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New largest bounds of (k, n)-arcs in PG(2,73)

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Abstract. A projective plane of order q consists of a set of $q^2 + q + 1$ points and a set of $q^2 + q + 1$ lines, there are exactly $q + 1$ points on each line and $q + 1$ lines pass through each point. A blocking set $\mathfrak{B}$ is a set of points such that each line contains at least t points of $\mathfrak{B}$ and some lines contains exactly t points of $\mathfrak{B}$. A blocking set $\mathfrak{B}$ is the complement of a (k, n)-arc K with $t = q + 1 - n$. In this paper, non-existence of some (k, n)-arcs are proved for $q = 73$.

Keywords: Finite projective plane · Arcs in P(2, q) · Lower and Upper bounds. Classification of (k, n)-arcs in P(2, q).

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1. Introduction

A projective plane of order q consists of a set of q^2+q+1 points and a set of q^2+q+1 lines, where each line contains exactly $q+1$ points. A t-fold blocking set $\mathfrak{B}$ in $PG(2,q)$ is a set of points such that each line contains at least t points of $\mathfrak{B}$ and some lines contains exactly t points of $\mathfrak{B}$. A blocking set in $PG(2,q)$ shall contain no line. However, a blocking set can be considered as the complete of a (k,n)-arc K in $PG(2,q)$ with $t=q+1-n$. The smallest blocking sets are just the lines, and any blocking set containing a line will called trivial.

Blocking set have been first studied in 1969 by Di Paola [11], where the author has calculated the minimum size of a non-trivial blocking set in $PG(3,q)$. In 1970 and 1971, Bruen [7],[8] proved that $|\mathfrak{B}| \geq \sqrt{q}(\sqrt{q}+1)+3$ for any non-trivial blocking set, respectively. In 1977, Bruen and Thas [9] showed that a blocking set with $\sqrt{q}(\sqrt{q}+1)+2$ points necessarily contain a Baer sub plane, $|\mathfrak{B}| \geq \sqrt{q}(\sqrt{q}+1)+3$ will hold. In 1981 [14], it is shown that, for even r, there are 2-blocking sets of size $3r$ and 3-blocking sets of size $4q$. In 1994 [5], the arcs (78,8) and (90,9) are the largest complete arcs in $PG(2,11)$, while for $PG(2,13)$, there exist no arcs of size (106,9), (110,10), (134,11). A new lower bound for $q<11$ is founded by Ball [6] in 1996. In 2004, Diskalov [10] investigated $PG(2,17)$. In 2011, Falih [12] found the classification and construction of $(k,3)$-arcs in $PG(2,8)$. After that Falih and Abbas [13] in 2013 found (k,n;f)-arcs of type (1,n) in PG(2,q), with q<8. In 2018, Alabdullah [1] established a new upper bound for the size of a (k,n)-arc in $PG(2,q)$. In 2019, Alabdullah and Hirschfeld [2] established a new lower bound in $PG(2,q)$. In 2021, Alabdullah and Hirschfeld [3] found new upper bounds $m_n(2,q) \leq \frac{(q+1)(2n-3)}{2}$ in $PG(2,q)$ for some values of q. In 2021, Alabdullah [4] found the classification of $(k,4)$-arcs in projective plane of order eight according to i-secant distribution. In this paper, A new upper bounds of (k,n)-arcs in $PG(2,73)$ have been founded.

2. Background

Definition 2.1. [15] A (k,n)-arc in $PG(2,q)$ is a set K of k points, no $n+1$ of which are collinear, but with at least one set of n points collinear. When $n=2$, a $(k,2)$-arc is called a k-arc.

Definition 2.2. [15] A (k,n)-arc is complete if it is not contained in a $(k+1,n)$-arc.

Theorem 2.1. Let K be (k,n)-arc in $PG(2,q)$, where q is prime.

1. If $t<\frac{q}{2}$, then $|\mathfrak{B}| \geq (t+\frac{1}{2})(q+1)$.
2. If $t>\frac{q}{2}$, then $|\mathfrak{B}| \geq (t+1)q$.

Theorem 2.2. (Ball [5]) Let $\mathfrak{B}$ be a t -fold blocking set in PG(2,q), q prime and $q>3$.

1. If $t<q/2$, then $|\mathfrak{B}| \geq (t+\frac{1}{2})(q+1)$.
2. If $t>q/2$, then $|\mathfrak{B}| \geq (t+1)q$.

Theorem 2.3. (Ball [5]) Let $\mathfrak{B}$ be a t -fold blocking set in PG(2,q) that contains a line.

3. If $(t-1,q)=1$, then $|\mathfrak{B}| \geq q(t+1)$.
4. If $(t-1,q)>1$ and $t \leq \frac{q}{2}+1$, then $|\mathfrak{B}| \geq tq+q-t+2$.
5. If $(t-1,q)>1$ and $t \geq \frac{q}{2}+1$, then $|\mathfrak{B}| \geq t(q+1)$.

