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(p, q) -Chebyshev polynomials for the families of biunivalent function associating a new integral operator with (p, q) -Hurwitz zeta function

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Abstract: In the present article, making use of the (p, q) -Hurwitz zeta function, we provide and investigate a new integral operator. Also, we define two families $\mathcal{SM}_{p,q}(\xi, \zeta, \delta, u, \tau)$ and $\mathcal{SC}_{p,q}(\lambda, \zeta, \vartheta, u, \tau)$ of biunivalent and holomorphic functions in the unit disc connected with (p, q) -Chebyshev Polynomials. Then we find coefficient estimates $|a_2|$ and $|a_3|$. Finally, we obtain Fekete-Szegő inequalities for these families.

Key words: Biunivalent function, (p, q) -Chebyshev polynomial, (p, q) -Hurwitz zeta function, a new integral operator, coefficient estimates, and Fekete-Szegő inequality

1. Introduction

Let Ω be the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (1.1)$$

where the class Ω is analytic in the open unit disk $\Sigma = \{z : |z| < 1\}$. We also define to \mathcal{P} as the class of all Ω functions that are univalent in Σ .

By the *Koebe One Quarter Theorem* [17] it is clear that the function $f \in \mathcal{P}$ has an inverse function f^{-1} , which is defined by

$$f^{-1}(f(z)) = z \quad \text{and} \quad f(f^{-1}(\tilde{\omega})) = \eta, \quad (|\tilde{\omega}| < r_0(f); r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(\tilde{\omega}) = \tilde{\omega} + \sum_{n=2}^{\infty} g_n \tilde{\omega}^n.$$

It is clear that

$$\tilde{\omega} = f(f^{-1}(\tilde{\omega})) = \tilde{\omega} + (g_2 + a_2)\tilde{\omega}^2 + (g_3 - 2a_2^2 + a_3)\tilde{\omega}^3 + (g_4 + 5a_2^3 - 5a_2a_3 + a_4)\tilde{\omega}^4 + \dots.$$

Assume that $g_2 = -a_2$, $g_3 = 2a_2^2 - a_3$ and $g_4 = -5a_2^3 + 5a_2a_3 - a_4$, we find that

$$f^{-1}(\tilde{\omega}) = \tilde{\omega} - a_2\tilde{\omega}^2 + (2a_2^2 - a_3)\tilde{\omega}^3 - (5a_2^3 - 5a_2a_3 + a_4)\tilde{\omega}^4 + \dots. \quad (1.2)$$

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If both f and f^{-1} are univalent in Ω , a function $f \in \Omega$ is said to be biunivalent in Ω . We denote the biunivalent function by Ξ . After that, many mathematicians have been interested in studying biunivalent functions and they have provided and investigated the bounds for the coefficients $|a_n|$ (see, [10, 11], [20, 21], [25], and [32]). The Fekete-Szegő inequality (problem), related to the Bieberbach conjecture, is an inequality for the coefficients of univalent analytic functions $f(z)$ in (1.1) discovered by Fekete and Szegő [19]. The Fekete-Szegő problem is the search for the estimate of the maximum value of the coefficient functional $|a_3 - \gamma a_2^2|$. Many authors have presented many subclasses of biunivalent analytic functions and examined applications of Fekete-Szegő inequality, such as Amourah et al. [4], Arikan et al. [6], Darus and Thomas [13], Deniz and Orhan [16], Yousef et al. ([43] and [44]), and Srivastava et al. ([34]–[38]).

The function $f(z)$ is called subordinate to $h(z)$, denoted by $f(z) \prec h(z)$; if $f(z)$ and $h(z)$ are two analytic functions in Ω and there is a Schwarz function φ with $\varphi(z) = 0$, $|\varphi(z)| < 1$ and $f(z) = h(\varphi(z))$. In addition, we get the following equivalence if the function h is univalent in Ω

$$f(z) \prec h(z) \Leftrightarrow f(0) = h(0) \text{ and } f(\Omega) \subset h(\Omega).$$

For any $j \geq 2$ and $0 < q < p \leq 1$, The second type of (p, q) -Chebyshev polynomial is known as the following relationship:

$$\mathcal{F}_j(x, \alpha, p, q) = (p^j + q^j) x \mathcal{F}_{j-1}(x, \alpha, p, q) + (pq)^{j-1} \alpha \mathcal{F}_{j-2}(x, \alpha, p, q) \tag{1.3}$$

with α is a variable and the initial values $\mathcal{F}_0(x, \alpha, p, q) = 1$ and $\mathcal{F}_1(x, \alpha, p, q) = (p + q)x$.

Depending on the values of p, q, s , and x , we get special polynomials from (p, q) -Chebyshev polynomial such as (Fibonacci polynomials, Pell polynomials, Jacobsthal polynomials, and Chebyshev polynomial of the second type), it was presented by many researchers (see, [1], [3, 4], [10, 11], and [18]).

In 2019, Kizilates et al. [27] provided and investigated the first and second types of (p, q) -Chebyshev polynomial and obtained derived formulas, generating functions, and some important results of this polynomial.

The generating function of the second type is defined as follows.

$$\mathcal{C}_{p,q}(z) = \frac{1}{1 - xpz\eta_p - xqz\eta_q - \alpha pqz^2\eta_{p,q}} = \sum_{j=0}^{\infty} \mathcal{F}_j(x, \alpha, p, q) z^j \quad (z \in \Sigma). \tag{1.4}$$

Let $\aleph(z)$ and $\varphi(\tilde{\omega})$ be two analytic functions in the unit disk Ω with $\aleph(0) = \varphi(0) = 0$, $|\aleph(z)| < 1, |\varphi(\tilde{\omega})| < 1$, and

$$\aleph(z) = \sigma_1 z + \sigma_2 z^2 + \sigma_3 z^3 + \dots, \text{ and } \varphi(\tilde{\omega}) = l_1 \tilde{\omega} + l_2 \tilde{\omega}^2 + l_3 \tilde{\omega}^3 + \dots \quad (z, \tilde{\omega} \in \Sigma). \tag{1.5}$$

The (p, q) -calculus denotes the possibility of extending the q -calculus to postquantum calculus. Chakrabarti and Jagannathan [12] proposed the (p, q) -calculus in quantum algebras to generalize the q -series, which has numerous applications in science and engineering. After that, many articles provided and investigated the application of (p, q) -calculus (see; [2], [7], [22], [23], [29], and [33]).

