# A GLOBAL IMPLICIT FUNCTION THEOREM AND ITS APPLICATIONS TO FUNCTIONAL EQUATIONS 

Dariusz Idczak<br>Faculty of Mathematics and Computer Science University of Lodz<br>Banacha 22, 90-238 Lodz, Poland


#### Abstract

The main result of the paper is a global implicit function theorem. In the proof of this theorem, we use a variational approach and apply Mountain Pass Theorem. An assumption guarantying existence of an implicit function on the whole space is a Palais-Smale condition. Some applications to differential and integro-differential equations are given.


1. Introduction. In the paper, we derive a global implicit function theorem for a map $F: X \times Y \rightarrow H$ where $X, Y$ are real Banach spaces and $H$ is a real Hilbert space. In the proof, we use a variational approach and apply Mountain Pass Theorem. Such a method has been used in paper [3] to prove a theorem on the global diffeomorphism between Banach and Hilbert spaces (global inverse function theorem). The main assumption guarantying the existence of an implicit function $\lambda: Y \rightarrow X$ described by the equation $F(x, y)=0$ is a Palais-Smale condition connected with $F$, with respect to $x \in X$.

In the literature, some extensions of the local implicit function theorem to the global ones are known. In [6], the case of Banach spaces $X, Y, H$ is considered. Author uses a concept of "continuation property" which is equivalent to the socalled "path-lifting property". In [2], some variants of global implicit function theorems in the case of Banach spaces $X, Y$ and $H=Y$, have been obtained. The main assumptions are some inequalities involving partial differentials $F_{x}(x, y)$, $F_{y}(x, y)$ and a function $\omega:[0, \infty) \rightarrow[0, \infty)$ such that $\int_{1}^{\infty} \frac{d s}{\omega(s)}=\infty$. In paper [8], the authors consider the case of $X=\mathbb{R}^{m}, Y=\mathbb{R}^{n}, H=\mathbb{R}^{m}$. As a condition guarantying existence of an implicit function on the whole space they propose a "lower bound" condition imposed on the Jacobian $F_{x}(x, y)$. The comparison of the above results to our ones seems to be not so easy and remains an open question.

Our paper consists in two parts. In the first part, we derive a global implicit function theorem. The second part is devoted to some applications. First, we study the classical nonlinear ordinary Cauchy problem involving a functional parameter $u$ (nonlinearly). Next, we analyze an integro-differential Cauchy problem of Volterra type involving two functional parameters $u$ (nonlinearly) and $v$ (linearly). Problem of such a type but without $u$, was investigated in [3]. In both cases, we obtain existence, uniqueness and global, differentiable dependence of solutions on parameters. Differentials of the mappings describing these dependencies are also given.

[^0]2. Local implicit function theorem. We have the following classical local infi-nite-dimensional implicit function theorem ([4]).
Theorem 2.1. Let $X, Y, Z$ be real Banach spaces. If $U \subset X \times Y$ is an open set, $F=F(x, y): U \rightarrow Z$ is of class $C^{1}, F(a, b)=0$ and

- differential $F_{x}(a, b): X \rightarrow Z$ is bijective,
then there exist balls $B(a, r), B(b, \rho)$ and a function $\lambda: B(b, \rho) \rightarrow B(a, r)$ such that $B(a, r) \times B(b, \rho) \subset U$ and
- equations $F(x, y)=0$ and $\lambda(y)=x$ are equivalent in the set $B(a, r) \times B(b, \rho)$
- function $\lambda$ is of class $C^{1}$ with differential $\lambda^{\prime}(y)$ given by

$$
\begin{equation*}
\lambda^{\prime}(y)=-\left[F_{x}(\lambda(y), y)\right]^{-1} \circ F_{y}(\lambda(y), y) \tag{1}
\end{equation*}
$$

for $y \in B(b, \rho)$.
3. Mountain Pass Theorem. Let $X$ be a real Banach space and $I: X \rightarrow \mathbb{R}$ - a functional of class $C^{1}$. A point $u^{*} \in X$ is called the critical point of $I$ if $I^{\prime}\left(u^{*}\right)=0$. In such a case $I\left(u^{*}\right)$ is called the critical value of $I$.

