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The Rational r- Powers of Bernstein-Kantorovich Sequence

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Abstract.

The rational r-powers of Bernstein-Kantorovich sequence $KB_{n,r}(f;\, x)$ is defined and studied in this paper. In the beginning the study shows that the sequence $KB_{n,r}\ (f;\, x)$ is converged to the function $f \in C[0,1]$ whenever $n \to \infty$. Next, for this sequence, the moments of order m and the Voronovskaja-type asymptotic formula are given. Also, a numerical application for the sequence $KB_{n,r}(f;\, x)$ is given for the test function $f(t) = \sin 10t$ to explain the convergence properties and compared with the numerical results of the classical Bernstein-Kantorovich sequence $KB_n(f; x)$. It shows that, the sequence $KB_{n,r}\ (f;\, x)$ has better numerical approximation properties than the sequence $KB_n(f; x)$.

Keywords: Voronovskaja-type asymptotic formula, Ordinary approximation, Rational Bernstein sequence.

MSC 2010: 41A25, 41A36.

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2.1 Introduction.

The well-known Bernstein sequence of order n:

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x)\, f\left(\frac{k}{n}\right),$$

where $f \in C[0,1]$, $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $x \in [0,1]$.

In 1930, Kantorovich [6], gave a modification to the sequence of Bernstein for a function $f \in C[0,1]$ as:

$$KB_n(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\ dt,$$

where $k = 0,1,2,\dots,n$.

After that, some other generalizations and modifications have been introduced by some researchers (see, [3], [4], [5]).

In 2000, Cal and Vall [2], introduced a class of sequences that are general to particular many cases.

In 2013, Mahmudov and Sabancigil [8], have introduced a generalization of q-type Bernstein-Kantorovich sequence as follows:

$$B_{n,q}(f;x) = \sum_{k=0}^{n} b_{n,k}(q,x) \int_{0}^{1} f\left(\frac{[k] + q^k t}{[n+1]}\right) d_q t.$$

where $f \in C[0,1]$, $0 < q < 1$.

In 2018, Acu, Manav and Sofonea [1], have studied the approximation properties and asymptotic type results concerning the Kantorovich variant of λ-Bernstein sequences.

$$KB_{n,\lambda}(f;x) = (n+1)\sum_{k=0}^{n} \hat{b}_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\ dt,$$

where $\lambda \in [-1,1]$ and $\hat{b}_{n,k}(\lambda;x) = b_{n,k}(x) + \lambda\left(\frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)\right)$.

In 2019, Karahan and Izgi [7], have studied the generalized of Bernstein-Kantorovich sequences for the function of two variables.

$$KB_{n,m}(f;x,y)=\frac{n+1}{2}\frac{m+1}{2}\sum_{k=0}^{n}\sum_{j=0}^{m}\varphi_{n,k}^{k,j}(x,y)\int_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1}\int_{2\frac{j}{m+1}-1}^{2\frac{j+1}{m+1}-1}f(t,u)\ dtdu\,,$$

where $f\in C(A)$, $A=[-1,1]\times[-1,1]$, $n,m\in\mathbb{N}$, $\varphi_{n,m}^{k,j}(x,y)=b_{n,k}(x)b_{m,j}(y)$, and $b_{n,k}(x)=\frac{1}{2^n}\binom{n}{k}(1+x)^k(1-x)^{n-k}$.

In 2020, Mohiuddine and Özger [10], construct Stancu-type Bernstein-Kantorovich sequences based on parameter α.

$$S_{n,\alpha}^{\theta,\beta}(f;y)=(n+\beta+1)\sum_{i=0}^{n}p_{n,j}^{(\alpha)}(y)\int_{\frac{i+\theta}{n+\beta+1}}^{\frac{j+\theta+1}{n+\beta+1}}f(s)ds,$$

where $p_{n,j}^{(\alpha)}(y)=\left[(1-\alpha)y\binom{n-2}{j}+(1-\alpha)(1-y)\binom{n-2}{j-2}+\alpha y(1-y)\binom{n}{j}\right]y^{j-1}(1-y)^{n-j-1}$ $n\geq 2$ and $\alpha,y\in[0,1]$.