Theorem 2.4. (Daskalov [10]). Let $\mathfrak{B}$ an $\{\ell,t\}$-blocking set in $PG(2,q)$, q prime.

1. If $t<\frac{q}{2}$, and $q>3$, then $\ell \geq n(q+1)+(q+1)/2$.
2. If $\ell=t(q+1)+(q+1)/2$, then
   a. Through each point of $\mathfrak{B}$ there are exactly $(q+3)/2$ lines that are not t-secants;
   b. Through each point of $\mathfrak{B}$ there are exactly $(q-3)/2$ lines that are t-secants;

c. The total number of t-secants is $\mu = \ell(q-1)/2t$.

Lemma 2.1. ([13], chapter 12). For any set of k points in $PG(2,q)$, the following holds:

$\sum_{i=0}^{q+1} \tau_i = q^2 + q + 1$ ....(1).

$\sum_{i=1}^{q+1} i\tau_i = |\mathfrak{B}|(q+1)$ ....(2).

$\sum_{i=2}^{q+1} i(i-1)\tau_i = |\mathfrak{B}|(|\mathfrak{B}|-1)$ ....(3).

Theorem 2.5. (Alabdullah [3]). For $(q+3/2) < n < q$, with r prime, $m_n(2,q) \leq \frac{(q+1)(2n-3)}{2}$.

Notation 2.1. For a (k,n)-arc K in $PG(2,q)$, let

$\tau_i =$ the total number of i-secants of K,

$\rho_i =$ the number of i-secants through a point P of K,

$m_n(2,q) =$ the maximum size of a (k,n)-arc in $PG(2,q)$.

3. New Largest Bound

In this section, there are some (k,n)-arcs proved not exist in $PG(2,73)$.

Theorem 3.1. In $PG(2,73)$, there exist no (k,n)-arc for the following values of k, giving corresponding upper bounds for $m_n(2,73)$.

3.1.1. Case I: Bounds for complete (k,n)-arcs when μ is a non-integer:

Table 1

k	2776	2850	2924	2998	3072	3146	3220	3294	3368
n	39	40	41	42	43	44	45	46	47
$m_n(2,73) \leq$	2775	2849	2923	2997	3071	3145	3219	3293	3367
k	3442	3516	3590	3664	3738	3812	3886	3960	4108
n	48	49	50	51	52	53	54	55	57
$m_n(2,73) \leq$	3441	3515	3589	3663	3737	3811	3885	3959	4107
k	4182	4256	4330	4404	4552	4626	4774	4848	4996
n	58	59	60	61	63	64	66	67	69
$m_n(2,73) \leq$	4181	4255	4329	4403	4551	4625	4773	4847	4995

Proof.

The above problems can be proved based on either Theorem 2.2, Theorem 2.3 and Theorem 2.4 or Theorem 2.5.

1. For $k = 2776$ and $n = 39$.

The largest size of $(2776,39)$-arc in $PG(2,73)$ is equivalent to find the largest 35-fold blocking set. Theorem 2.2 gives that $\mathfrak{B}$ must have at least 2627 points. Theorem 2.4 gives that the total number of 35-secants is $\mu = (2627 * 72)/70$, note that μ is not an integer number. Therefore a (2776,39)-arc does not exist and $m_{39}(2,73) \leq 2775$ □

The remaining cases are proved similarly.

3.1.2. Case II: Bounds for complete (k, n)-arcs when μ is a non-integer:

Table 2

k	4034	4478	4700	4922	5070	5144	5218
n	56	62	65	68	70	71	72
$m_n(2,73) \leq$	4033	4477	4699	4921	5069	5143	5217

Proof.

To finding the largest size of $(4034,56)$-arc is equivalent to finding the largest $\{1369,18\}$-blocking set $\mathfrak{B}$. Theorem 2.4 gives the total number of 18-secants which is 2738. Let r be the length of the longest secant. If $r = 74$, then S contains a line and Theorem 2.3 gives $|\mathfrak{B}| \geq 1387$, a contradiction. If $56 \leq r \leq 73$ then considering lines through a point on the longest secant, but not in $\mathfrak{B}$, so $\mathfrak{B}$ must have at least $18 * 73 + r$ points. This contradicts that $|\mathfrak{B}| = 1369$.

Consider the intersection of the 18-secants through $P \notin \mathfrak{B}$ with the longest secant. So,

$\tau_{18} \geq 36\, r + (74 - r)(q - i)$.................. 4.

The values of τ_{18} are calculated from Equation 4 for $i \leq 35$ and give the below table.

