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Optimal Control Problem to Robust Nonlinear Descriptor Control Systems with Matching Condition

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Abstract

In this paper, the solutions to class of robust non-linear semi-explicit descriptor control systems with matching condition via optimal control strategy are obtained. The optimal control strategy has been introduced and developed in the sense that, the optimal control solution is robust solution to the given non-linear uncertain semi-explicit descriptor control system. The necessary mathematical proofs and remarks as well as discussions are also discussed. The present approach is illustrated step-by-step by application example to show its effectiveness and efficiency to compensate the structure uncertainty in the given semi-explicit (descriptor) control system.

Keywords: Descriptor Systems, Matching condition, Robust Control Problem, Optimal Control Problem, Uncertainty.

مسألة سيطرة مثلى لنظم سيطرة وصفية رصينة غير خطية بوجود شرط المطابقة

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الخلاصة

في هذا البحث, تم الحصول على الحلول لصنف من أنظمة السيطرة الوصفية غير الخطية شبه الصريحة مع وجود شرط التطابق باستخدام أسلوبية السيطرة المثلى. تم تطوير هذه الطريقة بحيث ان الحل لمسائل السيطرة المثلى المكافئة هو حل لأنظمة السيطرة الوصفية غير الخطية والمقلقة بنيويا. كذلك تم عرض ومناقشة الجوانب الرياضية الضرورية لهذه المسالة. وقد تم توضيح هذه الأسلوبية بمثال توضيحي خطوة – خطوة لبيان فعاليتها وكفاءتها للتعامل مع نظم السيطرة الوصفية غير الخطية المقلقة بنيويا.

1. Introduction

A descriptor control system can be represented by differential and algebraic equations which is a generalized representation of the state-space system. The application of these systems can be found in electrical circuits, robots, etc. [1]. These systems are also referred to

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as singular systems, implicit systems, generalized state-space systems, semi-state systems, and differential-algebraic systems [2].
The solvability of linear descriptor systems may be found in [3], [4], [5], while nonlinear descriptor systems is discussed in [6], [7], [8] and one can also see [1]. Furthermore, Stability of linear and non-linear descriptor systems are studied in [9], [2], [10], [11], [12], [13]. The descriptor control uncertain system should preserve various system properties under some structure system (uncertainty) perturbation. The robustness may be defined to be insensitivity of system property due to inherent system uncertainty. The fact is that in many practical phenomena some parameters of system are not exactly known and there is only some information on the intervals to which they belong. Therefore, studying the robustness of such systems is an important field.

Recently, much attention has been given to the design of controllers, so that properties of a system are preserved under various classes of uncertainties that appear in the system. Such controllers are called robust controllers, and the resulting system is said to be robust control system. If the uncertainties are lying in the range of the input matrix (operator), they are called matched condition uncertainties. For state- space system and some class of control problem, matched condition have been discussed in [14]. If this condition is not satisfied, a decomposition approach may be used [15] and [16].

Due to the difficulty in solving general robust descriptor systems see [17], in this paper, robust control problem **(r. c. p)** is translated into a specific (equivalent) optimal control problem **(o. c. p.)**. The solution of **(o. c. p.)** is then a solution to the robust descriptor control problem based on the nominal system structure and the types of uncertainties. Descriptor systems, like other systems may contain many types of uncertainties. These uncertainties can be classified as with or without matching condition. The sterilizability and solvability of robust controller design semi explicit differential – algebraic control problems have presented and developed in [18]. The theoretical optimal control approach to solve some uncertain control system in state space form have been presented in [19]. The generalization of the work in [19] from state space approach to include a larger classes of descriptor systems (differential-algebraic control systems) with uncertainty of index one, have been discussed in [20].

In this paper, robust control with matching condition has been developed which is a generalization of the previous result [20]. In [20], the solution of the robust non-linear semi explicit descriptor uncertain system with matching condition is obtained. The algebraic constraints ate of linear type with rank deficient assumption. The optimal control approach to this differential algebraic system is applied in the sense that the optimal control solution is the robust controller to the original differential – algebraic uncertain system. This paper generalizes the work in [20] to include a large class of nonlinear uncertain differential - algebraic equation. The algebraic equation is completely nonlinear system of index one property. The implicit function theorem has been used to reduce the system into an equivalent state space uncertain nonlinear system. The theoretical and numerical results are also developed to cover the solvability and global asymptotic sterilizability of the given uncertain control problems with illustration.