The (p, q) -derivative operator is a kind of derivative operator defined by

$$d_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p - q)z} \quad (z \neq 0, 0 < q < p < 1, \text{ and } p \neq q).$$

It is clear that for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ and $z \in \Sigma$

$$d_{pq} \left\{ \sum_{j=1}^{\infty} a_j z^j \right\} = \sum_{j=1}^{\infty} [j]_{pq} z^{j-1},$$

where $[j]_{p,q} = \frac{p^j - q^j}{p - q} = p^{j-1} + p^{j-2}q + p^{j-3}q^2 + \dots + pq^{j-2} + q^{j-1}$ and $[0]_{p,q} = 0$.

This is a natural extension of the q -number ([24]),

$$\lim_{p \rightarrow 1} [j]_{p,q} = [j]_q = \frac{1 - q^j}{1 - q}, q \neq 1.$$

We introduce the p, q -Hurwitz zeta function, which is a generalization of q -Hurwitz zeta function (see, [30], [31], and [40]), motivated by the work cited above.

In the beginning, we define the (p, q) -Hurwitz zeta function $\zeta_{p,q}(u, \tau; z)$ by the following form

$$\zeta_{p,q}(u, \tau; z) = \sum_{j=0}^{\infty} \frac{z^j}{[j + u]_{p,q}^{\tau}},$$

where $u \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tau \in \mathbb{C}$, when $|z| < 1$, and $\text{Re}(\tau) > 1$ when $|z| = 1$.

Now, by the functions $f(z)$ in (1.1), we present the (p, q) -Srivastava-Attiya operator $\mathcal{J}_{p,q,\tau}^u f(z) : \Omega \rightarrow \Omega$ as follows

$$\mathcal{J}_{p,q,\tau}^u f(z) = (\Phi_{\tau}^u(p, q; z) * f(z)) (z \in \Sigma, u \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tau \in \mathbb{C}), \tag{1.6}$$

where

$$\Phi_{\tau}^u(p, q; z) = [1 + u]_{pq}^{\tau} \left[\zeta_{p,q}(u, \tau; z) - (u)_{pq}^{-\tau} \right]. \tag{1.7}$$

From (1.6) and (1.7), we note that

$$\mathcal{J}_{p,q,\tau}^u f(z) = z + \sum_{j=2}^{\infty} \left(\frac{[1 + u]_{pq}}{[j + u]_{pq}} \right)^{\tau} a_j z^j. \tag{1.8}$$

From (1.8), we observe that

1. When $p = 1$, we get a q -Srivastava-Attiya operator [39].

$$\mathcal{J}_{q,\tau}^b f(z) = z + \sum_{n=2}^{\infty} \left(\frac{[1 + u]_q}{[n + u]_q} \right)^{\tau} a_n z^n.$$

2. When $p = 1$ and $q \rightarrow 1$, we get a Srivastava-Attiya operator [34].

$$\mathcal{J}_{1s,\tau}^u f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1 + u}{j + u} \right)^{\tau} a_j z^j.$$

The aim of this work is to define a (p, q) -integral operator by using (p, q) -Hurwitz zeta function, which is a generalization of q -Srivastava-Attiya integral operator. After that, we determine initial coefficient bounds for some classes of analytic functions defined by subordination. Finally, our results deal with the Fekete-Szegő problem for (p, q) -Chebyshev polynomials.

2. New concepts and results

At the beginning, we introduce the concept of new families $\mathcal{SM}_{p,q}(\xi, \zeta, \delta, u, \tau)$ and $\mathcal{SC}_{p,q}(\lambda, \zeta, \vartheta, u, \tau)$.

Definition 2.1 For $0 \leq \xi \leq 1, \zeta \geq 0$ and $0 \leq \delta \leq 1$. A function $f \in \Xi$ is said to be in the family $\mathcal{SM}_{p,q}(\xi, \zeta, \delta, u, \tau)$ if the following subordinate conditions are satisfied:

$$(1 - \xi) \frac{z^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f(z))'}{[\mathcal{J}_{p,q,\tau}^u f(z)]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f(z))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f(z)}{z} \right)^\delta \prec \mathcal{C}_{p,q}(z) \tag{2.1}$$

and

$$(1 - \xi) \frac{\tilde{\omega}^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))'}{[\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})}{\tilde{\omega}} \right)^\delta \prec \mathcal{C}_{p,q}(\tilde{\omega}). \tag{2.2}$$

Remark 2.2 It is worth noting that the family $\mathcal{SM}_{p,q}(\xi, \zeta, \delta, u, \tau)$ contains many subfamilies, we mention them as follows:

1- We get the subfamily $\mathcal{SM}_{p,q}(\xi, \zeta, u, \tau)$, when $\delta = 0$; define as below

$$(1 - \xi) \frac{z^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f(z))'}{[\mathcal{J}_{p,q,\tau}^u f(z)]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f(z))' \prec \mathcal{C}_{p,q}(z),$$

and

$$(1 - \xi) \frac{\tilde{\omega}^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))'}{[\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' \prec \mathcal{C}_{p,q}(\tilde{\omega}).$$

2- We get the subfamily $\mathcal{SM}_{p,q}(\xi, \delta, u, \tau)$, when $\zeta = 1$; define as below

$$(1 - \xi) (\mathcal{J}_{p,q,\tau}^u f(z))' + \xi (\mathcal{J}_{p,q,\tau}^u f(z))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f(z)}{z} \right)^\delta \prec \mathcal{C}_{p,q}(z),$$

and

$$(1 - \xi) (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' + \xi (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})}{\tilde{\omega}} \right)^\delta \prec \mathcal{C}_{p,q}(\tilde{\omega}).$$

3- We get the subfamily $\mathcal{SM}_{p,q}(\xi, \zeta, u, \tau)$, when $\xi = 0$; define as below

$$(1 - \xi) \frac{z^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f(z))'}{[\mathcal{J}_{p,q,\tau}^u f(z)]^{1-\zeta}} \prec \mathcal{C}_{p,q}(z),$$

and

$$(1 - \xi) \frac{\tilde{\omega}^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))'}{[\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})]^{1-\zeta}} \prec \mathcal{C}_{p,q}(\tilde{\omega}).$$

Definition 2.3 For $\lambda \in \mathbb{C} \setminus \{0\}$, $\zeta \geq 0$ and $\vartheta \geq 0$. A function $f \in \Xi$ is said to be in the family $\mathcal{SC}_{p,q}(\lambda, \zeta, \vartheta, u, \tau)$ if satisfies the following subordinate conditions:

$$\frac{1}{\lambda} \left(\frac{(1 - \zeta + 2\vartheta)\mathcal{J}_{p,q,\tau}^u f(z) + \vartheta z^2(\mathcal{J}_{p,q,\tau}^u f(z))''}{z} + (\zeta - 2\vartheta)(\mathcal{J}_{p,q,\tau}^u f(z))' - 1 \right) \prec \mathcal{C}_{p,q}(z), \quad (2.3)$$

and

$$\frac{1}{\lambda} \left(\frac{(1 - \zeta + 2\vartheta)\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}) + \vartheta \tilde{\omega}^2(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))''}{\tilde{\omega}} + (\zeta - 2\vartheta)(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' - 1 \right) \prec \mathcal{C}_{p,q}(\tilde{\omega}). \quad (2.4)$$

