	1	2	9	30	32
D_3	1	2	3	53	37	38	S_4	24	4	3	36	30
D_4	1	2	3	53	4	11	I	1	1	12	27	33

3.5 The construction of the projectively distinct (7,3)-arcs

From (2.4) all the projectively distinct (6,3)-arcs D_i (i=1,2,3,4) are incomplete . So by adding one point of index zero to each of the D_i , i=1,2,3,4, we have five projectively distinct (7,3)-arcs $E_1 = D_1 \cup \{6\}$ $E_3 = D_1 \cup \{28\}$ $E_4 = D_4 \cup \{10\}$ $E_5 = D_4 \cup \{13\}$

Table (3.2)

Sy.	Distinct (7,3)-arc							G	|G|	r_3	r_2	r_1	r_0
E_1	1	2	3	53	37	5	6	I	1	4	9	33	27
E_2	1	2	3	53	37	5	11	I	1	3	12	30	28
E_3	1	2	3	53	37	5	28	Z_2	2	5	6	36	26
E_4	1	2	3	53	4	11	10	Z_2	2	2	15	27	29
E_5	1	2	3	53	4	11	13	I	1	1	18	24	30

3.6 The construction of the projectively distinct (8,3)-arcs

By the same way we get the following results :

Table (3.3)

Sy.	Distinct (8,3)-arc								G	|G|	r_3	r_2	r_1	r_0
F_1	1	2	3	53	37	5	6	7	I	1	6	10	34	23
F_2	1	2	3	53	37	5	6	11	I	1	4	16	28	25
F_3	1	2	3	53	37	5	6	15	I	1	5	13	31	24
F_4	1	2	3	53	37	5	11	39	I	1	3	19	25	26
F_5	1	2	3	53	4	11	10	44	Z_2	2	2	22	22	27

all the projectively distinct (8,3)-arcs F_i (i=1,2,…,5) are incomplete

3.7 The construction of the projectively distinct (9,3)-arcs

Table (3.4)

Sy.	Distinct (9,3)-arc									G	|G|	r_3	r_2	r_1	r_0
G_1	1	2	3	53	37	5	6	7	10	I	1	8	12	33	20
G_2	1	2	3	53	37	5	6	7	11	I	1	7	15	30	21
G_3	1	2	3	53	37	5	6	7	27	I	1	6	18	27	22
G_4	1	2	3	53	37	5	6	7	34	I	1	9	9	36	19
G_5	1	2	3	53	37	5	6	11	15	I	1	5	21	24	23
G_6	1	2	3	53	37	5	11	39	49	I	1	4	24	21	24
G_7	1	2	3	53	4	11	10	44	40	I	1	3	27	18	25

all the projectively distinct (9,3)-arcs G_i (i=1,2,…,7) are incomplete

The construction of the projectively distinct (10,3)-arcs

Table (3.5)

Sy.	Distinct (10,3)-arc										G	|G|	r_3	r_2	r_1	r_0
H_1	1	2	3	53	37	5	6	7	10	11	I	1	9	18	27	19
H_2	1	2	3	53	37	5	6	7	10	19	I	1	10	15	30	18
H_3	1	2	3	53	37	5	6	7	10	20	I	1	11	12	33	17
H_4	1	2	3	53	37	5	6	7	11	27	I	1	8	21	24	20
H_5	1	2	3	53	37	5	6	7	27	49	I	1	7	24	21	21
H_6	1	2	3	53	37	5	6	7	34	24	Z_2	2	12	9	36	16
H_7	1	2	3	53	37	5	11	39	49	13	I	1	6	27	18	22
H_8	1	2	3	53	4	11	20	44	40	48	I	1	4	33	12	24

all the projectively distinct (10,3)-arcs H_i (i=1,2,…,8) are incomplete

3.8 The construction of the projectively distinct (11,3)-arcs

Table (3.6)

Sy.	Distinct (11,3)-arc											G	|G|	r_3	r_2	r_1	r_0
I_1	1	2	3	53	37	5	6	7	10	11	20	I	1	13	16	28	16

Sy.	Distinct (11,3)-arc											G	$\vert G\vert$	r_3	r_2	r_1	r_0
I_2	1	2	3	53	37	5	6	7	10	11	24	I	1	12	19	25	17
I_3	1	2	3	53	37	5	6	7	10	11	27	I	1	11	22	22	18
I_4	1	2	3	53	37	5	6	7	10	19	44	I	1	14	13	31	15
I_5	1	2	3	53	37	5	6	7	10	20	44	I	1	15	10	34	14
I_6	1	2	3	53	37	5	6	7	11	27	46	I	1	10	25	19	19
I_7	1	2	3	53	37	5	11	39	49	13	18	I	1	8	31	13	21
I_8	1	2	3	53	37	5	11	39	49	13	21	I	1	9	28	16	20
I_9	1	2	3	53	4	11	10	44	40	48	61		8	5	40	4	24

all the projectively distinct (11,3)-arcs I_i (i=1,2,…,9) are incomplete

3.9 The construction of the projectively distinct (12,3)-arcs

Table (3.7)