A sequence $\left(u_{m}\right)$ satisfying conditions:

- $\left|I\left(u_{m}\right)\right| \leq M$ for all $m \in \mathbb{N}$ and some $M>0$,
- $I^{\prime}\left(u_{m}\right) \longrightarrow 0$,
is called the Palais-Smale (PS) sequence for functional $I$. We say that $I$ satisfies Palais-Smale (PS) condition if any (PS) sequence admits a convergent subsequence.

Let $d \neq 0$ be a point of $X$. By $W_{d}$ we denote the set

$$
W_{d}=\{U \subset X ; U \text { is open, } 0 \in U \text { and } d \notin \bar{U}\}
$$

We have ([1], [5])
Theorem 3.1 (Mountain Pass Theorem). Let $I: X \rightarrow \mathbb{R}$ be a functional of class $C^{1}$ satisfying (PS) condition and such that $I(0)=0$. If there exist constants $\rho$, $\alpha>0$ such that $\left.I\right|_{\partial B(0, \rho)} \geq \alpha$ and $I(e) \leq 0$ for some $e \in X \backslash \overline{B(0, \rho)}$, then

$$
b:=\sup _{U \in W_{e}} \inf _{u \in \partial U} I(u)
$$

is a critical value of $I$ and $b \geq \alpha$.
4. Global implicit function theorem. The main result of the paper is the following infinite-dimensional global implicit function theorem. The method of the proof is the same as in [3] in the case of the global inverse function theorem.

Theorem 4.1. Let $X, Y$ be real Banach spaces, $H$ - a real Hilbert space. If $F$ : $X \times Y \rightarrow H$ is of class $C^{1}$ and

- differential $F_{x}(x, y): X \rightarrow H$ is bijective for any $(x, y) \in X \times Y$
- the functional

$$
\begin{equation*}
\varphi: X \ni x \longmapsto(1 / 2)\|F(x, y)\|^{2} \in \mathbb{R} \tag{2}
\end{equation*}
$$

satisfies the $(P S)$ condition for any $y \in Y$,
then there exists a unique function $\lambda: Y \rightarrow X$ such that

- equations $F(x, y)=0$ and $\lambda(y)=x$ are equivalent in the set $X \times Y$.

This function is of class $C^{1}$ with differential $\lambda^{\prime}(y)$ given by (1) for $y \in Y$.

Proof. It is clear, in view of the local implicit function theorem, that it is sufficient to show that for any $y \in Y$ there exists exactly one $x \in X$ such that $F(x, y)=0$. So, let us fix a point $y \in Y$. Functional $\varphi$, being a superposition of two mappings of class $C^{1}$, is of the same type and its differential $\varphi^{\prime}(x)$ at $x \in X$ is given by

$$
\varphi^{\prime}(x) h=\left\langle F(x, y), F_{x}(x, y) h\right\rangle
$$

for $h \in X$. As a mapping of class $C^{1}$, bounded below and satisfying (PS) condition, $\varphi$ has a minimizer $x^{*}$ on $X([7$, Corollary 2.5]). Consequently,

$$
\left\langle F\left(x^{*}, y\right), F_{x}\left(x^{*}, y\right) h\right\rangle=0
$$

for $h \in X$. Since $F_{x}\left(x^{*}, y\right) X=H, F\left(x^{*}, y\right)=0$. Now, let us suppose that there exist $x_{1}, x_{2} \in X, x_{1} \neq x_{2}$, such that $F\left(x_{1}, y\right)=F\left(x_{2}, y\right)=0$. Let us put $e=x_{2}-x_{1}$ and

$$
g(x)=F\left(x+x_{1}, y\right)
$$

for $x \in X$. Of course,

$$
g(x)=g^{\prime}(0) x+o(x)=F_{x}\left(x_{1}, y\right) x+o(x)
$$

for $x \in X$, where $o(x) /\|x\|_{X} \rightarrow 0$ in $H$ when $x \rightarrow 0$ in $X$. So,