2. Problem Formulation

Consider a descriptor control nonlinear system described by the semi-explicit form:

$$\dot{x}_1 = \hat{F}(x_1, x_2, u) + G(x_1, x_2)u \tag{1a}$$

$$0 = F_1(x_1, x_2) \tag{1b}$$

Where $x = (x_1, x_2) \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^r$. Suppose that

$$\hat{F}(x_1, x_2, u) = F(x_1, x_2) + G(x_1, x_2)f(x_1, x_2) + G(x_1, x_2)\, h(u) \tag{2}$$

Which leads

$$\dot{x}_1 = F(x_1,x_2) + G(x_1,x_2)u + G(x_1,x_2)f(x_1,x_2) + G(x_1,x_2)\, h(u) \quad (3a)$$
$$0 = F_1(x_1,x_2) \quad (3b)$$

Where $F(x_1,x_2) \in C^1(D \times \mathbb{R}^{n_2};\mathbb{R}^{n_1})$, $G(x_1,x_2) \in C^1(D \times \mathbb{R}^{n_2};\mathbb{R}^{n\times r})$ and $F_1(x_1,x_2) \in C^1(D \times \mathbb{R}^{n_2};\mathbb{R}^{n_2})$ and $D \subset \mathbb{R}^{n_1}$ which is an open set, which represents the known functions of the system ,system (3), they are continuous. Suppose that:

$$F_1(0,0) = 0 \quad (4)$$

Note that $f:\mathbb{R}^n \longrightarrow \mathbb{R}^r$, and $h:\mathbb{R}^r \longrightarrow \mathbb{R}^r$ represent the uncertainties of the system (3). The following bounds in norm for the uncertainties are assumed as follows:

$$\|f(x)\|_R^2 \le \frac{1}{\rho} f_{max}^2(x) \quad (5)$$

$$\|h(u)\|_R^2 \le \frac{1}{\rho_1} h_{max}^2(x) \quad (6)$$

Where ρ and $\rho_1 > 1$ are positive constant real numbers with $\rho = \frac{\rho_1}{\rho_1 - 1}$. And for some R is symmetric positive definite matrix, $f_{max}(x)$ and $h_{max}(x)$ are positive continuous known functions such that $f(0,0) = 0, and \quad h(0) = 0$. The following assumptions are assumed to be satisfied.

Assumption A. There exists an open set $\widehat{\Omega}_x \subset D$ such that for all $\hat{x}_1 \in \widehat{\Omega}_x$, $F_1(\hat{x}_1,\hat{x}_2) = 0$ is solvable for $\hat{x}_2$. Define the solution manifold set as follows:

$$\widehat{\Omega} = \{x_1 \in \widehat{\Omega}_x\, , x_2 \in \mathbb{R}^{n_2} |\, F_1(x_1,x_2) = 0\, \}$$

One can also assume that the Jacobian matrix of (3b) with respect to x_2 , *i.e.*, $F_{1:x_2}(\hat{x}_1,\hat{x}_2) = 0$ is nonsingular for $(\hat{x}_1,\hat{x}_2) \in \widehat{\Omega}$. That means the rank of $F_{1:x_2}$ is assumed to be constant and full on $\widehat{\Omega}$.

On using the *implicit function theorem*, the assumption tells us that for every point $\hat{x}_1 \in \widehat{\Omega}_x$ there exist a neighborhood $O_{\hat{x}_1}$ of $\hat{x}_1$ and a corresp-ondding neighborhood $O_{\hat{x}_2}$ of $\hat{x}_2$ s. t. for all point $x_1 \in O_{\hat{x}_1,\hat{u}}$, A unique solution $x_2 \in O_{\hat{x}_2}$ exists where:

$$x_2 = \varphi_{\hat{x}_1}(x_1) \quad (7)$$

The subscript $\hat{x}_1$is included to clarify that the implicit theorem is only local. Define

$$\Omega = \{x_1 \in \Omega_x\, , x_2 \in \mathbb{R}^{n_2} |\, F_1(x_1,x_2) = 0\, \} \quad (8)$$

on which the implicit function solving $F_1(x_1,x_2) = 0$, is denoted

$$x_2 = \varphi(x_1), \quad \forall x_1 \in \Omega_x \quad (9)$$

Remarks 2.1

1. Assumption A, will help to reduce the system (3) into its equivalent state-space system. For $x_1 \in \Omega_x$, (3a) and (3b), can be rewritten as

$$\dot{x}_1 = F(x_1,\varphi(x_1)) + G(x_1,\varphi(x_1))\big(u + f\big(x_1,\varphi(x_1)\big) + h(u)\big) \quad (10)$$

2. Unlike the ODE, the consistent initial conditions should be chosen such that: $F_1\big(x_{1,0},x_{2,0}\big) = 0$, or $x_{2,0} = \varphi\big(x_{1,0}\big)$, $\forall x_{1,0} \in \Omega_x$.
3. The nominal system of (3) can be redefined as :

$$\dot{x}_1 = F(x_1,\varphi(x_1)) + G\big(x_1,\varphi(x_1)\big)u, \quad x_1(0) = x_{1,0} \in \Omega_x \quad (11)$$

Which depends only on the known parts of the system (10).

4. The system "Eq. (10)", is called closed loop system if $u = \mu(x_1)$ and has the form

$$\dot{x}_1 = F(x_1,\varphi(x_1)) + G(x_1,\varphi(x_1))\big(\mu + f\big(x_1,\varphi(x_1)\big) + h(\mu)\big) \quad (12)$$

And this is concerned to be the main part of robust control theory, i.e., robust control problem is to find a feedback control law $u = \mu(x_1)$ so that of the system (12), is globally asymptotically stable **(g. a. s)**.