Remark 2.4 It is worth noting that the family $\mathcal{SC}_{p,q}(\lambda, \zeta, \vartheta, u, \tau)$ contains many subfamilies, we mention them as follows:

1- We get the subfamily $\mathcal{SC}_{p,q}(\lambda, \vartheta, u, \tau)$, when $\zeta = 1 + 2\vartheta$;

$$\frac{1}{\lambda} \left(\vartheta z(\mathcal{J}_{p,q,\tau}^u f(z))'' + (\mathcal{J}_{p,q,\tau}^u f(z))' - 1 \right) \prec \mathcal{C}_{p,q}(z),$$

and

$$\frac{1}{\lambda} \left(\vartheta \tilde{\omega}(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))'' + (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' - 1 \right) \prec \mathcal{C}_{p,q}(\tilde{\omega}).$$

2- We get the subfamily $\mathcal{SC}_{p,q}(\lambda, \zeta, u, \tau)$, when $\vartheta = 0$;

$$\frac{1}{\lambda} \left((1 - \zeta) \frac{\mathcal{J}_{p,q,\tau}^u f(z)}{z} + \zeta(\mathcal{J}_{p,q,\tau}^u f(z))' - 1 \right) \prec \mathcal{C}_{p,q}(z),$$

and

$$\frac{1}{\lambda} \left((1 - \zeta) \frac{\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})}{\tilde{\omega}} + \zeta(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' - 1 \right) \prec \mathcal{C}_{p,q}(\tilde{\omega}).$$

3- We get the subfamily $\mathcal{SC}_{p,q}(\lambda, u, \tau)$, when $\zeta = 1$ and $\vartheta = 0$;

$$\frac{1}{\lambda} \left((\mathcal{J}_{p,q,\tau}^u f(z))' - 1 \right) \prec \mathcal{C}_{p,q}(z),$$

and

$$\frac{1}{\lambda} \left((\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' - 1 \right) \prec \mathcal{C}_{p,q}(\tilde{\omega}).$$

Theorem 2.5 Let $f(z) \in \Omega$ be in the family $\mathcal{SM}_{p,q}(\xi, \zeta, \delta, u, \tau)$, then

$$|a_2| \leq \frac{|(p+q)x|(2+u)_{pq}^\tau \sqrt{(p+q)x(3+u)_{pq}^\tau}}{\sqrt{2[\mathcal{H}_{p,q,x}(\xi, \zeta, \delta, u, \tau) + \mathcal{N}_{p,q,x}(\xi, \zeta, \delta, u, \tau) - \mathcal{Q}_{p,q,x}(\xi, \zeta, \delta, u, \tau)]}}, \quad (2.5)$$

and

$$|a_3| \leq \frac{|(p+q)x|(3+u)_{pq}^\tau}{[(1-\xi)(\zeta+1) + \xi\delta + 2](1+u)_{pq}^\tau} + \frac{(p+q)^2 x^2 (2+u)_{pq}^{2\tau}}{[(1-\xi)(\zeta+1) + \xi\delta + 1]^2 (1+u)_{pq}^{2\tau}}, \quad (2.6)$$

where $\mathcal{H}_{p,q,x}(\xi, \zeta, \delta, u, \tau) = (p+q)^2 x^2 ((1-\xi)(\zeta+1) + \xi\delta + 2)(1+u)_{pq}^\tau (2+u)_{pq}^{2\tau}$,

$\mathcal{N}_{p,q,x}(\xi, \zeta, \delta, u, \tau) = (p+q)^2 x^2 (\frac{1}{2}(\zeta - \xi\zeta)(\zeta + 3) + \xi(\delta(\delta + 1) - 2))(3+u)_{pq}^\tau (1+u)_{pq}^{2\tau}$,

and $\mathcal{Q}_{p,q,x}(\xi, \zeta, \delta, u, \tau) = (p^2 + q^2)(p+q)x^2 [(1-\xi)(\zeta+1) + \xi\delta + 1]^2 (3+u)_{pq}^\tau (1+u)_{pq}^{2\tau}$.

Proof Suppose that $f(z) \in \mathcal{SM}_{p,q}(\xi, \zeta, \delta, u, \tau)$, there are two analytic functions \aleph, φ as defined in (1.5), such that

$$(1 - \xi) \frac{z^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f(z))'}{[\mathcal{J}_{p,q,\tau}^u f(z)]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f(z))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f(z)}{z} \right)^\delta \prec \mathcal{C}_{p,q}(\aleph(z)), \tag{2.7}$$

and

$$(1 - \xi) \frac{\tilde{\omega}^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))'}{[\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})}{\tilde{\omega}} \right)^\delta \prec \mathcal{C}_{p,q}(\varphi(\tilde{\omega})). \tag{2.8}$$

From (2.7) and (2.8), we have

$$(1 - \xi) \frac{z^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f(z))'}{[\mathcal{J}_{p,q,\tau}^u f(z)]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f(z))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f(z)}{z} \right)^\delta = 1 + \mathcal{F}_1(x, \alpha, p, q) \aleph(z) + \mathcal{F}_2(x, \alpha, p, q) (\aleph(z))^2 + \dots \tag{2.9}$$

and

$$(1 - \xi) \frac{\tilde{\omega}^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))'}{[\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})}{\tilde{\omega}} \right)^\delta = 1 + \mathcal{F}_1(x, \alpha, p, q) \varphi(\tilde{\omega}) + \mathcal{F}_2(x, \alpha, p, q) (\varphi(\tilde{\omega}))^2 + \dots \tag{2.10}$$