In 2021. Mohammad and Abdul Samad [9], have introduced and studied the rational r- powers of the Bernstein sequence $B_{n,r}(f;x)$ for $f\in C[0,1]$ and $r\in\mathbb{N}$:={1, 2, …}as follows:

$$B_{n,r}(f;x)=\frac{\sum_{k=0}^{n}b_{n,k}^{r}(x)f\left(\frac{k}{n}\right)}{\sum_{k=0}^{n}b_{n,k}^{r}(x)}$$

In this paper, the rational r-power of the Bernstein-Kantorovich sequence $KB_{n,r}(f(t);x)$ is defined as:

$$KB_{n,r}(f(t);x)=\frac{(n+1)\sum_{k=0}^{n}b_{n,k}^{r}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(t)dt}{\sum_{k=0}^{n}b_{n,k}^{r}(x)},$$

Where $f\in C[0,1]$, $r\in\mathbb{N}$ and $b_{n,k}^{r}(x)=\left(\binom{n}{k}x^k(1-x)^{n-k}\right)^r$.

For the sequence $KB_{n,r}(f(t);x)$, the moment of order m, Korovkin theorem, and the Voronovskaja -type asymptotic formula is studied.

2. Preliminary Results.

Some preliminaries relative to the sequence r-th powers of the rational Bernstein-Kantorovich polynomials are introduced here.

Lemma 2.1. [9]

(i) $B_{n,r}(1;x) = 1;$

(ii) $B_{n,r}(t;x) \simeq \frac{(r(n+1)-1)x}{rn} + \frac{1-r}{2rn};$

(iii) $B_{n,r}(t^2;x) \simeq \frac{(r(n+1)-1)(r(n+1)-2)}{r^2n^2}x^2 + \frac{(2-r)(r(n+1)-1)}{r^2n^2}x + \frac{(1-r)^2}{4r^2n^2};$

(iv) $B_{n,r}(t^m;x) = \frac{(r(n+1)-1)!}{r^m n^m (r(n+1)-m-1)!}x^m + \frac{m(m-r)}{2}\frac{(r(n+1)-1)!}{r^m n^m (r(n+1)-m)!}x^{m-1} + TLP(x).$

Lemma 2.2.

For $x \in [0,1], m \in \mathbb{N}^0 := \{0,1,2,\ldots\}$, the following conditions are satisfied

(i) $KB_{n,r}(1;x) = 1;$

(ii) $KB_{n,r}(t;x) \simeq \frac{(r(n+1)-1)x}{r(n+1)} + \frac{1}{2r(n+1)};$

(iii) $KB_{n,r}(t^2;x) \simeq \frac{(r(n+1)-1)(r(n+1)-2)}{r^2(n+1)^2}x^2 + \frac{2(r(n+1)-1)}{r^2(n+1)^2}x + \frac{r^2+3}{12r^2(n+1)^2};$

(iv) $KB_{n,r}(t^m;x) = \frac{(r(n+1)-1)!}{r^m(n+1)^m((n+1)r-m-1)!}x^m + \frac{m(r(n+1)-1)!}{2r^m(n+1)^m((n+1)r-m)!}x^{m-1} + TLP(x).$

Where $TLP(x)$ means terms in lower powers of x.

Proof.

The proof of the above polynomials is going as:

By direct evaluation, one has

$KB_{n,r}(1;x) = 1.$

To prove (ii)

$$KB_{n,r}(t;x) = \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t\,dt$$

$$= \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} \left[\frac{t^2}{2}\right]_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} = \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)}{2\sum_{k=0}^{n} b_{n,k}^r(x)} \left[\left(\frac{k+1}{n+1}\right)^2 - \left(\frac{k}{n+1}\right)^2\right]$$

$$= \frac{2(n+1)\sum_{k=0}^{n} k b_{n,k}^r(x)}{2(n+1)^2\sum_{k=0}^{n} b_{n,k}^r(x)} + \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)}{2(n+1)^2\sum_{k=0}^{n} b_{n,k}^r(x)},$$