3. Robust descriptor control problem

Consider there is a feedback control law $u = \mu(x_1, x_2)$ such that the critical point $(x_1, x_2) = (0,0)$ of (12)

$$\dot{x}_1 = F(x_1, x_2) + G(x_1, x_2)\mu + G(x_1, x_2)f(x_1, x_2) + G(x_1, x_2)\, h(\mu) \quad (13a)$$

$$0 = F_1(x_1, x_2), \forall\, x_1 \in \Omega_x \quad (13b)$$

is (g. a. s) for all admissible perturbations $f(x_1, x_2)$, $h(\mu(x_1, x_2)\,))$. Then the system (13), is called ***robust control system***.

From Assumption A, and the discussions of remakes (2.1), the aim is then to find a feedback control law $u = \mu(x_1)$ such that $x_1 = 0$, for $x_1 \in \Omega_x$ and $x_2 = \varphi(x_1)$, of the system (12)

$$\left.\begin{array}{c} \dot{x}_1 = F(x_1, \varphi(x_1)) + G(x_1, \varphi(x_1))\big(\mu + f\big(x_1, \varphi(x_1)\big) + h(\mu(x_1))\big) \\ x_1(0) = x_{1,0} \in \Omega_x \end{array}\right] \quad (14)$$

$$\dot{x}_1 = F(x_1, \varphi(x_1)) + G(x_1, \varphi(x_1))\big(\mu + f\big(x_1, \varphi(x_1)\big) + h(\mu(x_1))\big)$$

with

$$x_1(0) = x_{1,0} \in \Omega_x \quad (14)$$

is (g. a. s.) for all admissible perturbations $f\big(x_1, \varphi(x_1)\big)$, $h(\mu(x_1))$.

To solve problem (14), the following strategy is suggested.

4. Optimal control equivalent problem

Now, the robust control problem Eq. (10) can be put in the equivalent quadratic optimal control problem for all $x_1 \in \Omega_x$, the nominal system will be

$$\dot{x}_1 = F(x_1, \varphi(x_1)) + G\big(x_1, \varphi(x_1)\big)u, \quad x_1(0) = x_{1,0} \in \Omega_x \quad (15a)$$

$$x_2 = \varphi(x_1) \quad (15b)$$

The considered class of performance index is

$$J = \int_0^\infty L(x_1, \varphi(x_1), u)dt \quad (16)$$

With $x_2 = \varphi(x_1)$ and

$$L(x_1, \varphi(x_1), u) = f_{max}^2\big(x_1, \varphi(x_1)\big) + x_1^T Q_1 x_1 + u^T R u + h_{max}^2\big(x_1, \varphi(x_1)\big) \quad (17)$$

The optimal control problem then defined as follows: Finding the optimal control $u \in \Delta$ the set of all admissible controllers, such that

$$V\big(\,x_1(0)\big) = \min_{u(\cdot)\in\Delta} J \triangleq \min_{u(\cdot)\in\Delta} \int_0^\infty L(x_1, \varphi(x_1), u)dt \quad (18)$$

subject to the dynamics Eq. (15), and the boundary conditions

$$x_1(0) = x_{1,0} \in \Omega_x$$

$$\lim_{t\to\infty} x_1(t) = 0$$

The minimization is done with respect to all $u \in \Delta$ the set of all admissible controllers and $f_{max}^2(x)$ is the upper bound of $f(x)$ and $h_{max}^2(x)$ is the upper bound of $h(u)$.

Lemma 4.1

Consider there is a positive definite continuously differentiable function $V(x_1)$ and $V(x_1) = \int_0^\infty L(x_1, \varphi(x_1), u)dt$, where $L(x_1, \varphi(x_1), u)$ is a positive semi-definite in x_1 and positive definite in u and $x_2 = \varphi(x_1)$, $for\ each\ x_1 \in \Omega_x$. Then the necessary condition for optimality is that $V(x_1)$ must satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \min_{u\in\Delta}\big[L(x_1, \varphi(x_1), u) + V_{x_1}^T\big(F(x_1, \varphi(x_1)) + G\big(x_1, \varphi(x_1)\big)u\big)\big] \quad (19)$$

Where $V_{x_1} = \frac{dV}{dx_1}$.

Proof: The proof is similar to that presented in [15] for descriptor system, which is the generalization of state-space approach in [21].

The first-order necessary condition for optimality of (19) yields the set of equations

$$0 = L_u + V_{x_1}^T G$$

$$0 = L + V_{x_1}^T(F + Gu)$$

$x_2 = \varphi(x_1)$
where the quantities in the right-hand sides are evaluated at $(x_1, \varphi(x_1), u)$. Since $x_2 = \varphi(x_1)$ is the unique solution of (1b), it is also possible to write these equations according to

$$0 = L_u + V_{x_1}^T G \tag{20a}$$

$$0 = L + V_{x_1}^T (F + Gu) \tag{20b}$$

$$0 = F_1(x_1, x_2) \tag{20c}$$

Theorem 4.2 (Equivalency theorem)
Consider the robust control problem as discussed in (3)

$$\dot{x}_1 = F(x_1, x_2) + G(x_1, x_2)u + G(x_1, x_2)f(x_1, x_2) + G(x_1, x_2)\, h(u) \tag{21a}$$