Comparing Equations (1.5) and (2.7)–(2.10), we get the following relationship

$$(1 - \xi) \frac{z^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f(z))'}{[\mathcal{J}_{p,q,\tau}^u f(z)]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f(z))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f(z)}{z} \right)^\delta = 1 + \mathcal{F}_1(x, \alpha, p, q) \sigma_1 z + [\mathcal{F}_1(x, s, p, q) \sigma_2 + \mathcal{F}_2(x, s, p, q) \sigma_1^2] z^2 + \dots \tag{2.11}$$

and

$$(1 - \xi) \frac{\tilde{\omega}^{1-\zeta} (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))'}{[\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})]^{1-\zeta}} + \xi (\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' \left(\frac{\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega})}{\tilde{\omega}} \right)^\delta = 1 + \mathcal{F}_1(x, \alpha, p, q) l_1 \tilde{\omega} + [\mathcal{F}_1(x, s, p, q) l_2 + \mathcal{F}_2(x, s, p, q) l_1^2] \tilde{\omega}^2 + \dots \tag{2.12}$$

Since $|\aleph(z)| < 1, |\varphi(\tilde{\omega})| < 1$, then

$$|\sigma_j| < 1 \text{ and } |l_j| < 1 \text{ for all } j \in \mathbb{N}. \tag{2.13}$$

By calculating the right-hand side of two conditions in (2.11) and (2.12), and comparing the coefficients, we get

$$[(1 - \xi) (\zeta + 1) + \xi \delta + 1] \frac{(1 + u)_{pq}^\tau}{(2 + u)_{pq}^\tau} a_2 = \mathcal{F}_1(x, \alpha, p, q) \sigma_1 \tag{2.14}$$

$$[(1 - \xi) (\zeta + 1) + \xi \delta + 2] \frac{(1 + u)_{pq}^\tau}{(3 + u)_{pq}^\tau} a_3 + \frac{[(\zeta - \xi \zeta)(\zeta + 3) + \xi(\delta(\delta + 1) - 2)](1 + u)_{pq}^{2\tau}}{2(2 + u)_{pq}^{2\tau}} a_2^2 = \mathcal{F}_1(x, s, p, q) \sigma_2 + \mathcal{F}_2(x, s, p, q) \sigma_1^2 \tag{2.15}$$

$$-[(1 - \xi)(\zeta + 1) + \xi\delta + 1] \frac{(1 + u)_{pq}^\tau}{(2 + u)_{pq}^\tau} a_2 = \mathcal{F}_1(x, \alpha, p, q) l_1 \tag{2.16}$$

and

$$\begin{aligned} [(1 - \xi)(\zeta + 1) + \xi\delta + 2] \frac{(1 + u)_{pq}^\tau}{(3 + u)_{pq}^\tau} (2a_2^2 - a_3) + \frac{[(\zeta - \xi\zeta)(\zeta + 3) + \xi(\delta(\delta + 1) - 2)](1 + u)_{pq}^{2\tau}}{2(2 + u)_{pq}^{2\tau}} a_2^2 \\ = \mathcal{F}_1(x, s, p, q) l_2 + \mathcal{F}_2(x, s, p, q) l_1^2. \end{aligned} \tag{2.17}$$

From (2.14) and (2.16), we have

$$\sigma_1 = -l_1 \tag{2.18}$$

and

$$\frac{2[(1 - \xi)(\zeta + 1) + \xi\delta + 1]^2 (1 + u)_{pq}^{2\tau}}{(2 + u)_{pq}^{2\tau}} a_2^2 = \mathcal{F}_1^2(x, \alpha, p, q) (\sigma_1^2 + l_1^2). \tag{2.19}$$

Now, by adding (2.15) to (2.17), we obtain

$$\begin{aligned} [2((1 - \xi)(\zeta + 1) + \xi\delta + 2) \frac{(1 + u)_{pq}^\tau}{(3 + u)_{pq}^\tau} + \frac{2[\frac{1}{2}(\zeta - \xi\zeta)(\zeta + 3) + \xi(\delta(\delta + 1) - 2)](1 + u)_{pq}^{2\tau}}{(2 + u)_{pq}^{2\tau}}] a_2^2 \\ = \mathcal{F}_1(x, s, p, q) (\sigma_2 + l_2) + \mathcal{F}_2(x, s, p, q) (\sigma_1^2 + l_1^2). \end{aligned} \tag{2.20}$$

By (2.19) we substitute the value of $(\sigma_1^2 + l_1^2)$ into Equation (2.20), we get

$$\begin{aligned} [2\mathcal{F}_1^2(x, \alpha, p, q) ((1 - \xi)(\zeta + 1) + \xi\delta + 2) (1 + u)_{pq}^\tau (2 + u)_{pq}^{2\tau} \\ + 2\mathcal{F}_1^2(x, \alpha, p, q) (\frac{1}{2}(\zeta - \xi\zeta)(\zeta + 3) + \xi(\delta(\delta + 1) - 2)) (3 + u)_{pq}^\tau (1 + u)_{pq}^{2\tau} \\ - 2\mathcal{F}_2(x, s, p, q) [(1 - \xi)(\zeta + 1) + \xi\delta + 1]^2 (3 + u)_{pq}^\tau (1 + u)_{pq}^{2\tau}] a_2^2 \\ = \mathcal{F}_1^3(x, \alpha, p, q) (2k + u)^{2\tau} (3k + u)^\tau (\sigma_2 + l_2). \end{aligned} \tag{2.21}$$

From (1.3), (2.13), and (2.21), we have

$$|a_2| \leq \frac{|(p + q)x| (2 + u)_{pq}^\tau \sqrt{(p + q)x(3 + u)_{pq}^\tau}}{\sqrt{2[\mathcal{H}_{p,q,x}(\xi, \zeta, \delta, u, \tau) + \mathcal{N}_{p,q,x}(\xi, \zeta, \delta, u, \tau) - \mathcal{Q}_{p,q,x}(\xi, \zeta, \delta, u, \tau)]}}.$$

Now, to get the bound on $|a_3|$, using (2.18) and subtracting (2.17) from (2.15)

$$2[(1 - \xi)(\zeta + 1) + \xi\delta + 2] \frac{(1 + u)_{pq}^\tau}{(3 + u)_{pq}^\tau} (a_3 - a_2^2) = \mathcal{F}_1(x, s, p, q) (\sigma_2 - l_2) + \mathcal{F}_2(x, s, p, q) (\sigma_1^2 - l_1^2). \tag{2.22}$$

According to (2.18) and (2.22), we get

$$a_3 = \frac{\mathcal{F}_1(x, s, p, q) (3 + u)_{pq}^\tau (\sigma_2 - l_2)}{2[(1 - \xi)(\zeta + 1) + \xi\delta + 2] (1 + u)_{pq}^\tau} + \frac{\mathcal{F}_1^2(x, \alpha, p, q) (2 + u)_{pq}^{2\tau} (\sigma_1^2 + l_1^2)}{2[(1 - \xi)(\zeta + 1) + \xi\delta + 1]^2 (1 + u)_{pq}^{2\tau}}.$$