Applying Lemma 2.1, one has

$$KB_{n,r}(t;x) \simeq \frac{(r(n+1)-1)x}{r(n+1)} + \frac{1-r}{2r(n+1)} + \frac{1}{2(n+1)}$$

$$\simeq \frac{(r(n+1)-1)x}{r(n+1)} + \frac{1}{2r(n+1)}.$$

To prove (iii)

$$KB_{n,r}(t^2;x) = \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t^2 dt$$

$$= \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)}{\sum_{k=0}^{n} b_{n,k}^{r}(x)} \left[\frac{t^3}{3}\right]_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} = \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)}{3\sum_{k=0}^{n} b_{n,k}^{r}(x)} \left[\left(\frac{k+1}{n+1}\right)^3 - \left(\frac{k}{n+1}\right)^3\right]$$

$$= \frac{3\sum_{k=0}^{n} k^2 b_{n,k}^{r}(x)}{3(n+1)^2\sum_{k=0}^{n} b_{n,k}^{r}(x)} + \frac{3\sum_{k=0}^{n} k b_{n,k}^{r}(x)}{3(n+1)^2\sum_{k=0}^{n} b_{n,k}^{r}(x)} + \frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)}{3(n+1)^2\sum_{k=0}^{n} b_{n,k}^{r}(x)},$$

Applying Lemma 2.1, one has

$$\simeq \frac{(r(n+1)-1)(r(n+1)-2)}{r^2(n+1)^2}x^2 + \frac{(2-r)(r(n+1)-1)}{r^2(n+1)^2}x + \frac{(1-r)^2}{4r^2(n+1)^2}$$

$$+ \frac{(r(n+1)-1)}{r(n+1)^2}x + \frac{1-r}{2r(n+1)^2} + \frac{1}{3(n+1)^2}$$

$$\simeq \frac{(r(n+1)-1)(r(n+1)-2)}{r^2(n+1)^2}x^2 + \frac{(r(n+1)-1)(2-r+r)}{r^2(n+1)^2}x$$

$$+ \frac{(1-r)^2}{4r^2(n+1)^2} + \frac{1-r}{2r(n+1)^2} + \frac{1}{3(n+1)^2}.$$

$$KB_{n,r}(t^2;x) \simeq$$

$$\frac{(r(n+1)-1)(r(n+1)-2)}{r^2(n+1)^2}x^2 + \frac{2(r(n+1)-1)}{r^2(n+1)^2}x + \frac{r^2+3}{12r^2(n+1)^2}.$$

In general,

$$KB_{n,r}(t^m;x) = \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)}{\sum_{k=0}^{n} b_{n,k}^{r}(x)} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t^m dt$$

$$= \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)}{\sum_{k=0}^{n} b_{n,k}^{r}(x)} \left[\frac{t^{m+1}}{m+1}\right]_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} = \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)}{m+1\sum_{k=0}^{n} b_{n,k}^{r}(x)} \left[\left(\frac{k+1}{n+1}\right)^{m+1} - \left(\frac{k}{n+1}\right)^{m+1}\right]$$

$$= \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)}{(n+1)^{m+1}(m+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)}\left[(k+1)^{m+1} - k^{m+1}\right]$$

$$= \frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)}{(n+1)^m(m+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$\times\left\{k^{m+1} + (m+1)k^m + \frac{m(m+1)}{2}k^{m-1} + \cdots + (m+1)k + 1 - k^{m+1}\right\}$$

$$= \frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)}{(m+1)(n+1)^m\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$\times\left\{(m+1)k^m + \frac{m(m+1)}{2}k^{m-1} + \cdots + (m+1)k\right\} + \frac{1}{(n+1)^m(m+1)}$$

$$= \frac{(m+1)\sum_{k=0}^{n} k^m b_{n,k}^{r}(x)}{(n+1)^m(m+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)} + \frac{m(m+1)\sum_{k=0}^{n} k^{m-1} b_{n,k}^{r}(x)}{2(n+1)^m(m+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)} + TLP(x)$$

$$=\frac{\sum_{k=0}^{n}k^m b_{n,k}^r(x)}{(n+1)^m\sum_{k=0}^{n}b_{n,k}^r(x)}+\frac{m\sum_{k=0}^{n}k^{m-1}b_{n,k}^r(x)}{2(n+1)^m\sum_{k=0}^{n}b_{n,k}^r(x)}+TLP(x),$$