$$0 = F_1(x_1, x_2) \tag{21b}$$

Where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^r$, $u \in R^r$,
$F(x_1, x_2) \in C^1(D \times \mathbb{R}^{n_2}; \mathbb{R}^{n_1})$, $G(x_1, x_2) \in C^1(D \times \mathbb{R}^{n_2}; \mathbb{R}^{n \times r})$ and $F_1(x_1, x_2) \in C^1(D \times \mathbb{R}^{n_2}; \mathbb{R}^{n_2})$, $D \subset \mathbb{R}^{n_1}$ is an open set, which represents the known continuous functions of the system (21). Suppose that:

$$F_1(0,0) = 0 \tag{22}$$

Note that $f: \mathbb{R}^n \longrightarrow \mathbb{R}^r$, and $h: \mathbb{R}^r \longrightarrow \mathbb{R}^r$ which represent the uncertainties of the system (21), with

$$\|f(x)\|_R^2 \leq \frac{1}{\rho} f_{max}^2(x) \tag{23a}$$

$$\|h(u)\|_R^2 \leq \frac{1}{\rho_1} h_{max}^2(x) \tag{23b}$$

Where ρ and $\rho_1 > 1$ are positive constant real numbers with $\rho = \frac{\rho_1}{\rho_1 - 1}$. Where the matrix $R > 0$ is symmetric positive definite matrix, $f_{max}(x)$ and $h_{max}(x)$ are positive continuous known functions and $f(0) = 0$, $h(0) = 0$, $h(0) = 0$, and $f_{max}(0) = 0$.
The optimal control problem becomes

$$V(x_1(0)) = \min_{u \in \Delta} \int_0^\infty (f_{max}^2(x) + x_1^T Q_1 x_1 + u^T R u + h_{max}^2(x))dt \tag{24a}$$

subject to the dynamics

$$\dot{x}_1 = F(x_1, \varphi(x_1)) + G(x_1, \varphi(x_1))u, \quad x_1(0) = x_{1,0} \in \Omega_x \tag{24b}$$

Where the matrix $Q_1 \geq 0$.
Then, the solution of the o. c. p. of system (24), is the solution of the robust control problem (21).

proof

From assumption A, equation (21) is equivalent to

$$\dot{x}_1 = F(x_1, \varphi(x_1)) + G(x_1, \varphi(x_1))\big(u + f(x_1, \varphi(x_1)) + h(u)\big) \tag{25}$$

For all $x_1 \in \Omega_x$. Set:

$$V(x_1) = \min_{u \in \Delta} \int_{t_0}^{t_f} \big(f_{max}^2(x_1, \varphi(x_1)) + x_1^T Q_1 x_1 + u^T R u + h_{max}^2(x_1, \varphi(x_1))\big)dt$$

For all $x_1 \in \Omega_x$ and $u(0) = 0, u \in \Delta$ to be the minimum cost of bringing the system (24),where Δ is the set of all admissible controllers from $x_1(t_0) = x_{1,0}$ to $x_1 = 0$, and from lemma (4.1), $V(x_1)$ must satisfy Hamiltonian-Jacobean- Bellman equation (H-J-B)

$$0 = \min_{u \in \Delta}\big[L(x_1, \varphi(x_1), u) + V_{x_1}^T\big(F(x_1, \varphi(x_1)) + G(x_1, \varphi(x_1))u\big)\big] \tag{26}$$

Now if $u = \mu(x_1)$ is the solution of the optimal control problem (24), then

$$f_{max}^2(x) + x_1^T Q_1 x_1 + \mu^T R \mu + h_{max}^2(x) + V_{x_1}^T (F + G\mu) = 0 \tag{27a}$$

$$2\mu^T R + V_{x_1}^T G = 0 \tag{27b}$$

$$F_1(x_1, x_2) = 0 \tag{27c}$$

For all $x_1 \in \Omega_x$.
Now, we will show that $u = \mu(x_1)$ of (27) is the solution of the robust control problem (25), i.e., $x_1 = 0$ of (25), is g. a. s. for all admissible uncertainty $f(x)$ and $h(\mu(x_1))$.

To do so, we finally show that $V(x_1)$ is a Lyapunov function of the system (25).

1. Since $u(0) = 0, f_{max}(0) = 0,\ \ h_{max}(0) = 0$ then $V(0) = 0$,
2. And since $f_{max}^2(x) > 0\ ,\ h_{max}^2(x) > 0\ , x_1^T Q_1 x_1 > 0\ , u^T R u > 0$,
   $\forall\ x = \big(x_1, \varphi(x_1)\big) \neq (0,0)$ then $V(x_1) > 0$, for all $x_1 \neq 0$.

$$\begin{aligned}\dot{V}(x_1) &= V_{x_1}^T \dot{x}_1\\ &= V_{x_1}^T[F + G\mu + Gf + Gh(\mu)]\\ &= V_{x_1}^T[F + G\mu] + V_{x_1}^T G[f + h(\mu)]\end{aligned}$$

From (27b), (24), one gets

$$\dot{V}(x_1) = -[f_{max}^2(x) + x_1^T Q_1 x_1 + \mu^T R \mu + h_{max}^2(x)] - 2\mu^T R[f + h(\mu)]$$