$$
\beta\|x\|_{X} \leq\left\|F_{x}\left(x_{1}, y\right) x\right\|_{H} \leq\|g(x)\|_{H}+\|o(x)\|_{H} \leq\|g(x)\|_{H}+(1 / 2) \beta\|x\|_{X}
$$

for sufficiently small $\|x\|_{X}$ and some $\beta>0$ (existence of such an $\beta$ follows from the bijectivity of $\left.F_{x}(x, y)\right)$. Thus, there exists $\rho>0$ such that

$$
(1 / 2) \beta\|x\|_{X} \leq\|g(x)\|_{H}
$$

for $x \in \overline{B(0, \rho)}$. Without loss of the generality one may assume that $\rho<\|e\|_{X}$. Put

$$
\psi(x)=(1 / 2)\|g(x)\|_{H}^{2}=(1 / 2)\left\|F\left(x+x_{1}, y\right)\right\|_{H}^{2}=\varphi\left(x+x_{1}\right)
$$

for $x \in X$. Of course, $\psi$ is of class $C^{1}$ and

$$
\psi^{\prime}(x)=\varphi^{\prime}\left(x+x_{1}\right)
$$

Consequently, since $\varphi$ satisfies (PS) condition, $\psi$ has this property, too. Moreover, $\psi(0)=\psi(e)=0, e \notin \overline{B(0, \rho)}$ and $\psi(x) \geq \alpha$ for $x \in \partial B(0, \rho)$ with $\alpha=(1 / 8) \beta^{2} \rho^{2}>$ 0 . Thus, $\psi: X \rightarrow \mathbb{R}$ satisfies assumptions of the Mountain Pass Theorem. So, $b=\sup _{U \in W_{e}} \inf _{x \in \partial U} \psi(x)$ is a critical value of $\psi$ and $b \geq \alpha$, i.e. there exists a point $x^{*} \in X$ such that $\psi\left(x^{*}\right)=b>0$ and

$$
\psi^{\prime}\left(x^{*}\right) h=\left\langle F\left(x^{*}+x_{1}, y\right), F_{x}\left(x^{*}+x_{1}, y\right) h\right\rangle=0
$$

for $h \in X$. The first condition means that $F\left(x^{*}+x_{1}, y\right) \neq 0$. The second one and equality $F_{x}\left(x^{*}+x_{1}, y\right) X=H$ imply that $F\left(x^{*}+x_{1}, y\right)=0$. The obtained contradiction completes the proof.

Remark 1. The first assumption can be replaced by a less restrictive one, namely: "differential $F_{x}(x, y): X \rightarrow H$ is bijective for any $(x, y) \in X \times Y$ such that $F(x, y)=$ 0 and $F(x, y) \in F_{x}(x, y) X$ for any $(x, y) \in X \times Y$ such that $F(x, y) \neq 0$ ".
Remark 2. When $X=\mathbb{R}^{n}, Y=\mathbb{R}^{m}$ and $H=\mathbb{R}^{n}$, assumption on $\varphi$ can be replaced by the following one: " $\varphi$ is coercive for any $y \in \mathbb{R}^{m}$, i.e. $\varphi(x) \rightarrow \infty$ when $|x| \rightarrow \infty$.
5. Applications. In this section, we give two examples illustrating the obtained results. First of them concerns a nonlinear differential Cauchy problem containing a functional parameter.