Then

$$|a_3| \leq \frac{|(p+q)x|(3+u)_{pq}^\tau}{[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^\tau} + \frac{(p+q)^2x^2(2+u)_{pq}^{2\tau}}{[(1-\xi)(\zeta+1)+\xi\delta+1]^2(1+u)_{pq}^{2\tau}}.$$

The proof is complete. □

For the special case $\delta = 0$, Theorem 2.5 becomes:

Corollary 2.6 *Let $f(z) \in \Omega$ be in the family $\mathcal{SM}_{p,q}(\xi, \zeta, u, \tau)$, then*

$$|a_2| \leq \frac{|(p+q)x|(2+u)_{pq}^\tau \sqrt{(p+q)x(3+u)_{pq}^\tau}}{\sqrt{2} \left[\tilde{N}_{p,q,x}(\xi, \zeta, u, \tau) + \tilde{K}_{p,q,x}(\xi, \zeta, u, \tau) - \tilde{L}_{p,q,x}(\xi, \zeta, u, \tau) \right]},$$

$$|a_3| \leq \frac{|(p+q)x|(3+u)_{pq}^\tau}{[\zeta(1-\xi)+\xi+3](1+u)_{pq}^\tau} + \frac{(p+q)^2x^2(2+u)_{pq}^{2\tau}}{[\zeta(1-\xi)+\xi+2]^2(1+u)_{pq}^{2\tau}},$$

where $\tilde{N}_{p,q,x}(\xi, \zeta, u, \tau) = (p+q)^2x^2(\zeta(1-\xi)+\xi+3)(1+u)_{pq}^\tau(2+u)_{pq}^{2\tau}$,
 $\tilde{K}_{p,q,x}(\xi, \zeta, u, \tau) = (p+q)^2x^2\left(\frac{1}{2}(\zeta-\xi\zeta)(\zeta+3)+4\xi\right)(3+u)_{pq}^\tau(1+u)_{pq}^{2\tau}$, and
 $\tilde{L}_{p,q,x}(\xi, \zeta, u, \tau) = (p^2+q^2)(p+q)x^2(\zeta(1-\xi)+\xi+2)^2(3+u)_{pq}^\tau(1+u)_{pq}^{2\tau}$.

For the special case $\zeta = 1$, Theorem 2.5 becomes:

Corollary 2.7 *Let $f(z) \in \Omega$ be in the family $\mathcal{SM}_{p,q}(\xi, \delta, u, \tau)$, then*

$$|a_2| \leq \frac{|(p+q)x|(2+u)_{pq}^\tau \sqrt{(p+q)x(3+u)_{pq}^\tau}}{\sqrt{2} \left[\tilde{V}_{p,q,x}(\xi, \delta, u, \tau) + \tilde{U}_{p,q,x}(\xi, \delta, u, \tau) - \tilde{O}_{p,q,x}(\xi, \delta, u, \tau) \right]},$$

$$|a_3| \leq \frac{|(p+q)x|(3+u)_{pq}^\tau}{(\xi(1-\delta)+3)(1+u)_{pq}^\tau} + \frac{(p+q)^2x^2(2+u)_{pq}^{2\tau}}{(\xi(\delta-1)+2)^2(1+u)_{pq}^{2\tau}},$$

where $\tilde{V}_{p,q,x}(\xi, \delta, u, \tau) = (p+q)^2x^2(\xi(1-\delta)+3)(1+u)_{pq}^\tau(2+u)_{pq}^{2\tau}$,
 $\tilde{U}_{p,q,x}(\xi, \delta, u, \tau) = (p+q)^2x^2\left(\frac{1}{2}(\xi(\delta(\delta+1)-2))\right)(3+u)_{pq}^\tau(1+u)_{pq}^{2\tau}$, and
 $\tilde{O}_{p,q,x}(\xi, \delta, u, \tau) = (p^2+q^2)(p+q)x^2(\xi(\delta-1)+2)^2(3+u)_{pq}^\tau(1+u)_{pq}^{2\tau}$.

Theorem 2.8 *Let $f(z) \in \Omega$ be in the family $\mathcal{SC}_{p,q}(\lambda, \zeta, \vartheta, u, \tau)$, then*

$$|a_2| \leq \frac{|(p+q)x\lambda|(2+u)_{pq}^\tau \sqrt{(p+q)x(3+u)_{pq}^\tau}}{\sqrt{2} [Q_{p,q,x}(\lambda, \zeta, \vartheta, k, u, \tau) - M_{p,q,x}(\lambda, \zeta, \vartheta, k, u, \tau)]},$$

and

$$|a_3| \leq |a_2|^2 + \frac{(p+q)x(3+u)_{pq}^\tau}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau},$$

where $Q_{p,q,x}(\lambda, \zeta, \vartheta, u, \tau) = \lambda(1 + 2\zeta + 2\vartheta)(1 + u)_{pq}^\tau(2 + u)_{pq}^{2\tau}(p + q)^2x^2$
 and $M_{p,q,x}(\lambda, \zeta, \vartheta, u, \tau) = (p^2 + q^2)(p + q)x^2(1 + \zeta)^2(3 + u)_{pq}^\tau(1 + u)_{pq}^{2\tau}$.

Proof Suppose that $f(z) \in \mathcal{SC}_{p,q}(\lambda, \zeta, \vartheta, u, \tau)$. There are two analytic functions \aleph, φ as defined in (1.5), such that

$$\frac{1}{\lambda} \left(\frac{(1 - \zeta + 2\vartheta)\mathcal{J}_{p,q,\tau}^u f(z) + \vartheta z^2(\mathcal{J}_{p,q,\tau}^u f(z))''}{z} + (\zeta - 2\vartheta)(\mathcal{J}_{p,q,\tau}^u f(z))' - 1 \right) \prec \mathcal{C}_{p,q}(\aleph(z)), \tag{2.23}$$

and

$$\frac{1}{\lambda} \left(\frac{(1 - \zeta + 2\vartheta)\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}) + \vartheta \tilde{\omega}^2(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))''}{\tilde{\omega}} + (\zeta - 2\vartheta)(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' - 1 \right) \prec \mathcal{C}_{p,q}(\varphi(\tilde{\omega})). \tag{2.24}$$