Sy.	Distinct (12,3)-arc												G	$\vert G\vert$	r_3	r_2	r_1	r_0
J_1	1	2	3	53	37	5	6	7	10	11	20	26	I	1	16	18	24	15
J_2	1	2	3	53	37	5	6	7	10	11	20	44	I	1	17	15	27	14
J_3	1	2	3	53	37	5	6	7	10	11	24	26	I	1	15	21	12	16
J_4	1	2	3	53	37	5	6	7	10	11	27	46	I	1	14	24	18	17
J_5	1	2	3	53	37	5	6	7	10	19	44	20	I	1	18	12	30	13
J_6	1	2	3	53	37	5	11	39	49	13	18	7	I	1	13	27	15	18
J_7	1	2	3	53	37	5	11	39	49	13	18	21	I	1	12	30	12	19
J_8	1	2	3	53	4	11	10	44	40	48	61	13	Z_2	2	10	36	6	21

all the projectively distinct (12,3)-arcs J_i (i=1,2,…,8) are incomplete

3.10 The construction of the projectively distinct (13,3)-arcs

Table (3.8)

Sy.	Distinct (13,3)-arc													G	$\vert G\vert$	r_3	r_2	r_1	r_0
K_1	1	2	3	53	37	5	6	7	10	11	20	26	43	I	1	20	18	21	14
K_2	1	2	3	53	37	5	6	7	10	11	20	44	52	I	1	21	15	24	13
K_3	1	2	3	53	37	5	6	7	10	11	24	26	27	I	1	19	21	18	15
K_4	1	2	3	53	37	5	6	7	10	19	44	20	52	I	1	22	12	27	12
K_5	1	2	3	53	37	5	11	39	49	13	18	7	27	I	1	18	24	15	16
K_6	1	2	3	53	37	5	11	39	49	13	18	7	43	I	1	17	27	12	17
K_7	1	2	3	53	37	5	11	39	49	13	18	7	59	I	1	16	30	9	18
K_8	1	2	3	53	37	5	11	39	49	13	18	21	59	I	1	15	33	6	14

all the projectively distinct (13,3)-arcs K_i (i=1,2,…,8) are incomplete except i= 2,4 which are a complete (13,4)-arcs.

3.12 The construction of the projectively distinct (14,3)-arcs

Table (3.9)

Sy.	Distinct (14,3)-arc														G	$\vert G\vert$	r_3	r_2	r_1	r_0
L_1	1	2	3	53	37	5	6	7	10	11	20	26	43	52	I	1	25	16	19	13
L_2	1	2	3	53	37	5	11	39	49	13	18	7	27	51	I	1	24	19	16	14
L_3	1	2	3	53	37	5	11	39	49	13	18	7	43	46	I	1	22	25	10	16
L_4	1	2	3	53	37	5	11	39	49	13	18	7	43	59	I	1	21	28	7	17
L_5	1	2	3	53	37	5	11	39	49	13	18	7	43	70	I	1	23	22	13	15

Sy.	Distinct (14,3)-arc														G	|G|	r_3	r_2	r_1	r_0
L_6	1	2	3	53	37	5	11	39	49	13	18	21	59	46	I	1	20	31	4	18

all the projectively distinct (14,3)-arcs L_i (i=1,2,…,6) are incomplete except i= 1,5 which are a complete (14,4)-arcs.

3.13 The construction of the projectively distinct (15,3)-arcs

Table (3.10)

Sy.	Distinct (15,3)-arc															G	|G|	r_3	r_2	r_1	r_0
M_1	1	2	3	53	37	5	11	39	49	13	18	7	27	51	62	I	1	31	12	18	12
M_2	1	2	3	53	37	5	11	39	49	13	18	7	43	46	59	I	1	27	24	6	16
M_3	1	2	3	53	37	5	11	39	49	13	18	21	59	46	68		12	25	30	0	18

all the projectively distinct (15,3)-arcs M_i (i=1,2,3) are a complete (15,4)-arcs.

3.14 Conclusion : The maximum value $m(3)_{8,2}$ for which (k,3)-arcs is not exist

4. <u>Theorem</u>: In PG(2,8), a complete (k,3)-arc does not exist for 3≤k≤8.