Example 1. Let us consider the following control system

$$
\begin{equation*}
x^{\prime}(t)=f(t, x(t), u(t)), t \in J=[0,1] \text { a.e. } \tag{3}
\end{equation*}
$$

where $f: J \times \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}, x \in A C_{0}^{2}=A C_{0}^{2}\left(J, \mathbb{R}^{n}\right)=\left\{x: J \rightarrow \mathbb{R}^{n} ; x\right.$ is absolutely continuous, $\left.x(0)=0, x^{\prime} \in L^{2}\left(J, \mathbb{R}^{n}\right)\right\}, u \in L^{\infty}\left(J, \mathbb{R}^{m}\right)$. On the function $f$ we assume that
(A1) $f(\cdot, x, u)$ is measurable on $J$ for any $x \in \mathbb{R}^{n}, u \in \mathbb{R}^{m} ; f(t, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^{n} \times \mathbb{R}^{m}$ for $t \in J$ a.e.
(A2) there exist functions $a, b \in L^{2}\left(J, \mathbb{R}_{0}^{+}\right), \gamma \in C\left(\mathbb{R}_{0}^{+}, \mathbb{R}_{0}^{+}\right)$such that

$$
|f(t, x, u)| \leq a(t)|x|+b(t) \gamma(|u|)
$$

for $t \in J$ a.e., $x \in \mathbb{R}^{n}, u \in \mathbb{R}^{m}$ and

$$
\int_{0}^{1}(a(t))^{2} t d t<1 / 8
$$

(A3) there exist functions $c, d \in L^{2}\left(J, \mathbb{R}_{0}^{+}\right), \alpha, \beta \in C\left(\mathbb{R}_{0}^{+}, \mathbb{R}_{0}^{+}\right)$such that

$$
\left|f_{x}(t, x, u)\right|,\left|f_{u}(t, x, u)\right| \leq c(t) \alpha(|x|)+d(t) \beta(|u|)
$$

for $t \in J$ a.e., $x \in \mathbb{R}^{n}, u \in \mathbb{R}^{m}$.
We shall check that the mapping

$$
\begin{gathered}
F: A C_{0}^{2} \times L^{\infty}\left(J, \mathbb{R}^{m}\right) \rightarrow L^{2}\left(J, \mathbb{R}^{n}\right) \\
F(x, u)=x^{\prime}(t)-f(t, x(t), u(t))
\end{gathered}
$$

satisfies assumptions of global implicit function theorem with $X=A C_{0}^{2}, Y=$ $L^{\infty}\left(J, \mathbb{R}^{m}\right), H=L^{2}\left(J, \mathbb{R}^{n}\right)$. In a standard way one can check that it is of class $C^{1}$ and the differential with respect to $x$

$$
F_{x}(x, u): A C_{0}^{2} \ni h \longmapsto h^{\prime}(\cdot)-f_{x}(\cdot, x(\cdot), u(\cdot)) h(\cdot) \in L^{2}\left(J, \mathbb{R}^{n}\right)
$$

is "one-one" and "onto" (cf. [4, Theorem 0.4.1]).
Now, let us fix a function $u \in L^{\infty}\left(J, \mathbb{R}^{m}\right)$ and consider the functional

$$
\varphi: A C_{0}^{2} \ni x \longmapsto(1 / 2)\|F(x, u)\|^{2}=(1 / 2) \int_{0}^{1}\left|x^{\prime}(t)-f(t, x(t), u(t))\right|^{2} \in \mathbb{R}
$$

It is easy to see that, for any $x \in A C_{0}^{2}$,

$$
|\varphi(x)| \geq(1 / 2)\|x\|_{A C_{0}^{2}}^{2}-\int_{0}^{1} x^{\prime}(t) f(t, x(t), u(t)) d t
$$

and

$$
\left|\int_{0}^{1} x^{\prime}(t) f(t, x(t), u(t)) d t\right| \leq\|x\|_{A C_{0}^{2}}\left(\int_{0}^{1}|f(t, x(t), u(t))|^{2} d t\right)^{(1 / 2)}
$$