Table 3

r	55	54	53	52	51	50	49	48	47	46	45	44
i	0	1	2	3	4	5	6	7	8	9	10	11
τ_{18}	3367	3384	3399	3412	3423	3432	3439	3444	3447	3448	3447	3444
r	43	42	41	40	39	38	37	36	35	34	33	32
i	12	13	14	15	16	17	18	19	20	21	22	23
τ_{18}	3439	3432	3423	3412	3399	3384	3367	3348	3327	3304	3279	3252
r	31	30	29	28	27	26	25	24	23	22	21	20
i	24	25	26	27	28	29	30	31	32	33	34	35
τ_{18}	3223	3192	3159	3124	3087	3048	3007	2964	2919	2872	2823	2772

Table (3) illustrates that all values of r for $r = 20, 21, ..., 55$ give a contradiction. This is because the total number of 18-seants is 74. However, for $r = 19$ and 18, now by using Equations 1, 2 and 3 of Lemma 2.1 get the following

$\tau_{18} + \tau_{19} = 5403;$

$18\, \tau_{18} + 19\tau_{19} = 101306;$

$\tau_{18} + \tau_{19} = 1872792.$

This system has no solution, so there is no (4034,56)-arc exists and $m_{56}(2,73) \leq 4033$. □

The others cases can be proved in the same method.

4. Conclusion

In this paper, new some $m_n(2, q)$ are proved to the largest (k, n)-arcs in $PG(2,73)$.

References

[1] Alabdullah, S. A. F. (2018). *Classification of arcs in finite geometry and applications to operational research* (Doctoral dissertation, University of Sussex).

[2] Alabdullah, S., & Hirschfeld, J. W. P. (2019). A new lower bound for the smallest complete (k, n)-arc in PG (2, q). *Designs, Codes and Cryptography*, *87*(2), 679-683.

[3] Alabdullah, S., & Hirschfeld, J. W. P. (2021). A new upper bound for the largest complete (k, n)-arc in PG (2, q). *American mathematical society,Providence, Rl*, Vol. *770*, pp. 1-9.

[4] Alabdullah, S. (2021). Classification of (k,4)-arcs in projective plane of order eight according to i-secant distribution. *Iraq University College Journal of Engineering and Applied Sciences,* Vol. *1*, No. 1.

[5] Ball, S. M. (1994). On sets of points on finite planes (Doctoral dissertation, University of Sussex).

[6] Ball, S. (1996). On the Size of a Triple Blocking Set inPG (2, q). European Journal of Combinatorics, 17(5), 427-435.

[7] Bruen, A. (1970). Baer subplanes and blocking sets. Bulletin of the American Mathematical Society, 76(2), 342-344.

[8] Bruen, A. (1971). Blocking sets in finite projective planes. SIAM Journal on Applied Mathematics, 21(3), 380-392.

[9] Bruen, A. A., & Thas, J. A. (1977). Blocking sets. Geometriae Dedicata, 6(2), 193-203.

[10] Daskalov, R. (2004, June). On the existence and the nonexistence of some (k, r)-arcs in PG (2, 17). In Proc. of Ninth International Workshop on Algebraic and Combinatorial Coding Theory (pp. 19-25).

[11] Di Paola, J. W. (1969). On minimum blocking coalitions in small projective plane games. SIAM Journal on Applied Mathematics, 17(2), 378-392.

[12] Falih, S. (2011). On complete (k,3)-arcs in PG (2,8). Journal of Basrah Researches ((Sciences)), Vol. 37, Number 4.

[13] Falih, S. & Abbas . (2013). (k,n;f)-arcs of type (1,n) in PG(2,q), with q<8. International Mathematical Forum, Vol. 8, 2013, no. 6, 289 - 300

[14] Hill, R., & Mason, J. R. M. On (fc, n)-arcs and the falsity of the Lunelli-Sce conjecture'. Finite geometries and designs, 153-168.

[15] Hirschfeld, J. W. P. (1998). Projective geometries over finite fields. Oxford mathematical monographs. New York: Oxford University Press.

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مجلة كلية العراق الجامعة للهندسة والعلوم التطبيقية

أكبر الحدود الجديدة للاقواس (k,n) في $PG(2,73)$

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الملخص. يتالف المستوي الاسقاطي من مجموعة نقاط عددها q^2+q+1 ومجموعة مستقيمات عددها q^2+q+1 حيث يقع $q+1$من النقاط على كل مستقيم، وفي كل نقطة يمر $q+1$ من المستقيمات. المجموعة القالبية $\mathfrak{B}$ هي مجموعة من النقاط حيث كل مستقيم يحتوي على الاقل t من النقاط وبعض المستقيمات تحتوي فقط t من النقاط. تعتبر المجموعة القالبية متممة للقوس (k,n)-والذي يسمى K حيث $t=q+1-n$. في هذا البحث تم اثبات عدم وجود بعض اقواس (k,n) في المستوي الاسقاطي من الرتبة الثالثة والسبعون.

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