$$=\frac{1}{(n+1)^m}\left(\frac{(r(n+1)-1)!}{r^m(r(n+1)-m-1)!}x^m+\frac{m(m-r)}{2}\frac{(r(n+1)-1)!}{r^m(r(n+1)-m)!}x^{m-1}+TLP(x)\right)$$

$$+\frac{m}{2(n+1)^m}\left(\frac{(r(n+1)-1)!}{r^{m-1}((n+1)r-m)!}x^{m-1}\right.$$

$$\left.+\frac{(m-1)(m-1-r)}{2}\frac{(r(n+1)-1)!}{r^{m-1}(r(n+1)-m+1)!}x^{m-2}+TLP(x)\right)$$

$$=\frac{(r(n+1)-1)!}{r^m(n+1)^m(r(n+1)-m-1)!}x^m$$

$$+\left(\frac{m(m-r)}{2}\frac{(r(n+1)-1)!}{r^m(n+1)^m(r(n+1)-m)!}x^{m-1}+\frac{m}{2(n+1)^m}\frac{(r(n+1)-1)!}{r^{m-1}(r(n+1)-m)!}x^{m-1}\right)+TLP(x)$$

$$=\frac{((n+1)r-1)!}{r^m(n+1)^m(r(n+1)-m-1)!}x^m$$

$$+\left(\frac{m((n+1)r-1)!}{2r^{m-1}(n+1)^m(r(n+1)-m)!}x^{m-1}\left(\frac{m-r+r}{r}\right)\right)+TLP(x)$$

$KB_{n,r}(t^m;x)$

$$=\frac{(r(n+1)-1)!}{r^m(n+1)^m(r(n+1)-m-1)!}x^m+\frac{m(r(n+1)-1)!}{2r^m(n+1)^m(r(n+1)-m)!}x^{m-1}$$

$+TLP(x)$. ■

For $m\in\mathbb{N}^0$, we define the following:

The m-th order moment $\mu_{n,m}^r(x)$ of the polynomials $KB_{n,r}(f(t);x)$

$$\mu_{n,m}^r(x)=\frac{(n+1)\sum_{k=0}^{n}b_{n,k}^r(x)}{\sum_{k=0}^{n}b_{n,k}^r(x)}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^m dt.$$

The two functions $\omega_{n+1,m+1}(x)$ and $\varphi_{n+1,m+1}(x)$

$$\omega_{n+1,m+1}(x)=\frac{\sum_{k=0}^{n}b_{n,k}^r(x)\left(\frac{k+1}{n+1}-x\right)^{m+1}}{\sum_{k=0}^{n}b_{n,k}^r(x)}$$

$$\varphi_{n+1,m+1}(x)=\frac{\sum_{k=0}^{n}b_{n,k}^r(x)\left(\frac{k}{n+1}-x\right)^{m+1}}{\sum_{k=0}^{n}b_{n,k}^r(x)}.$$

The next lemma shows the relation between the two functions above and the function $\mu_{n,m}^r(x)$.

Lemma 2.3.

For $m \in \mathbb{N}^0$, the functions $\mu^r_{n,m}(x)$ have

$$\mu^r_{n,m}(x) = \frac{n+1}{m+1}\Big(\omega_{n+1,m+1}(x) - \varphi_{n+1,m+1}(x)\Big).$$

Proof.

$$\mu^r_{n,m}(x) = \frac{(n+1)\sum_{k=0}^{n} b^r_{n,k}(x)}{\sum_{k=0}^{n} b^r_{n,k}(x)} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^m dt$$

$$= \frac{(n+1)\sum_{k=0}^{n} b^r_{n,k}(x)}{\sum_{k=0}^{n} b^r_{n,k}(x)} \left[\frac{(t-x)^{m+1}}{m+1}\right]_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}$$

$$= \frac{(n+1)\sum_{k=0}^{n} b^r_{n,k}(x)}{(m+1)\sum_{k=0}^{n} b^r_{n,k}(x)} \left[\left(\frac{k+1}{n+1} - x\right)^{m+1} - \left(\frac{k}{n+1} - x\right)^{m+1}\right]$$