From (2.23) and (2.24), we have

$$\begin{aligned} \frac{1}{\lambda} \left(\frac{(1 - \zeta + 2\vartheta)\mathcal{J}_{p,q,\tau}^u f(z) + \vartheta z^2(\mathcal{J}_{p,q,\tau}^u f(z))''}{z} + (\zeta - 2\vartheta)(\mathcal{J}_{p,q,\tau}^u f(z))' - 1 \right) \\ = 1 + \mathcal{F}_1(x, \alpha, p, q)\aleph(z) + \mathcal{F}_2(x, \alpha, p, q)(\aleph(z))^2 + \dots \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} \frac{1}{\lambda} \left(\frac{(1 - \zeta + 2\vartheta)\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}) + \vartheta \tilde{\omega}^2(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))''}{\tilde{\omega}} + (\zeta - 2\vartheta)(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' - 1 \right) \\ = 1 + \mathcal{F}_1(x, \alpha, p, q)\varphi(\tilde{\omega}) + \mathcal{F}_2(x, \alpha, p, q)(\varphi(\tilde{\omega}))^2 + \dots \end{aligned} \tag{2.26}$$

Comparing Equations (1.5) and (2.23)–(2.26), we get the following relationship

$$\begin{aligned} \frac{1}{\lambda} \left(\frac{(1 - \zeta + 2\vartheta)\mathcal{J}_{p,q,\tau}^u f(z) + \vartheta z^2(\mathcal{J}_{p,q,\tau}^u f(z))''}{z} + (\zeta - 2\vartheta)(\mathcal{J}_{p,q,\tau}^u f(z))' - 1 \right) \\ = 1 + \mathcal{F}_1(x, \alpha, p, q)\sigma_1 z + [\mathcal{F}_1(x, s, p, q)\sigma_2 + \mathcal{F}_2(x, s, p, q)\sigma_1^2]z^2 + \dots \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} \frac{1}{\lambda} \left(\frac{(1 - \zeta + 2\vartheta)\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}) + \vartheta \tilde{\omega}^2(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))''}{\tilde{\omega}} + (\zeta - 2\vartheta)(\mathcal{J}_{p,q,\tau}^u f^{-1}(\tilde{\omega}))' - 1 \right) \\ = 1 + \mathcal{F}_1(x, \alpha, p, q)l_1\tilde{\omega} + [\mathcal{F}_1(x, s, p, q)l_2 + \mathcal{F}_2(x, s, p, q)l_1^2]\tilde{\omega}^2 + \dots \end{aligned} \tag{2.28}$$

Since $|\aleph(z)| < 1, |\varphi(\tilde{\omega})| < 1$, then

$$|\sigma_j| < 1 \text{ and } |l_j| < 1 \text{ for all } j \in \mathbb{N}. \tag{2.29}$$

By calculating the right-hand side of two conditions in (2.27) and (2.28), and comparing the coefficients, we get

$$\frac{1}{\lambda}(1 + \zeta) \frac{(1 + u)_{pq}^\tau}{(2 + u)_{pq}^\tau} a_2 = \mathcal{F}_1(x, \alpha, p, q)\sigma_1, \tag{2.30}$$

$$\frac{1}{\lambda} (1 + 2\zeta + 2\vartheta) \frac{(1 + u)_{pq}^\tau}{(3 + u)_{pq}^\tau} a_3 = \mathcal{F}_1(x, s, p, q) \sigma_2 + \mathcal{F}_2(x, s, p, q) \sigma_1^2, \tag{2.31}$$

$$-\frac{1}{\lambda} (1 + \zeta) \frac{(1 + u)_{pq}^\tau}{(2 + u)_{pq}^\tau} a_2 = \mathcal{F}_1(x, \alpha, p, q) l_1, \tag{2.32}$$

and

$$\frac{1}{\lambda} (1 + 2\zeta + 2\vartheta) \frac{(1 + u)_{pq}^\tau}{(3 + u)_{pq}^\tau} (2a_2^2 - a_3) = \mathcal{F}_1(x, s, p, q) l_2 + \mathcal{F}_2(x, s, p, q) l_1^2. \tag{2.33}$$

From (2.30) and (2.32), we have

$$\sigma_1 = -l_1 \tag{2.34}$$

and

$$\frac{2}{\lambda^2} (1 + \zeta)^2 \frac{(1 + u)_{pq}^{2\tau}}{(2 + u)_{pq}^{2\tau}} a_2^2 = \mathcal{F}_1^2(x, \alpha, p, q) (\sigma_1^2 + l_1^2). \tag{2.35}$$

Now, by adding (2.31) to (2.33), we obtain

$$\frac{2}{\lambda} (1 + 2\zeta + 2\vartheta) \frac{(1 + u)_{pq}^\tau}{(3 + u)_{pq}^\tau} a_2^2 = \mathcal{F}_1(x, s, p, q) (\sigma_2 + l_2) + \mathcal{F}_2(x, s, p, q) (\sigma_1^2 + l_1^2). \tag{2.36}$$

By (2.35) we substitute the value of $(\sigma_1^2 + l_1^2)$ into Equation (2.36), we get

$$\begin{aligned} 2 \left[\lambda (1 + 2\zeta + 2\vartheta) (1 + u)_{pq}^\tau (2 + u)_{pq}^{2\tau} \mathcal{F}_1^2(x, \alpha, p, q) - \mathcal{F}_2(x, s, p, q) (1 + \zeta)^2 (3 + u)_{pq}^\tau (1 + u)_{pq}^{2\tau} \right] a_2^2 \\ = \mathcal{F}_1^3(x, \alpha, p, q) \lambda^2 (2k + u)^{2\tau} (3k + u)^\tau (\sigma_2 + l_2). \end{aligned} \tag{2.37}$$

From (1.3), (2.29) and (2.37), we have

$$|a_2| \leq \frac{|(p + q)x\lambda| (2 + u)_{pq}^\tau \sqrt{(p + q)x(3 + u)_{pq}^\tau}}{\sqrt{2 [Q_{p,q,x}(\lambda, \zeta, \vartheta, u, \tau) - M_{p,q,x}(\lambda, \zeta, \vartheta, u, \tau)]}}.$$

Now, to get the bound on $|a_3|$, using (2.34) and subtracting (2.33) from (2.31)

$$\frac{2}{\lambda} (1 + 2\zeta + 2\vartheta) (1 + u)_{pq}^\tau a_3 = \frac{2}{\lambda} (1 + 2\zeta + 2\vartheta) (1 + u)_{pq}^\tau a_2^2 + (3 + u)_{pq}^\tau \mathcal{F}_1(x, s, p, q) (\sigma_2 - l_2). \tag{2.38}$$

According to (2.34) and (2.38), we get

$$\frac{2}{\lambda} (1 + 2\zeta + 2\vartheta) (1 + u)_{pq}^\tau |a_3| \leq \frac{2}{\lambda} (1 + 2\zeta + 2\vartheta) (1 + u)_{pq}^\tau |a_2|^2 + 2(3 + u)_{pq}^\tau \mathcal{F}_1(x, s, p, q).$$

