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Classification of $(k, 4)$-arcs in projective plane of order eight according to i-secant distribution

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Abstract. A projective plane of order q consists of a set of $q^2 + q + 1$ points and a set of lines $q^2 + q + 1$, there are exactly $q + 1$ points on each line and $q + 1$ lines pass through each point. A (k, n)-arc is a set of k points, such that there is some n but no $n + 1$ are collinear, where $n \geq 2$ and a (k, n)-arc is complete if there is no $(k + 1, n)$-arc containing it. In this paper the classification of (k, n)-arcs in $PG(2, q)$ for the projective plane of order eight has been done using different methods.

Keywords: Finite projective plane · Arcs in $PG(2, q)$ · Lower and Upper bounds. Classification of (k, n)-arcs in $PG(2, q)$.

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1. Introduction

A projective plane of order q consists of a set of q^2+q+1 points and a set of lines q^2+q+1, where each line contains exactly $q+1$ points. In the last 1950's, Segre [10] introduced the notations of arcs and complete arcs. An arc in a plane is a set of points with no three are collinear and maximal arcs under the set inclusion are called complete arcs. A line containing two points of an arc is called a secant. An arc is complete if and only if its secants cover the whole plane. The main research topics in finite geometry is: what is the maximum and the minimum sizes of complete arcs?

A (k,n)-arc is a set of k points, such that there is some n but no $n+1$ are collinear, where $n \geq 2$ and a (k,n)-arc is complete if there is no $(k+1,n)$-arc containing it. Let K be a (k,n)-arc in $PG(2,q)$, the projective plane over the Galois field $GF(q)$ of q elements. The maximum value of k for which a $(k,4)$-arc exist in $PG(2,5)$ has been proved by Barlotti [4] to be sixteen. Sadah [9] have shown the classification and construction of k-arcs over the Galois field $GF(q)$ with $q \leq 11$. The full classification of k-arcs in $PG(2,q)$ for $q \leq 19$ is shown in [11]. Sticker [5],[6] obtained the full classification of k-arcs in $PG(2,23)$, $PG(2,25)$ and $PG(2,27)$. Coolsact [7] obtained the classification of k-arcs in $PG(2,31)$, in 2014. The classification and construction of $(k,3)$-arcs in $PG(2,8)$ were given by Falih [8]. In 2018, Alabdullah [1] calculated some largest size of complete (k,n)-arcs in $PG(2,q)$ for some q. A new lower bound is proved for smallest size of complete (k,n)-arcs is founded by Alabdullah [2] in 2019. A new largest upper bound of $m_n(2,q) \leq \frac{(q+1)(2n-3)}{2}$ in $PG(2,q)$ is founded by Alabdullah and Hirschfeld [3] in 2021. The main purpose of this paper is to construct and classify the distinct (k,n)-arcs in $PG(2,q)$ for $q=8$ based on i-secant distribution using different methods and Fortran programs.

2. Background

Definition 2.1. [8] A (k,n)-arc in $PG(2,q)$ is a set K of k points, no $n+1$ of which are collinear, but with at least one set of n points collinear. When $n=2$, a $(k,2)$-arc is called a k-arc.

Definition 2.2. [8] A (k,n)-arc is complete if it is not contained in a $(k+1,n)$-arc.

Definition 2.3. [8] The i-secant distribution of K is the $(n+1)$-tuple $(\tau_n, \tau_{n-1}, \dots, \tau_1, \tau_0)$.

Definition 2.3. (Companion matrix [8]) Let $f(x)$ be a monic polynomial in $F[x]$:

$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

The companion matrix $C(f)$ is $n \times n$ matrix given by

$$C(f) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}$$

In $PG(2,q)$, let

$f(x) = x^3 + a_2x^2 + a_1x + a_0$.

The companion matrix $C(f)$ is 3×3 matrix given by

$$C(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}.$$

Theorem 2.1. (Hirschfeld [8])

$$m_2(2,q) = \begin{cases} q+2, & for\ q\ even \\ q+1, & for\ q\ odd \end{cases}$$

Theorem 2.2. (Hirschfeld [8])

1. $m_2(2,q)\begin{cases} = (n-1)q+n, & for\ q\ even\ and\ n|q \\ < (n-1)q+n\ , & for\ q\ odd \end{cases}$.

2. A (k,n)-arc K is maximal if and only if every line in $PG(2,q)$ is either an n-secant or an external line.

Lemma 2.1. (Hirschfeld [8]) For a (k,n)-arc K, the following equations hold.

$\sum_{i=0}^{n}\beta_i = q^2+q+1$; .... (2.1)

$\sum_{i=1}^{n} i\beta_i = k(q+1)$; .... (2.2)

$\sum_{i=2}^{n} i(i-1)\beta_i = k(k-1)$; .... (2.3)

Notation 2.1. For a (k,n)-arc K in $PG(2,q)$, let
β_i = the total number of i-secants of K,
ρ_i = the number of i-secants through a point P of K,
$m_n(2,q)$ = the maximum size of a (k,n)-arc in $PG(2,q)$.