$$\mu^r_{n,m}(x) = \frac{n+1}{m+1}\left[\frac{\sum_{k=0}^{n} b^r_{n,k}(x)\left(\frac{k+1}{n+1} - x\right)^{m+1}}{\sum_{k=0}^{n} b^r_{n,k}(x)} - \frac{\sum_{k=0}^{n} b^r_{n,k}(x)\left(\frac{k}{n+1} - x\right)^{m+1}}{\sum_{k=0}^{n} b^r_{n,k}(x)}\right]$$

$$\mu^r_{n,m}(x) = \frac{n+1}{m+1}\Big(\omega_{n+1,m+1}(x) - \varphi_{n+1,m+1}(x)\Big) \blacksquare$$

Lemma 2.4.

The functions $\omega_{n+1,m+1}(x)$ and $\varphi_{n+1,m+1}(x)$ have the following recurrence relations

(i) $\omega_{n+1,m+2}(x) = \frac{x(1-x)}{r(n+1)}\Big(\omega'_{n+1,m+1}(x) + (m+1)\omega_{n+1,m}(x)\Big) + \omega_{n+1,m+1}(x)\omega_{n+1,1}(x),$

where $\omega_{n+1,1}(x) \simeq \frac{(1-2x)+r}{2r(n+1)}$, $\omega_{n+1,2}(x) \simeq \frac{(2-r(n+1))x^2}{r^2(n+1)^2} - \frac{(2-rn)x}{r^2(n+1)^2} + \frac{1}{4r^2(n+1)^2}.$

(ii) $\varphi_{n+1,m+2}(x) = \frac{x(1-x)}{r(n+1)}\Big(\varphi'_{n+1,m+1}(x) + (m+1)\varphi_{n+1,m}(x)\Big) + \varphi_{n+1,m+1}(x)\varphi_{n+1,1}(x)$

where $\varphi_{n+1,1}(x) \simeq \frac{(1-2x)-r}{2r(n+1)}$, $\varphi_{n+1,2}(x) \simeq \frac{(2-r(n+1))}{r^2(n+1)^2}x^2 - \frac{(2-r(n+2))}{r^2(n+1)^2}x + \frac{(1-r)^2}{4r^2(n+1)^2}.$

Proof.

The proof of the consequence (i) is going as:

$$\omega_{n+1,1}(x) = \frac{\sum_{k=0}^{n} b^r_{n,k}(x)\left(\frac{k+1}{n+1} - x\right)}{\sum_{k=0}^{n} b^r_{n,k}(x)} = \frac{\sum_{k=0}^{n} k b^r_{n,k}(x)}{(n+1)\sum_{k=0}^{n} b^r_{n,k}(x)} + \frac{1}{n+1} - x$$

$$\simeq \frac{(r(n+1)-1)x}{r(n+1)} + \frac{1-r}{2r(n+1)} + \frac{1}{(n+1)} - x$$

$$\simeq \frac{2rnx + 2rx - 2x + 1 - r + 2r - 2rnx - 2rx}{2r(n+1)} \simeq \frac{(1-2x)+r}{2r(n+1)}.$$

$$\omega_{n+1,2}(x) = \frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)\left(\frac{k+1}{n+1} - x\right)^2}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$\simeq \frac{\sum_{k=0}^{n} k^2 b_{n,k}^{r}(x)}{(n+1)^2 \sum_{k=0}^{n} b_{n,k}^{r}(x)} + 2\frac{\sum_{k=0}^{n} k b_{n,k}^{r}(x)}{(n+1)^2 \sum_{k=0}^{n} b_{n,k}^{r}(x)} + \frac{1}{(n+1)^2} - 2x\frac{\sum_{k=0}^{n} k b_{n,k}^{r}(x)}{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$-\frac{2x}{(n+1)} + x^2$$

$$\simeq \frac{(r(n+1)-1)(r(n+1)-2)}{r^2(n+1)^2}x^2 + \frac{(2-r)(r(n+1)-1)}{r^2(n+1)^2}x + \frac{(1-r)^2}{4r^2(n+1)^2}$$