Then

$$|a_3| \leq |a_2|^2 + \frac{|(p + q)x| (3 + u)_{pq}^\tau}{\lambda (1 + 2\zeta + 2\vartheta) (1 + u)_{pq}^\tau}.$$

Here, the proof is complete. □

For the special case $\zeta = 1 + 2\vartheta$, Theorem 2.8 becomes:

Corollary 2.9 Let $f(z) \in \Omega$ be in the family $\mathcal{SC}_{p,q}(\lambda, \vartheta, u, \tau)$, then

$$|a_2| \leq \frac{|(p+q)x\lambda|(2+u)_{pq}^\tau \sqrt{(p+q)x(3+u)_{pq}^\tau}}{\sqrt{2 \left[\tilde{Q}_{p,q,x}(\lambda, \vartheta, u, \tau) - \tilde{M}_{p,q,x}(\lambda, \vartheta, u, \tau) \right]}}$$

$$|a_3| \leq |a_2|^2 + \frac{|(p+q)x|(3+u)_{pq}^\tau}{\lambda(1+2\zeta)(1+u)_{pq}^\tau},$$

where $\tilde{Q}_{p,q,x}(\lambda, \vartheta, u, \tau) = 3\lambda(1+2\vartheta)(1+u)_{pq}^\tau(2+u)_{pq}^{2\tau}(p+q)^2x^2$
 and $\tilde{M}_{p,q,x}(\lambda, \vartheta, u, \tau) = 2(p^2+q^2)(p+q)x^2(1+\zeta)^2(3+u)_{pq}^\tau(1+u)_{pq}^{2\tau}$.

For the special case $\vartheta = 0$, Theorem 2.8 becomes:

Corollary 2.10 Let $f(z) \in \Omega$ be in the family $\mathcal{SC}_{p,q}(\lambda, \zeta, u, \tau)$, then

$$|a_2| \leq \frac{|(p+q)x\lambda|(2+u)_{pq}^\tau \sqrt{(p+q)x(3+u)_{pq}^\tau}}{\sqrt{2 \left[\tilde{D}_{p,q,x}(\lambda, \zeta, u, \tau) - \tilde{F}_{p,q,x}(\lambda, \zeta, u, \tau) \right]}}$$

$$|a_3| \leq |a_2|^2 + \frac{|(p+q)x|(3+u)_{pq}^\tau}{\xi(1+2\zeta)(1+u)_{pq}^\tau},$$

where $\tilde{D}_{p,q,x}(\lambda, \zeta, u, \tau) = \lambda(1+2\zeta)(1+u)_{pq}^\tau(2+u)_{pq}^{2\tau}(p+q)^2x^2$
 and $\tilde{F}_{p,q,x}(\lambda, \zeta, u, \tau) = (p^2+q^2)(p+q)x^2(1+\zeta)^2(3+u)_{pq}^\tau(1+u)_{pq}^{2\tau}$.

3. Fekete-Szegö inequality

Theorem 3.1 Let $f(z)$ be in the family $\mathcal{SM}_{p,q}(\xi, \zeta, \delta, u, \tau)$ and $\gamma \in \mathbb{R}$, then

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{|(p+q)x|(3+u)_{pq}^\tau}{[(1-\xi)(\zeta+1) + \xi\delta + 2](1+u)_{pq}^\tau}, & \text{if } |\gamma - 1| \leq \chi_1, \\ \frac{|(p+q)^3x^3|(2+u)_{pq}^{2\tau}(3+u)_{pq}^\tau|1-\gamma|}{|[(p+q)(\mathcal{A}(\xi, \zeta, \delta, u, \tau) + \mathcal{B}(\xi, \zeta, \delta, u, \tau)) - (p^2+q^2)\mathcal{G}(\xi, \zeta, \delta, u, \tau)](p+q)x^2 - pq\alpha\mathcal{G}(\xi, \zeta, \delta, u, \tau)|}, & \text{if } |\gamma - 1| \geq \chi_1 \end{cases}$$

where $\mathcal{A}(\xi, \zeta, \delta, u, \tau) = ((1-\xi)(\zeta+1) + \xi\delta + 2)(1+u)_{pq}^\tau(2+u)_{pq}^{2\tau}$,
 $\mathcal{B}(\xi, \zeta, \delta, u, \tau) = (\frac{1}{2}(\zeta - \xi\zeta)(\zeta+3) + \xi(\delta(\delta+1) - 2))(3+u)_{pq}^\tau(1+u)_{pq}^{2\tau}$,
 $\mathcal{G}(\xi, \zeta, \delta, u, \tau) = [(1-\xi)(\zeta+1) + \xi\delta + 1]^2(3+u)_{pq}^\tau(1+u)_{pq}^{2\tau}$,
 and $\chi_1 = \frac{[(p+q)(\mathcal{A}(\xi, \zeta, \delta, u, \tau) + \mathcal{B}(\xi, \zeta, \delta, u, \tau)) - (p^2+q^2)\mathcal{G}(\xi, \zeta, \delta, u, \tau)](p+q)x^2 - pq\alpha\mathcal{G}(\xi, \zeta, \delta, u, \tau)}{(p+q)^2x^2[(1-\xi)(\zeta+1) + \xi\delta + 2](2+u)_{pq}^{2\tau}(1+u)_{pq}^\tau}$.

Proof By Equations (2.21) and (2.22), it follows that

$$\begin{aligned} a_3 - \gamma a_2^2 &= \frac{\mathcal{F}_1(x, s, p, q) (3+u)_{pq}^\tau (\sigma_2 - l_2)}{2[(1-\xi)(\zeta+1) + \xi\delta + 2] (1+u)_{pq}^\tau} + (1-\gamma) a_2^2 \\ &= \frac{\mathcal{F}_1(x, s, p, q) (3+u)_{pq}^\tau (\sigma_2 - l_2)}{2[(1-\xi)(\zeta+1) + \xi\delta + 2] (1+u)_{pq}^\tau} + \\ &\quad \frac{\mathcal{F}_1^3(x, \alpha, p, q) (3+u)_{pq}^\tau (2+u)_{pq}^{2\tau} (\sigma_2 + l_2) (1-\gamma)}{\mathcal{F}_1^2(x, \alpha, p, q) (\mathcal{A}(\xi, \zeta, \delta, u, \tau) + \mathcal{B}(\xi, \zeta, \delta, u, \tau)) - \mathcal{F}_2(x, s, p, q) \mathcal{G}(\xi, \zeta, \delta, u, \tau)} \\ &= \mathcal{F}_1(x, \alpha, p, q) \left[\left(\tilde{\mathcal{G}}(\xi, \zeta, \delta, \gamma) + \frac{(3+u)_{pq}^\tau}{2[(1-\xi)(\zeta+1) + \xi\delta + 2] (1+u)_{pq}^\tau} \right) \sigma_2 + \left(\tilde{\mathcal{G}}(\xi, \zeta, \delta, \gamma) - \frac{(3+u)_{pq}^\tau}{2[(1-\xi)(\zeta+1) + \xi\delta + 2] (1+u)_{pq}^\tau} \right) l_2 \right], \end{aligned}$$