By adding and subtracting the terms $\rho f^T R f$, $\rho_1 h^T R h$, we get that

$$\begin{aligned}\dot{V}(x_1) = &-(f_{max}^2(x) - \rho f^T R f) - (h_{max}^2(x) - \rho_1 h^T R h) - x_1^T Q_1 x_1 - \mu^T R \mu - 2\mu^T R f -\\ &2\mu^T R h - \rho f^T R f - \rho_1 h^T R h\\ = &-(f_{max}^2(x) - \rho f^T R f) - (h_{max}^2(x) - \rho_1 h^T R h) - x_1^T Q_1 x_1 - \frac{1}{\rho_1}\mu^T R \mu - \frac{1}{\rho}\mu^T R \mu -\\ &\mu^T R f - \mu R f^T - \mu^T R h - \mu R h^T - \rho f^T R f - \rho_1 h^T R h\\ = &-(f_{max}^2(x) - \rho f^T R f) - (h_{max}^2(x) - \rho_1 h^T R h) - x_1^T Q_1 x_1 - \Big(\frac{1}{\rho}\mu^T R \mu + \mu^T R f +\\ &\mu R f^T + \rho f^T R f\Big) - \Big(\frac{1}{\rho_1}\mu^T R \mu + \mu^T R h + \mu R h^T + \rho_1 h^T R h\Big)\\ = &-(f_{max}^2(x) - \rho f^T R f) - (h_{max}^2(x) - \rho_1 h^T R h) - x_1^T Q_1 x_1 -\\ &\left(\frac{1}{\sqrt{\rho}}\mu + \sqrt{\rho} f\right)^T R \left(\frac{1}{\sqrt{\rho}}\mu + \sqrt{\rho} f\right) - \left(\frac{1}{\sqrt{\rho_1}}\mu + \sqrt{\rho_1} h\right)^T R \left(\frac{1}{\sqrt{\rho_1}}\mu + \sqrt{\rho_1} h\right)\\ = &-(f_{max}^2(x) - \rho \|f(x)\|_R^2) - (h_{max}^2(x) - \rho_1 \|h(x)\|_R^2) - \left\|\frac{1}{\sqrt{\rho}}\mu + \sqrt{\rho} f\right\|_R^2\\ &- \left\|\frac{1}{\sqrt{\rho_1}}\mu + \sqrt{\rho_1} h\right\|_R^2 - \|x_1\|_{Q_1}^2\end{aligned}$$

From the conditions (5) and (6), we conclude that

$\|f(x)\|_R^2 \leq \frac{1}{\rho} f_{max}^2(x)\ \Rightarrow f_{max}^2(x) - \rho \|f(x)\|_R^2 \geq 0$ and

$\|h(x)\|_R^2 \leq \frac{1}{\rho_1} h_{max}^2(x)\ \Rightarrow h_{max}^2(x) - \rho_1 \|h(x)\|_R^2 \geq 0$ and from properties of the norm that

$\left\|\frac{1}{\sqrt{\rho}}\mu + \sqrt{\rho} f\right\|_R^2 \geq 0,\ \ \left\|\frac{1}{\sqrt{\rho_1}}\mu + \sqrt{\rho_1} h\right\|_R^2 \geq 0,$

Therefore, $\quad \dot{V}(x_1) \leq -\|x_1\|_{Q_1}^2 < 0$

Thus, the condition of asymptotically stable is implemented, and there exists a neighbourhood $N_\epsilon = \{x_1 \in \Omega_x\ ;\ \|x_1\|_{Q_1} < \epsilon\}$ for some $\epsilon > 0$. Such that if $x_1(t)$ enters N_ϵ then $\lim_{t\to\infty} \|x_1(t)\|_{Q_1} = 0$. But $x_1(t)$ cannot remains forever outside N_ϵ, otherwise $\|x_1(t)\|_{Q_1} > \epsilon$ for all $t > 0$, therefore

$$\begin{aligned}V(x_1(t)) - V(x_1(0)) &= \int_0^t \dot{V}(x_1(s))ds\\ &\leq -\int_0^t \|x_1(s)\|_{Q_1}^2 ds\\ &\leq -\int_0^t \epsilon^2 ds\\ &= -\epsilon^2 t\end{aligned}$$

$V(x_1(t)) \leq V(x_1(0)) - \epsilon^2 t$

Letting $\to \infty$, we have $V(x_1(t)) \to -\infty$ which contradicts the fact that $V(x_1(t)) > 0$ for all $x_1 \in \Omega_x$. Therefore $\lim_{t\to\infty} \|x_1(t)\|_{Q_1} = 0$.

But $x_2(t) = \varphi(x_1(t))$ such that $x(t) = (x_1(t), x_2(t)) \in \widetilde{\Omega}$.