Since

$$
\begin{aligned}
& \int_{0}^{1}|f(t, x(t), u(t))|^{2} d t \leq \int_{0}^{1}(a(t)|x(t)|+b(t) \gamma(|u(t)|))^{2} d t \\
& \leq 2 \int_{0}^{1}\left((a(t))^{2}|x(t)|^{2}+(b(t))^{2}(\gamma(|u(t)|))^{2}\right) d t \\
& \leq 2 \int_{0}^{1}(a(t))^{2} t d t \int_{0}^{1}\left|x^{\prime}(t)\right|^{2} d t+2 \int_{0}^{1}(b(t))^{2}(\gamma(|u(t)|))^{2} d t \\
& \quad \leq 2 \int_{0}^{1}(a(t))^{2} t d t\|x\|_{A C_{0}^{2}}^{2}+2 \int_{0}^{1}(b(t))^{2}(\gamma(|u(t)|))^{2} d t
\end{aligned}
$$

therefore

$$
\begin{aligned}
|\varphi(x)| & \geq(1 / 2)\|x\|_{A C_{0}^{2}}^{2}-\|x\|_{A C_{0}^{2}}\left(2 \int_{0}^{1}(a(t))^{2} t d t\|x\|_{A C_{0}^{2}}^{2}+2 \int_{0}^{1}(b(t))^{2}(\gamma(|u(t)|))^{2} d t\right)^{(1 / 2)} \\
& \geq\left(\left((1 / 4)-2 \int_{0}^{1}(a(t))^{2} t d t\right)\|x\|_{A C_{0}^{2}}^{4}-2 \int_{0}^{1}(b(t))^{2}(\gamma(|u(t)|))^{2} d t\|x\|_{A C_{0}^{2}}^{2}\right) \\
& /\left((1 / 2)\|x\|_{A C_{0}^{2}}^{2}+\|x\|_{A C_{0}^{2}}\left(2 \int_{0}^{1}(a(t))^{2} t d t\|x\|_{A C_{0}^{2}}^{2}+2 \int_{0}^{1}(b(t))^{2}(\gamma(|u(t)|))^{2} d t\right)^{(1 / 2)}\right)
\end{aligned}
$$

for $x \in A C_{0}^{2}$. This means that $\varphi$ is coercive.
In a standard way, we check that the differential $\varphi^{\prime}(x)$ of $\varphi$ at $x$ is given by

$$
\varphi^{\prime}(x) h=\int_{0}^{1}\left(x^{\prime}(t)-f(t, x(t), u(t))\right)\left(h^{\prime}(t)-f_{x}(t, x(t), u(t)) h(t)\right) d t
$$

for $h \in A C_{0}^{2}$. Consequently, for any $x_{m}, x_{0} \in A C_{0}^{2}$,

$$
\begin{equation*}
\left(\varphi^{\prime}\left(x_{m}\right)-\varphi^{\prime}\left(x_{0}\right)\right)\left(x_{m}-x_{0}\right)=\left\|x_{m}-x_{0}\right\|_{A C_{0}^{2}}^{2}+\sum_{i=1}^{5} \psi_{i}\left(x_{m}\right) \tag{4}
\end{equation*}
$$

where

$$
\begin{gathered}
\psi_{1}\left(x_{m}\right)=\int_{0}^{1} x_{m}^{\prime}(t)\left(f_{x}\left(t, x_{m}(t), u(t)\right)\left(x_{0}(t)-x_{m}(t)\right)\right) d t \\
\psi_{2}\left(x_{m}\right)=\int_{0}^{1} x_{0}^{\prime}(t)\left(f_{x}\left(t, x_{0}(t), u(t)\right)\left(x_{m}(t)-x_{0}(t)\right)\right) d t \\
\psi_{3}\left(x_{m}\right)=\int_{0}^{1} f\left(t, x_{m}(t), u(t)\right)\left(f_{x}\left(t, x_{m}(t), u(t)\right)\left(x_{m}(t)-x_{0}(t)\right)\right) d t \\
\psi_{4}\left(x_{m}\right)=\int_{0}^{1} f\left(t, x_{0}(t), u(t)\right)\left(f_{x}\left(t, x_{0}(t), u(t)\right)\left(x_{0}(t)-x_{m}(t)\right)\right) d t \\
\psi_{5}\left(x_{m}\right)=\int_{0}^{1}\left(f\left(t, x_{0}(t), u(t)\right)-f\left(t, x_{m}(t), u(t)\right)\right)\left(x_{m}^{\prime}(t)-x_{0}^{\prime}(t)\right) d t
\end{gathered}
$$