<u>Proof:</u> For $3 \le k \le 8$ the equations (4) and (5) of lemma (1.1) become
$R_1 + R_2 + R_3 = 9$
$R_2 + 2 R_3 = k-1$
Let m = [(k-1) / 2] , where [(k-1) / 2] is the integral part of (k-1) / 2 .
So the maximum value of R_3 can accure is m. Assume that r_i= [(k-1-2i)] , i=0,1,…,m . It is clear that m is positive for $k \ge 3$.
Suppose α_m denoted the number of points of PG (2,8) of type ($R_1,\ldots,r_m$-j, m) , j=0,1,…,r_m
According to equation (1) and (2) of lemma (1.2) we have,
$m\alpha_m + (m-1)\alpha_{m-1} + \cdots + \alpha_1 = 3r_3$ …(*),
where r_3 is the total number of 3-secants of (k,3)-arc in PG(2,8) , with $3 \le k \le 8$.
Since $m \ge 0$, for $k \ge 3$, we obtain
$\alpha_m + \alpha_{m-1} + \cdots + \alpha_1 = m(\sum_{k=0}^{m} \alpha_k)$… (**) is bigger than;
$m\alpha_m + (m-1)\alpha_{m-1} + \cdots + \alpha_1 = \sum_{k=0}^{m} k\alpha_k$.
Therefore, $m(\sum_{k=0}^{m} k\alpha_k) = mk > (\sum_{k=0}^{m} k\alpha_k) = 3\, r_3$.
This implies $mk > 3r_3$ or, $r_3 < mk/3$. Furthermore,
Since $m \le (k-1)/2$, then we have $r_3 < k(k-1)/6$ ……………..(1)
On the other hand if the (k,3)-arc K is complete for $3 \le k \le 8$, then
according to lemma (1.4), we have $6\, r_4 \ge 73 - k$ or $r_3 \ge (73 - k)/6$ ………..(2)
Now, for k=3 we obtain from the equations (1) and (2)
$r_4 < 1$ and $r_3 > 11$, which is impossible. So a complete (3,3)-arc does not exist in PG(2,8).
for k=8 , we obtain from equations (1) and (2)
$r_3 < 9$ and $r_3 > 10$ which is impossible , so a complete (8,3)-arc does not exist in PG(2,8).□

References

[1] Abdul-Hussain, M. A., "Classification of (k,4)-arcs in the projective plane of order five", M. Sc. thesis, University of Basrah, Iraq, (1997).

[2] Abood. H. M., "Classification of (k,4)-arcs in the projective plane of order three", J. Basrah Researches, Vol. B, Part1, (1997).

[3] Ball S. and Hirschfeld, J.W.P., "Bounds on (n, r)-arcs and their application to linear codes", J. Geom, 1-11, (2005).

[4] Barlotti A., "Su {k; n}-archi di un piano lineare finito", Boll. Un. Mat. Ital., 11, 553-556, (1956).

[5] Bierbrauer J., "(k, n)-arcs of maximal size in the plane of order 8",unpublished manuscript (1988).

[6] Bortun David, "Introduction to modern abstract algebraic", Addison Wesley, University of New Hampshire, London, (1967).

[7] Bose R. C., "Mathematical theory of the symmetrical factorial design", Sankyha, 8, 107-166, (1947).

[8] Coolsaet K., Sticker H., "A full classification of the complete k-arcs in PG(2, 27) ", Ghent University, Belgium, (2009).

[9] Daskalov R.N., "On the maximum size of some (k, r) −arcs in PG(2, q)", University of Gabrovo, Bulgaria, (2007).

[10] Falih S.A. "On complete (k,4)-arcs in projective plane of order eight", M. Sc. thesis, University of Basrah, Iraq, (2009).

[11] Fralergh J.B.,"A first course in abstract algebra", Seventh Edition, Addison Wesley, North-Holland, (2003).

[12] Haimulin J.N. "Some properties of $\{k, n\}_q$ − arcs in Galois planes", Soviet Math. Dokl. 7,1100-1103, (1966).

[13] Hameed F. K." (k,n)-arcs in the projective plane PG(2,q) ", M. Sc. thesis, University of Sussex, UK,(1984).

[14] Hirschfeld J. W. P., "Cyclic projectivity in PG(n,q) ", Teovie combinatoric, volume I, Accad.Naz. dei Linei, 201-211, (1979).

[15] Hirschfeld J. W. P., "Projective Geometries over Finite Fields",Second Edition, Oxford University Press, Oxford, xiv + 555 pp., (1998).

[16] Hirschfeld J.W.P., " Maximum sets in a finite projective space", 19th June (2008).

[17] Hirschfeld ,J.W.P. and Storme ,L., "The packing problem in statistics, coding theory and finite projective spaces", update 2001 in: Finite Geometries, Developments in Mathematics 3, Kluwer , 201–246, (2001).

[28] Ibrahim M. A. "Classification of (k,4)-arcs and (k,3)-arcs in the projective plane of order seven", M. Sc. thesis, University of Basrah, Iraq, (2003).

[19] Khalid M. Sh. "Classification of complete (k,4)-arcs in the projective plane of order eleven", M. Sc. thesis, University of Basrah, Iraq, (2010).

[20] Sadeh A.R., " The classification of k-arcs and cubic surfaces with twenty seven lines over the field of eleven elements ", M. Sc. thesis,University of Sussex ; UK , (1984).

[21] Segre B., "Sulle ovali nei piani lineari finiti", Atti Accad. Naz. Lincei Rend, 17, 1-2, (1954).

[22] Segre B., "Ovals in a finite projective plane", Canad. J. Math., 7, 414-416, (1955).

[23] Segre B., "Sui k-archi nei piani finiti di caratteristica due", Rev. Math. Pures Appl., 2, 289-300, (1957).

[24] Singer J., "A theorem in finite projective geometry and some applications to number theory" Trans. Amer. Math. Soc. 43, 377-385, (1938).