Then

$$\lim_{t\to\infty}\|x_2(t)\| = \left\|\lim_{t\to\infty} x_2(t)\right\|$$
$$= \|\lim_{t\to\infty} \varphi(x_1(t))\ \|,\ x_1 \in \Omega_x$$
$$= \|\varphi(\lim_{t\to\infty} x_1(t))\|,\ x_1 \in \Omega_x$$
$$= \|\varphi(0)\| = 0$$

So $\lim_{t\to\infty}\|x(t)\| = \|\lim_{t\to\infty}(x_1(t), x_2(t))\| = \|(0,0)\| = 0.$ For all $x(t) \in \widetilde{\Omega}$

6. Illustration

Consider the robust descriptor system

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = A\begin{pmatrix}x_1\\ x_2\end{pmatrix} + Bu + Bf(x) + Bh(u) \tag{28a}$$

$$0 = F(x)\,,\quad x^T = (x_1, x_2, x_3, x_4) \tag{28b}$$

Where $A = \begin{pmatrix}-2 & 1.73\\ 1.73 & 0\end{pmatrix}$, $B = \begin{pmatrix}0\\ -1\end{pmatrix}$, $F(x) = \begin{pmatrix}f_1(x)\\ f_2(x)\end{pmatrix}$,

$f(x) = q_1 x_1 \sin(x_1 x_3)\,, q_1 \in [-1,1]\,, h(u) = q_2 x_2 \sin(x_4 u)\,,\ q_2 \in [-1,1],$

$f_1(x) = x_1^2 + x_2^2 + (x_3+1)^2 + (x_4+1)^2 - 2$

$f_2(x) = x_1^2 - x_2^2 + (x_3+1)^2 - (x_4+1)^2$; with $\rho = \rho_1 = 2, R = 10,$

$Q_1 = \begin{pmatrix}5 & 10\\ 10 & 20.1\end{pmatrix}$ with eigenvalues $\sigma(Q_1) = [0.02, 25.08]$

Step (1): $f(x) = q_1 x_1 \sin(x_1 x_3)\,, q_1 \in [-1,1]$, ,

$$\|f(x)\|_R^2 = 10\, q_1^2 x_1^2 \sin^2(x_1 x_3) \le 10x_1^2 = \tfrac{1}{2} f_{max}^2(x)$$

$\Rightarrow f_{max}^2(x) = 20x_1^2 = x_1^T M x_1$ where $M = \begin{pmatrix}20 & 0\\ 0 & 0\end{pmatrix}$; Similarly,

$h_{max}^2(u) = 20x_1^2 = x_1^T N x_1$ where $N = \begin{pmatrix}0 & 0\\ 0 & 20\end{pmatrix}$

Step (2): Consistent initial condition: The solution manifold is defined as

$\widetilde{\Omega} = \{(x_1, x_2)^T \in \widetilde{\Omega}_x \subseteq D = [-1,1]\times[-1,1] \subset \mathbb{R}^2, (x_3, x_4) \in \mathbb{R}^2 | (f_1(x), f_2(x) = (0,0)\,\}$

$$\widetilde{\Omega} = \left\{\begin{matrix}(x_1, x_2)^T \in \widetilde{\Omega}_x \subseteq D = [-1,1]\times[-1,1], (x_3, x_4)^T \in \mathbb{R}^2 |\ x_3 = -1 + \sqrt{1 - x_1^2},\\ x_4 = -1 + \sqrt{1 - x_2^2}\end{matrix}\right\}$$

Therefore, the initial condition $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0})$ is consistent if and only if

$x_{3,0} = -1 + \sqrt{1 - x_{1,0}^2}$ and $x_{4,0} = -1 + \sqrt{1 - x_{2,0}^2}$ for a given

$(x_{1,0}, x_{2,0}) \in \widetilde{\Omega}_x \subseteq D = [-1,1]\times[-1,1] \subset \mathbb{R}^2.$

Step (3): The nominal system will be

$\dot{X}_1 = AX_1 + Bu$ where $X_1 = (x_1, x_2)^T$

Step (4): Check controllability

$(B\!:\!AB) = \begin{pmatrix}0 & -1.73\\ -1 & 0\end{pmatrix}$, $Rank(B\!:\!AB) = 2$, the system is controllable.

Step (5): The optimal control problem which is equivalent to (28), For all $(x_1, x_2)^T \in \widetilde{\Omega}_x$, we have

$$\min_{u(\cdot)} \int_0^\infty (x_1^T M x_1 + x_1^T Q_1 x_1 + 10u^2 + x_1^T N x_1)dt$$
$$= \int_0^\infty (x_1^T (M + N + Q_1) x_1 + 10u^2)dt$$
$$= \int_0^\infty \left(x_1^T \begin{pmatrix}25 & 10\\ 10 & 41.1\end{pmatrix} x_1 + 10u^2\right)dt$$

Subject to $\dot{X}_1 = AX_1 + Bu$.

Step (6):

Suppose $P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and solve Riccati Equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

$$\begin{pmatrix} 0 & 1.73 \\ 1.73 & -2 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} -2 & 1073 \\ 1.73 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} - \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} 25 & 10 \\ 10 & 41.1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

On solving this algebraic Riccati equation for $a, b, c,$ $and\ d$, we have got that:

$a = 13.7316$, $b = 17.1366$, $c = 31.6848$ and $P = \begin{pmatrix} 13.7316 & 17.1366 \\ 17.1366 & 31.6848 \end{pmatrix}$.