$$+2\left(\frac{(r(n+1)-1)x}{r(n+1)^2} + \frac{1-r}{2r(n+1)^2}\right) + \frac{1}{(n+1)^2}$$

$$-2x\left(\frac{(r(n+1)-1)x}{r(n+1)} + \frac{1-r}{2r(n+1)}\right) - \frac{2x}{(n+1)} + x^2$$

$$\simeq \left(\frac{r^2(n+1)^2 - 3r(n+1) + 2}{r^2(n+1)^2} - \frac{2r(n+1)-2}{r(n+1)} + 1\right)x^2 +$$

$$\left(\frac{2r(n+1)-2}{r^2(n+1)^2} - \frac{r^2(n+1)-r}{r^2(n+1)^2} + \frac{2r(n+1)-2}{r(n+1)^2} - \frac{1-r}{r(n+1)} - \frac{2x}{(n+1)}\right)x$$

$$+\frac{(1-r)^2}{4r^2(n+1)^2} + \frac{1-r}{r(n+1)^2} + \frac{1}{(n+1)^2}$$

$$\omega_{n+1,2}(x) \simeq \frac{\left(2-r(n+1)\right)x^2}{r^2(n+1)^2} - \frac{(2-rn)x}{r^2(n+1)^2} + \frac{(1+r)^2}{4r^2(n+1)^2}.$$

Now,

$$\omega_{n+1,m+1}(x) = \frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)\left(\frac{k+1}{n+1} - x\right)^{m+1}}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$\omega'_{n+1,m+1}(x) = \frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)\sum_{k=0}^{n} b_{n,k}^{r}(x)\left((-1)(m+1)\left(\frac{k+1}{n+1} - x\right)^{m}\right)}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)^2}$$

$$+\frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)\sum_{k=0}^{n}\left(b_{n,k}^{r}(x)\right)'\left(\frac{k+1}{n+1} - x\right)^{m+1}}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)^2}$$

$$-\frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)\left(\frac{k+1}{n+1} - x\right)^{m+1}\sum_{k=0}^{n}\left(b_{n,k}^{r}(x)\right)'}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)^2}$$

$$= -(m+1)\omega_{n+1,m}(x) + \frac{\sum_{k=0}^{n}\left(b_{n,k}^{r}(x)\right)'\left(\frac{k+1}{n+1} - x\right)^{m+1}}{\sum_{k=0}^{n} b_{n,k,}^{r}(x)}$$

$$-\frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)\left(\frac{k+1}{n+1}-x\right)^{m+1}\sum_{k=0}^{n}\left(b_{n,k}^{r}(x)\right)'}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)^{2}}$$

$$x(1-x)\left(\omega_{n+1,m+1}'(x)+(m+1)\omega_{n+1,m}(x)\right)$$

$$=\frac{\sum_{k=0}^{n} x(1-x)\left(b_{n,k}^{r}(x)\right)'\left(\frac{k+1}{n+1}-x\right)^{m+1}}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$-\frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)\left(\frac{k+1}{n+1}-x\right)^{m+1}\sum_{k=0}^{n} x(1-x)\left(b_{n,k}^{r}(x)\right)'}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)^{2}},$$

by using the fact $x(1-x)\left(b_{n,k}^{r}(x)\right)'=(kr-nrx)b_{n,k}^{r}(x)$, one has

$$=\frac{r\sum_{k=0}^{n} b_{n,k}^{r}(x)(k-nx+x-x+1-1)\left(\frac{k+1}{n+1}-x\right)^{m+1}}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$-\frac{\sum_{k=0}^{n} b_{n,k}^{r}(x)\left(\frac{k+1}{n+1}-x\right)^{m+1} r\sum_{k=0}^{n} b_{n,k}^{r}(x)(k-nx+x-x+1-1)}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)^{2}}$$