where $\tilde{\mathcal{G}}(\xi, \zeta, \delta, \gamma) = \frac{\mathcal{F}_1^2(x, \alpha, p, q) (3+u)_{pq}^\tau (2+u)_{pq}^{2\tau} (1-\gamma)}{\mathcal{F}_1^2(x, \alpha, p, q) (\mathcal{A}(\xi, \zeta, \delta, u, \tau) + \mathcal{B}(\xi, \zeta, \delta, u, \tau)) - \mathcal{F}_2(x, s, p, q) \mathcal{G}(\xi, \zeta, \delta, u, \tau)}$.

Thus, according to (p, q) -Chebyshev polynomials in (1.3), we conclude that

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{(p+q)x(3+u)_{pq}^\tau}{[(1-\xi)(\zeta+1) + \xi\delta + 2] (1+u)_{pq}^\tau}, & 0 \leq |\tilde{\mathcal{G}}(\xi, \zeta, \delta, \gamma)| \leq \frac{(3k+u)^\tau}{2[(1-\xi)(\zeta+1) + \xi\delta + 2] (1+u)_{pq}^\tau} \\ 2|(p+q)x| \left| \tilde{\mathcal{G}}(\xi, \zeta, \delta, \gamma) \right|, & \text{for } |\tilde{\mathcal{G}}(\xi, \zeta, \delta, \gamma)| \geq \frac{(3+u)_{pq}^\tau}{2[(1-\xi)(\zeta+1) + \xi\delta + 2] (1+u)_{pq}^\tau} \end{cases}$$

After making several simplifications, the proof of Theorem 3.1 is complete. □

Theorem 3.2 Let $f(z)$ be in the family $\mathcal{SC}_{p,q}(\lambda, \zeta, \vartheta, u, \tau)$ and $\gamma \in \mathbb{R}$, then

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{|(p+q)x|(3+u)_{pq}^\tau}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau} & \text{if } |\gamma - 1| \leq \chi_2, \\ \frac{|(p+q)^3 x^3| \lambda^2 (3+u)_{pq}^\tau (2+u)_{pq}^{2\tau} |1-\gamma|}{\left| [(p+q)\tilde{\mathcal{A}}(\lambda, \zeta, \delta, u, \tau) - (p^2+q^2)\tilde{\mathcal{B}}(\lambda, \zeta, \delta, u, \tau)](p+q)x^2 - pq\alpha\tilde{\mathcal{B}}(\lambda, \zeta, \delta, u, \tau) \right|}, & \text{if } |\gamma - 1| \geq \chi_2 \end{cases}$$

where $\tilde{\mathcal{A}}(\lambda, \zeta, \delta, u, \tau) = \lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau (2+u)_{pq}^{2\tau}$,

$\tilde{\mathcal{B}}(\lambda, \zeta, \delta, u, \tau) = (1+\zeta)^2 (3+u)_{pq}^\tau (1+u)_{pq}^{2\tau}$,

and $\chi_2 = \frac{[(p+q)\tilde{\mathcal{A}}(\lambda, \zeta, \delta, u, \tau) - (p^2+q^2)\tilde{\mathcal{B}}(\lambda, \zeta, \delta, u, \tau)](p+q)x^2 - pq\alpha\tilde{\mathcal{B}}(\lambda, \zeta, \delta, u, \tau)}{(p+q)^2 x^2 \lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau (2+u)_{pq}^{2\tau}}$.

Proof By Equations (2.37) and (2.38), it follows that

$$\begin{aligned} a_3 - \gamma a_2^2 &= \frac{(3+u)_{pq}^\tau \mathcal{F}_1(x, s, p, q) (\sigma_2 - l_2)}{2\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau} + (1-\gamma) a_2^2 \\ &= \frac{\mathcal{F}_1(x, \alpha, p, q)}{2} \left[\left(\Upsilon(\gamma, \zeta, \lambda, \vartheta) + \frac{(3+u)_{pq}^\tau}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau} \right) \sigma_2 \right] \end{aligned}$$

$$+ \left(\Upsilon(\gamma, \zeta, \lambda, \vartheta) - \frac{(3+u)_{pq}^\tau}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau} \right) l_2]$$

where

$$\Upsilon(\gamma, \zeta, \xi, \vartheta) = \frac{\mathcal{F}_1^2(x, \alpha, p, q) \xi^2 (3+u)_{pq}^\tau (2+u)_{pq}^{2\tau} (1-\gamma)}{\mathcal{F}_1^2(x, \alpha, p, q) \tilde{\mathcal{A}}(\xi, \zeta, \delta, u, \tau) - \mathcal{F}_2(x, s, p, q) \tilde{\mathcal{B}}(\xi, \zeta, \delta, u, \tau)}.$$

Thus, according to (p, q) -Chebyshev polynomials in (1.3), we conclude that

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{|(p+q)x|(3+u)_{pq}^\tau}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau}, & 0 \leq |\Upsilon(\gamma, \zeta, \lambda, \vartheta)| \leq \frac{(3+u)_{pq}^\tau}{\xi(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau}, \\ 2(p+q)x|\Upsilon(\gamma, \zeta, \lambda, \vartheta)|, & \text{for } |\Upsilon(\gamma, \zeta, \lambda, \vartheta)| \geq \frac{(3+u)_{pq}^\tau}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^\tau} \end{cases}$$

After making several simplifications, the proof of Theorem 3.2 is complete. □

4. Conclusion

In this article, we introduced a new integral operator defined by (p, q) -Hurwitz zeta function, which is a generalization of the q -Srivastava-Attiya operator. We also provided two families $\mathcal{SM}_{p,q}(\xi, \zeta, \delta, u, \tau)$ and $\mathcal{SC}_{p,q}(\lambda, \zeta, \vartheta, u, \tau)$ of biunivalent and holomorphic functions in the unit disk, which is defined by (p, q) -Chebyshev polynomials, and we obtained Fekete-Szegő inequalities for these families. We also think that this construction has many applications.

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