The eigenvalues of P are (3.3628 , 42.0536). Then P is positive definite symmetric matrix.

Step (7): Setting the optimal control law $u = -KX_1$, where $K = R^{-1}B^T P$,

$$K = 0.1(0, -1)\begin{pmatrix} 13.7316 & 17.1366 \\ 17.1366 & 31.6848 \end{pmatrix} = (-1.7137\ , -3.1685).$$

$$u = -KX_1 \Rightarrow \ u = 1.7137x_1 + 3.1685x_2\ .$$

Step (8): The nominal feedback control system for all $X_1 \in \widetilde{\Omega}_x$ will be

$$\dot{X}_1 = AX_1 - BKX_1 \Rightarrow \dot{X}_1 = (A - BK)X_1$$

$$\dot{X}_1 = \begin{pmatrix} -2 & 1.73 \\ 0.0163 & -3.1685 \end{pmatrix} X_1 \Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1.73 \\ 0.0163 & -3.1685 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Stability is being checked by using the eigenvalues of $-BK$, which are (-3.1922 , -1.9763), therefore the system $\dot{X}_1 = (A - BK)X_1$ is stable.

Step (9): The robust feedback system is

$$\dot{X}_1 = (A - BK)X_1 + B(q_1 x_1 \sin(x_1 x_3) + q_2 x_2 \sin(1.7137x_1 x_4 + 3.1685x_2 x_4)$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1.73 \\ 0.0163 & -3.1685 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}(q_1 x_1 \sin\left(x_1(-1 + \sqrt{1 - x_1^2})\right) + q_2 x_2 \sin\left((-1 + \sqrt{1 - x_1^2})(1.7137x_1 + 3.1685x_2)\right)$$

This system is stable by using the theorem 1 for all $q_1, q_2 \in [-1,1]$.

Step (10):

The nominal feedback control system for all $x^T = (x_1, x_2, x_3, x_4) \in \widetilde{\Omega}$ is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1.73 \\ 0.0163 & -3.1685 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \frac{-x_1}{x_3 + 1}(-2x_1 + 1.73x_2) \\ \frac{-x_2}{x_4 + 1}(0.0163x_1 - 3.1685x_2) \end{pmatrix}$$

Step (11):

The robust feedback system for all $x^T = (x_1, x_2, x_3, x_4) \in \widetilde{\Omega}$ is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1.73 \\ 0.0163 & -3.1685 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}(q_1 x_1 \sin(x_1 x_3) + q_2 x_2 \sin(1.7137x_1 x_4 + 3.1685x_2 x_4)$$

$$\begin{pmatrix}\dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{pmatrix} \dfrac{-x_1}{x_3+1}(-2x_1+1.73x_2) \\ \dfrac{-x_2}{x_4+1}((0.0163x_1-3.1685x_2)-(q_1x_1\sin(x_1x_3)-q_2x_2\sin(1.7137x_1x_4+3.1685x_2x_4)) \end{pmatrix}$$

The robust control system solution with different system uncertainty and its equivalent to optimal control system solution, are shown in the following figures.

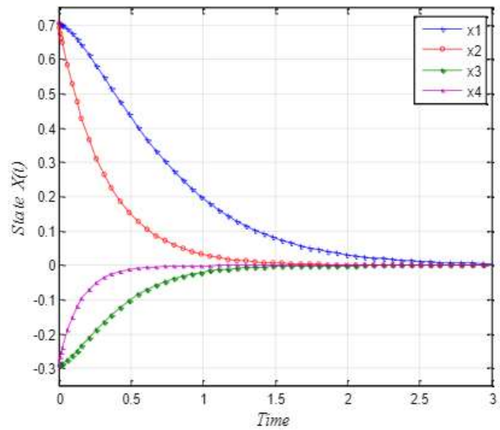


Figure 1-Robust solution (x_1, x_2, x_3, x_4) with $q_1 = 0.5,\ q_2 = 0.5$, $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}) \in \widetilde{\Omega}$

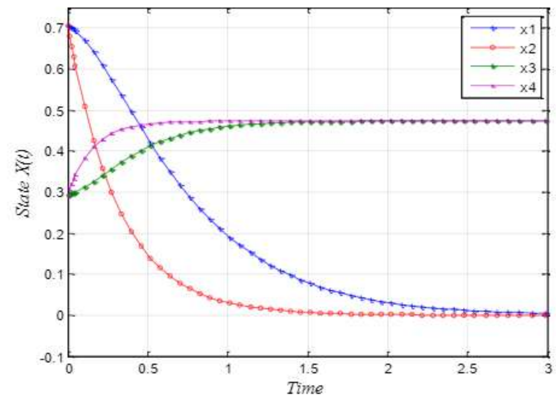


Figure 2-optimal solution (x_1, x_2, x_3, x_4) *with* $q_1 = 0.5,\ q_2 = 0.5$, and $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}) \notin \widetilde{\Omega}$