Now, we are in a position to prove that $\varphi$ satisfies (PS) condition. Indeed, if $\left(x_{m}\right)$ is a (PS) sequence for $\varphi$, then the coercivity of $\varphi$ implies its boundedness. Consequently, there exists a subsequence $\left(x_{m_{k}}\right)$ which is weakly convergent in $A C_{0}^{2}$ to some $x_{0}$ (we recall that the weak convergence of functions in $A C_{0}^{2}$ implies the
uniform convergence of these functions and weak convergence of their derivatives in $\left.L^{2}\left(I, \mathbb{R}^{n}\right)\right)$. Let us observe that

$$
\begin{array}{r}
\left|\psi_{1}\left(x_{m_{k}}\right)\right| \leq\left(\int_{0}^{1}\left|x_{m_{k}}^{\prime}(t)\right|^{2} d t\right)^{(1 / 2)}\left(\int_{0}^{1}\left|f_{x}\left(t, x_{m_{k}}(t), u(t)\right)\right|^{2}\left|x_{m_{k}}(t)-x_{0}(t)\right|^{2} d t\right)^{(1 / 2)} \\
\leq\left(\int_{0}^{1}\left|x_{m_{k}}^{\prime}(t)\right|^{2} d t\right)^{(1 / 2)}\left(\int_{0}^{1}\left(c(t) \alpha\left(\left|x_{m_{k}}(t)\right|\right)+d(t) \beta(|u(t)|)\right)^{2}\left|x_{m_{k}}(t)-x_{0}(t)\right|^{2} d t\right)^{(1 / 2)} \\
\leq C \max \left\{\left|x_{m_{k}}(t)-x_{0}(t)\right|^{2} ; t \in[0,1]\right\}
\end{array}
$$

where $C>0$ is a constant depending on the sequence $\left(x_{m_{k}}\right)$ and control $u$. Thus, from the uniform convergence of $\left(x_{m_{k}}\right)$ to $x_{0}$ the convergence of $\left(\psi_{1}\left(x_{m_{k}}\right)\right)$ to 0 follows. In the same way, one can check that sequences $\left(\psi_{i}\left(x_{m_{k}}\right)\right), i=2,3,4$, converge to 0 . Convergence of the sequence $\left(\psi_{5}\left(x_{m_{k}}\right)\right)$ to 0 follows from the weak convergence of $\left(x_{m_{k}}\right)$ to $x_{0}$ in $L^{2}\left(J, \mathbb{R}^{n}\right)$ and from the convergence of the sequence $\left(f\left(t, x_{m_{k}}(t), u(t)\right)\right)$ to $f\left(t, x_{0}(t), u(t)\right)$ in $L^{2}\left(J, \mathbb{R}^{n}\right)$ (the last convergence follows from the Lebesgue dominated convergence theorem). Thus, from (4) it follows that ( $x_{m_{k}}$ ) converges to $x_{0}$ in $A C_{0}^{2}$, i.e. $\varphi$ satisfies (PS) condition.

So, all assumptions of the global implicit function theorem are satisfied. Consequently, for any $u \in L^{\infty}\left(J, \mathbb{R}^{m}\right)$ there exists a unique solution $x_{u} \in A C_{0}^{2}$ of the equation (3) and the mapping

$$
\lambda: L^{\infty}\left(J, \mathbb{R}^{m}\right) \ni u \longmapsto x_{u} \in A C_{0}^{2}
$$

is of class $C^{1}$ with the differential $\lambda^{\prime}(u)$ at a point $u \in L^{\infty}\left(J, \mathbb{R}^{m}\right)$ given by

$$
L^{\infty}\left(J, \mathbb{R}^{m}\right) \ni g \longmapsto z_{g} \in A C_{0}^{2}
$$

where $z_{g}$ is such that

$$
z_{g}^{\prime}(t)-f_{x}\left(t, x_{u}(t), u(t)\right) z_{g}(t)=f_{u}\left(t, x_{u}(t), u(t)\right) g(t)
$$

a.e. on $J$.