3. Projective Plane of Order Eight

The projective plane of order eight contains 73 points and 73 lines as shown in Table (1) and Table (2) respectively. Every line contains 9 points and through every point there pass 9 lines.

Let $f(x) = x^3 + x + \delta^4$ be an irreducible polynomial over $GF(8)$, then the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \delta^4 & 1 & 0 \end{bmatrix}$$

is cyclic projectivity which is given by right multiplication on the points of $GF(8)$.

Let the point P_i be represented by the vector $(1,0,0)$, then $P_1M^i = P_i, i = 1,2,\cdots 73$. The 73 points are shown in Table (1).

Let ℓ_i be the line which contains the points $\{P_1, P_2, P_4, P_8, P_{16}, P_{32}, P_{37}, P_{55}, P_{64}\}$, then let $\ell_i M^i = \ell_i, i = 1,2,\cdots 73$ are the lines of $GF(8)$. The 73 lines are given in Table (2).

Note that $F_8 = \{0,1,\delta,\delta^2,\delta^3,\delta^4,\delta^5,\delta^6 : \delta^3+\delta^2+1 = 2 = 0\}$.

Table 1. Points of $GF(8)$

$P_1(1,0,0)$	$P_2(0,1,0)$	$P_3(0,0,1)$	$P_4(1,\delta^3,0)$	$P_5(0,1,\delta^3)$	$P_6(1,\delta^3,1)$
$P_7(0,1,\delta^6)$	$P_8(1,\delta^6,0)$	$P_9(0,1,\delta^6)$	$P_{10}(1,\delta^3,\delta^4)$	$P_{11}(1,\delta^4,\delta^2)$	$P_{12}(1,1,\delta^5)$
$P_{13}(1,\delta^2,\delta^5)$	$P_{14}(1,\delta^2,1)$	$P_{15}(1,0,\delta^5)$	$P_{16}(1,\delta^2,0)$	$P_{17}(1,0,\delta^2)$	$P_{18}(1,\delta^3,\delta)$
$P_{19}(1,\delta^5,\delta^5)$	$P_{20}(1,\delta^2,\delta^3)$	$P_{21}(1,\delta,\delta^2)$	$P_{22}(1,1,\delta^2)$	$P_{23}(1,1,\delta)$	$P_{24}(1,\delta^5,\delta^2)$
$P_{25}(1,1,\delta^6)$	$P_{26}(1,\delta^6,\delta^4)$	$P_{27}(1,\delta^4,\delta^5)$	$P_{28}(1,\delta^2,\delta^2)$	$P_{29}(1,1,\delta^3)$	$P_{30}(1,\delta,1)$
$P_{31}(1,0,\delta^4)$	$P_{32}(1,\delta^4,0)$	$P_{33}(0,1,\delta^4)$	$P_{34}(1,\delta^3,\delta^6)$	$P_{35}(1,\delta^6,1)$	$P_{36}(1,0,\delta^2)$
$P_{37}(1,1,0)$	$P_{38}(0,1,1)$	$P_{39}(1,\delta^3,\delta^3)$	$P_{40}(1,\delta,\delta^3)$	$P_{41}(1,\delta,\delta)$	$P_{42}(1,\delta^5,\delta^3)$
$P_{43}(1,\delta,\delta^5)$	$P_{44}(1,\delta^2,\delta^6)$	$P_{45}(1,\delta^6,\delta^6)$	$P_{46}(1,\delta^6,\delta^3)$	$P_{47}(1,\delta,\delta^6)$	$P_{48}(1,\delta^6,\delta^5)$
$P_{49}(1,\delta^2,\delta^4)$	$P_{50}(1,\delta^4,\delta)$	$P_{51}(1,\delta^5,\delta^6)$	$P_{52}(1,\delta^6,\delta^2)$	$P_{53}(1,1,1)$	$P_{54}(1,0,\delta^3)$
$P_{55}(1,\delta,0)$	$P_{56}(0,1,\delta)$	$P_{57}(1,\delta^3,\delta^2)$	$P_{58}(1,1,\delta^4)$	$P_{59}(1,\delta^4,\delta^6)$	$P_{60}(1,\delta^6,\delta)$
$P_{61}(1,\delta^5,\delta)$	$P_{62}(1,\delta^5,1)$	$P_{63}(1,0,\delta)$	$P_{64}(1,\delta^5,0)$	$P_{65}(0,1,\delta^5)$	$P_{66}(1,\delta^3,\delta^5)$
$P_{67}(1,\delta^2,\delta)$	$P_{68}(1,\delta^5,\delta^4)$	$P_{69}(1,\delta^4,\delta^4)$	$P_{70}(1,\delta^4,\delta^3)$	$P_{71}(1,\delta,\delta^4)$	$P_{72}(1,\delta^4,1)$
$P_{73}(1,0,1)$					