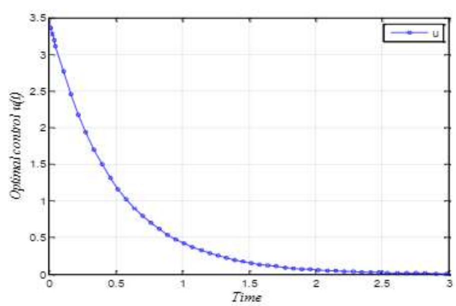


Figure 3- *The optimal control* $u(t)$ $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}-1, \frac{1}{\sqrt{2}}-1\right) \in \widetilde{\Omega}$, $q_1 = 0.5\ , q_2 = 0.5$

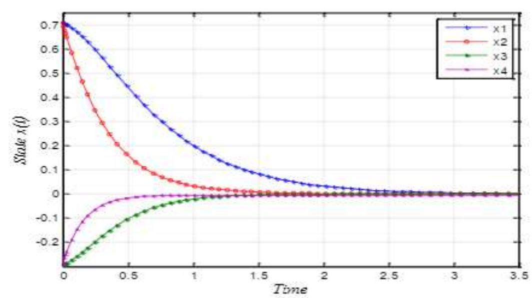
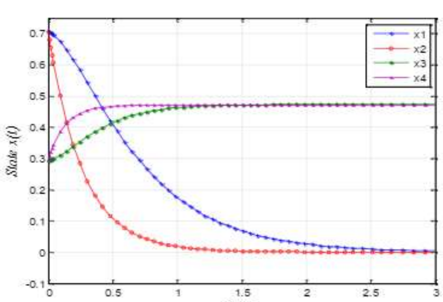


Figure 4-*optimal solution* (x_1, x_2, x_3, x_4) $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}-1, \frac{1}{\sqrt{2}}-1\right) \in \widetilde{\Omega}$ $q_1 = 0.5\ , q_2 = 0.5$



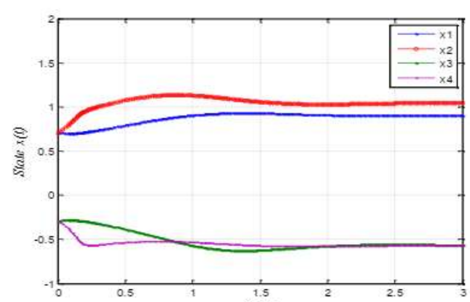


Figure 5-*Robust solution represents* (x_1, x_2, x_3, x_4), $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}) \notin \widetilde{\Omega}$ $q_1 = 0.5, q_2 = 0.5$

Figure 6- *Robust solution represents* (x_1, x_2, x_3, x_4) $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}-1, \frac{1}{\sqrt{2}}-1) \in \widetilde{\Omega}$ $q_1 = 4, q_2 = 4$

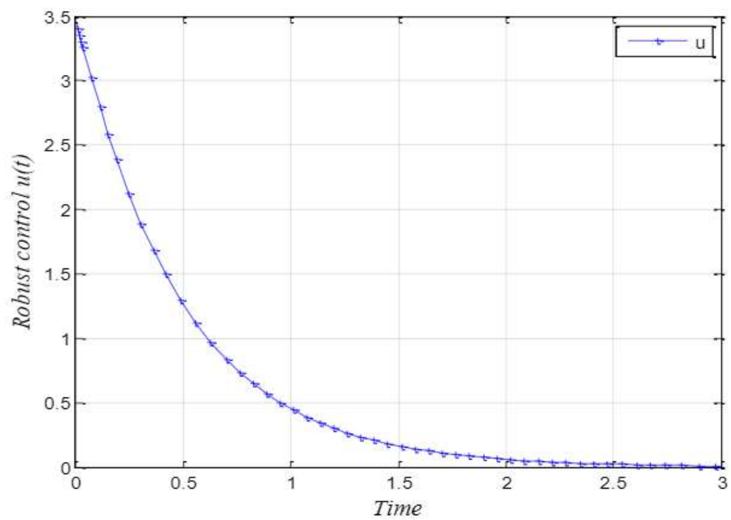


Figure 7-*The robust control represents* (x_1, x_2, x_3, x_4) *with* $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}-1, \frac{1}{\sqrt{2}}-1) \in \widetilde{\Omega}$, $q_1 = 0.5, q_2 = 0.5$

Figures 1 and Figure 4 show the robust and optimal solution with consistent initial conditions where the uncertain parameters are in the given range [-1,1]. While figure-2 and Figure 5 have shown the divergence of the solutions when the initial conditions are selected out of consistent set range. Out the range of the uncertain parameters, the solution will also no longer converges as on can see this fact form Figure-6. Figure 7 represents the smoothness and convergence the robust control within the class of consistent initial conditions and uncertain parameters.

7. Conclusions

The optimal control approach for solving nonlinear uncertain differential-algebraic system have been modified to solve some classes of uncertain control system when the perturbed nonlinear functions satisfy some matching conditions. This approach will help to solve large classes of problems without caring on the nature of system uncertainty. The only requirement is a good understanding of a linear –quadratic optimal control problems and application of implicit function theorems to reduce the differential-algebraic uncertain control systems into its standard control systems. This approach not only helps for solving the given problem, but also it helps to design a globally asymptotically stabilizing optimal (robust) controller which is very important in the real-life applications.

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