An example of a function satisfying conditions (A1), (A2), (A3) is the function $f(t, x, u)=\frac{1}{3} t^{5} \sin x+\sqrt{t}\left(\sin ^{2} x\right) u$.

Second example concerns a nonlinear integro-differential Cauchy problem containing two functional parameters. It is a generalization of the Cauchy problem considered in [3] and illustrating the global inverse function theorem.

Example 2. Let us consider a nonlinear integro-differential control system of Volterra type

$$
\begin{equation*}
x^{\prime}(t)+\int_{0}^{t} \Phi(t, \tau, x(\tau), u(\tau)) d \tau=v(t), t \in J \text { a.e., } \tag{5}
\end{equation*}
$$

where $\Phi: P_{\Delta} \times \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}\left(P_{\Delta}=\{(t, \tau) \in J \times J ; \tau \leq t\}\right), x \in A C_{0}^{2}$, $u \in L^{2}\left(J, \mathbb{R}^{m}\right), v \in L^{2}\left(J, \mathbb{R}^{n}\right)$. On the function $\Phi$ we assume that
(B1) $\Phi(\cdot, \cdot, x, u)$ is measurable on $P_{\Delta}$ for any $x \in \mathbb{R}^{n}, u \in \mathbb{R}^{m} ; \Phi(t, \tau, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^{n} \times \mathbb{R}^{m}$ for $(t, \tau) \in P_{\Delta}$ a.e.
(B2) there exist a function $a \in L^{2}\left(P_{\Delta}, \mathbb{R}_{0}^{+}\right)$and a constant $b>0$ such that

$$
|\Phi(t, \tau, x, u)| \leq a(t, \tau)|x|+b|u|
$$

for $(t, \tau) \in P_{\Delta}$ a.e., $x \in \mathbb{R}^{n}, u \in \mathbb{R}^{m}$ and

$$
\int_{P_{\Delta}} a^{2}(t, \tau) d t d \tau<1 / 2
$$

(B3) there exist functions $c, e \in L^{2}\left(P_{\Delta}, \mathbb{R}_{0}^{+}\right), \alpha, \beta \in C\left(\mathbb{R}_{0}^{+}, \mathbb{R}_{0}^{+}\right)$and constants $d$, $p, C>0$ such that

$$
\begin{gathered}
\left|\Phi_{x}(t, \tau, x, u)\right| \leq c(t, \tau) \alpha(|x|)+d|u| \\
\left|\Phi_{u}(t, \tau, x, u)\right| \leq e(t, \tau) \beta(|x|)+p|u|
\end{gathered}
$$

for $(t, \tau) \in P_{\Delta}$ a.e., $x \in \mathbb{R}^{n}, u \in \mathbb{R}^{m}$ and

$$
\int_{0}^{t} c^{2}(t, \tau) d \tau \leq C, t \in J \text { a.e. }
$$

We shall show that the mapping

$$
\begin{gathered}
F: A C_{0}^{2} \times L^{2}\left(J, \mathbb{R}^{m+n}\right) \rightarrow L^{2}\left(J, \mathbb{R}^{n}\right) \\
F(x, u, v)=x^{\prime}(t)+\int_{0}^{t} \Phi(t, \tau, x(\tau), u(\tau)) d \tau-v(t)
\end{gathered}
$$

satisfies assumptions of the global implicit function theorem with $X=A C_{0}^{2}, Y=$ $L^{2}\left(J, \mathbb{R}^{m+n}\right), H=L^{2}\left(J, \mathbb{R}^{n}\right)$.