Table 2. Lines of $GF(8)$

ℓ_1	P_1	P_2	P_4	P_8	P_{16}	P_{32}	P_{37}	P_{55}	P_{64}
ℓ_2	P_2	P_3	P_5	P_9	P_{17}	P_{33}	P_{38}	P_{56}	P_{65}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ℓ_{73}	P_{73}	P_1	P_3	P_7	P_{15}	P_{31}	P_{36}	P_{54}	P_{63}

4. Classification of (k,n)-arcs in $GF(q)$.

To find the classification of (k,n)-arcs in $GF(q)$ need to use the method which based on the type of i-secant distribution and is used to find the large complete (k,n)-arcs. To explain this method, the classification of $(k,4)$-arcs in $GF(8)$ is used The Equations (2.1), (2.2) and (2.3) of Lemma 2.1 are used here.

4.1 The construction of the distinct (4,4)-arcs

Let $\mu = \{1,2,4,37\}$ be a (4,4)-arcs in $GF(8)$. A (4,4)-arc has the same type of i-secant distribution as $\mathcal{A}$. Therefore, there is only one (4,4)-arc in $GF(8)$ based on the type of i-secant distribution. This can be calculated from the following equations:

$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 73,$
$\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 = 36,$
$\beta_2 + 3\beta_3 + 6\beta_4 = 6.$

Since $\beta_4 = 1,\ \beta_3 = 0,\ \beta_2 = 0$, so the only type of (4,4)-arc is (1,0,0,32,40).

4.2 The construction of the distinct (5,4)-arcs

From Section 4.1, there is only one (4,4)-arc $\mathcal{A}$, and there are 64 pints of index zero which do not lie on 4-secant of $\mathcal{A}$. So, by adding one point of the points of index zero to $\mathcal{A}$, then there is only one type of (5,4)-arc denoted by $\mathcal{B}$, satisfying the following:

$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 73,$
$\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 = 45,$
$\beta_2 + 3\beta_3 + 6\beta_4 = 10.$

Since $\beta_4 = 1,\ \beta_3 = 0$, so the only type of (5,4)-arc is (1,0,4,33,35).

4.3 The construction of the distinct (6,4)-arcs

From Section 4.2, there is only one $(5,4)$-arc $\mathcal{B}$, and there are 63 pints of index zero. So, by adding one point of the points of index zero to $\mathcal{B}$, two distinct (6,4)-arcs $\mathcal{C}_1$ and $\mathcal{C}_2$ are obtained. Where $\mathcal{C}_1$ is of type (1,0,9,14,7) and $\mathcal{C}_2$ is of type (1,1,6,17,6).

4.4 The construction of the distinct (k,4)-arcs, $k = 7, 8, \cdots, 28$

Table (3) illustrates the number of i-secant distribution of $(k,4)$-arcs in $GF(8)$. Here, γ is the number of distinct $(k,4)$-arcs according to i-secant distribution.

Table 3. The i-secant distribution of $(k,4)$-arcs in $GF(8)$

γ	(k,4)-arcs	γ	(k,4)-arcs	γ	(k,4)-arcs	γ	(k,4)-arcs
9	(7,4)-arcs	108	(13,4)-arcs	94	(19,4)-arcs	2	(25,4)-arcs
20	(8,4)-arcs	118	(14,4)-arcs	71	(20,4)-arcs	1	(26,4)-arcs
32	(9,4)-arcs	128	(15,4)-arcs	45	(21,4)-arcs	1	(27,4)-arcs

52	(10,4)-arcs	130	(16,4)-arcs	32	(22,4)-arcs	1	(28,4)-arcs
75	(11,4)-arcs	127	(17,4)-arcs	15	(23,4)-arcs		
95	(12,4)-arcs	119	(18,4)-arcs	6	(24,4)-arcs		

5. Conclusion

In this paper, the classification of (k,4)-arcs in $GF(8)$ is calculated and $m_{28}(2,8)$ is 28 and the only type of (28,4)-arc is (63,0,0,0,10).

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مجلة كلية العراق الجامعة للهندسة والعلوم التطبيقية

تصنيف الاقواس الرباعي في المستوي الاسقاطي من الرتبة الثامنة بالاعتماد على توزيع عدد القواطع I

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الملخص. يتالف المستوي الاسقاطي من مجموعة نقاط عددها q^2+q+1 ومجموعة مستقيمات عددها q^2+q+1 حيث يقع q +1 من النقاط على كل مستقيم، وفي كل نقطة يمر $q+1$ من المستقيمات. يعرف القوس (k,n) على انه مجموعة k من النقاط بحيث n فقط تكون على استقامة واحدة ولا يوجد $n+1$ على استقامة واحدة. يقال عن القوس (k,n) على انه قوس تام اذا لم يوجد قوس $(k+1,n)$ يحويه. في هذا البحث تم ايجاد تصنيف الاقواس $(k,4)$ في المستوي الاسقاطي من الرتبة 8 بالاعتماد على عدد القواطع (i-secants)