In a standard way, one can check that $F$ is of class $C^{1}$ and the mappings

$$
\begin{gathered}
F_{x}(x, u, v): A C_{0}^{2} \rightarrow L^{2}\left(J, \mathbb{R}^{n}\right), \\
F_{x}(x, u, v) h=h^{\prime}(t)+\int_{0}^{t} \Phi_{x}(t, \tau, x(\tau), u(\tau)) h(\tau) d \tau \\
F_{u, v}(x, u, v): L^{2}\left(J, \mathbb{R}^{m}\right) \times L^{2}\left(J, \mathbb{R}^{n}\right) \rightarrow L^{2}\left(J, \mathbb{R}^{n}\right), \\
F_{u, v}(x, u, v)(f, g)=\int_{0}^{t} \Phi_{u}(t, \tau, x(\tau), u(\tau)) f(\tau) d \tau-g(t)
\end{gathered}
$$

are the differentials of $F$ in $x$ and $(u, v)$, respectively.
Let us fix a function $(u, v) \in L^{2}\left(J, \mathbb{R}^{m+n}\right)$. The mapping

$$
\begin{gathered}
\widetilde{\Phi}: P_{\Delta} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \\
\widetilde{\Phi}(t, \tau, x)=\Phi(t, \tau, x, u(t))
\end{gathered}
$$

satisfies assumptions of Theorem 4.1 from the paper [3]. Since

$$
h^{\prime}(t)+\int_{0}^{t} \Phi_{x}(t, \tau, x(\tau), u(\tau)) h(\tau) d \tau=h^{\prime}(t)+\int_{0}^{t} \widetilde{\Phi}_{x}(t, \tau, x(\tau)) h(\tau) d \tau
$$

therefore, just as in [3], one can show that the mapping $F_{x}(x, u, v)$ is "one-one" and "onto" for any $x \in A C_{0}^{2}$.

Moreover, in the same way as in the proof of Theorem 4.1 from [3], one can show that, for any fixed $(u, v) \in L^{2}\left(J, \mathbb{R}^{m+n}\right)$, the functional

$$
\varphi: A C_{0}^{2} \rightarrow \mathbb{R}
$$

$\varphi(x)=(1 / 2)\|F(x, u, v)\|_{L^{2}\left(J, \mathbb{R}^{n}\right)}^{2}=(1 / 2) \int_{0}^{1}\left|x^{\prime}(t)+\int_{0}^{t} \widetilde{\Phi}(t, \tau, x(\tau)) d \tau-v(t)\right|^{2} d t$ satisfies (PS) condition.

Thus, from the global implicit function theorem it follows that for any $(u, v) \in$ $L^{2}\left(I, \mathbb{R}^{m+n}\right)$ there exists a unique solution $x_{u, v} \in A C_{0}^{2}$ of the equation (5) and the mapping

$$
\lambda: L^{2}\left(J, \mathbb{R}^{m+n}\right) \ni(u, v) \longmapsto x_{u, v} \in A C_{0}^{2}
$$

is of class $C^{1}$ with the differential $\lambda^{\prime}(u, v)$ at a point $(u, v) \in L^{2}\left(J, \mathbb{R}^{m+n}\right)$ of the form

$$
L^{2}\left(J, \mathbb{R}^{m+n}\right) \ni(f, g) \longmapsto z_{f, g} \in A C_{0}^{2}
$$

where $z_{f, g}$ is such that

$$
\begin{aligned}
& z_{f, g}^{\prime}(t)+\int_{0}^{t} \Phi_{x}\left(t, \tau, x_{u, v}(\tau), u(\tau)\right) z_{f, g}(\tau) d \tau \\
& =-\int_{0}^{t} \Phi_{u}\left(t, \tau, x_{u, v}(\tau), u(\tau)\right) f(\tau) d \tau+g(t)
\end{aligned}
$$

a.e. on $J$.

An example of a function satisfying conditions (B1), (B2), (B3) is the function $\Phi(t, \tau, x, u)=\frac{1}{3} t^{5} \sqrt{\tau} \sin x+\sqrt{t} \tau^{3}\left(\sin ^{2} x\right) \sin u$.

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E-mail address: idczak@math.uni.lodz.pl


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