$$=\frac{r(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)\left(\left(\frac{k+1}{n+1}-x\right)+\frac{(x-1)}{n+1}\right)\left(\frac{k+1}{n+1}-x\right)^{m+1}}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$-\frac{r(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x)\left(\frac{k+1}{n+1}-x\right)^{m+1}\sum_{k=0}^{n} b_{n,k}^{r}(x)\left(\left(\frac{k+1}{n+1}-x\right)+\frac{(x-1)}{n+1}\right)}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)^{2}}$$

$$=r(n+1)\left(\omega_{n+1,m+2}(x)\right)+r(x-1)\omega_{n+1,m+1}(x)-r(n+1)\omega_{n+1,m+1}(x)\omega_{n+1,1}(x)$$

$$-r(x-1)\omega_{n+1,m+1}(x)$$

$$=r(n+1)\omega_{n+1,m+2}(x)-r(n+1)\omega_{n+1,m+1}(x)\omega_{n+1,1}(x),$$

hence,

$$\omega_{n+1,m+2}(x)=\frac{x(1-x)}{r(n+1)}\left(\omega_{n+1,m+1}'(x)+(m+1)\omega_{n+1,m}(x)\right)+\omega_{n+1,m+1}(x)\omega_{n+1,1}(x).$$

Using similar steps, we have

$$\varphi_{n+1,1}(x)\simeq\frac{(1-2x)-r}{2r(n+1)};$$

$$\varphi_{n+1,2}(x)\simeq\frac{(2-r(n+1))}{r^{2}(n+1)^{2}}x^{2}-\frac{(2-r(n+2))}{r^{2}(n+1)^{2}}x+\frac{(1-r)^{2}}{4r^{2}(n+1)^{2}};$$

$$\varphi_{n+1,m+2}(x)=\frac{x(1-x)}{r(n+1)}\left(\varphi'_{n+1,m+1}(x)+(m+1)\varphi_{n+1,m}(x)\right)+\varphi_{n+1,m+1}(x)\varphi_{n+1,1}.\ \blacksquare$$

Lemma 2.5.

The moment $\left(\mu^r_{n,m}(x)\right)$ of order m has the recurrence relation

$$\mu^r_{n,m+1}(x)=\frac{(m+1)x(1-x)}{r(m+2)(n+1)}\left(\left(\mu^r_{n,m}(x)\right)'+m\mu^r_{n,m-1}(x)\right)+\frac{(m+1)}{(m+2)}\frac{(1-2x)}{r(n+1)}\mu^r_{n,m}(x)+$$
$$\frac{1}{2(m+2)}\left(\omega_{n+1,m+1}(x)+\varphi_{n+1,m+1}\right),$$

where $\mu^r_{n,0}(x)=1$, and $\mu^r_{n,1}(x)\simeq\frac{(1-2x)}{2r(n+1)}$.

Proof.

From Lemma 2.3, one gets

$$\frac{1}{n+1}\mu^r_{n,0}(x)=\omega_{n+1,1}(x)-\varphi_{n+1,1}(x)$$
$$\frac{1}{n+1}\mu^r_{n,0}(x)\simeq\frac{(1-2x)+r}{2r(n+1)}-\frac{(1-2x)-r}{2r(n+1)}\simeq\frac{2r}{2r(n+1)}\simeq\frac{1}{n+1}$$

$\mu^r_{n,0}(x)=1$, and

$$\frac{2}{n+1}\mu^r_{n,1}(x)=\omega_{n+1,2}(x)-\varphi_{n+1,2}(x)$$
$$\frac{2}{n+1}\mu^r_{n,1}(x)\simeq\left(\frac{\left(2-r(n+1)\right)}{r^2(n+1)^2}x^2-\frac{(2-rn)}{r^2(n+1)^2}x+\frac{(1+r)^2}{4r^2(n+1)^2}\right)$$
$$-\left(\frac{\left(2-r(n+1)\right)}{r^2(n+1)^2}x^2-\frac{\left(2-r(n+2)\right)}{r^2(n+1)^2}x+\frac{(1-r)^2}{4r^2(n+1)^2}\right)$$
$$\simeq\left(-\frac{(2-rn)}{r^2(n+1)^2}x+\frac{(1+r)^2}{4r^2(n+1)^2}\right)-\left(\frac{-\left(2-r(n+2)\right)}{r^2(n+1)^2}x+\frac{(1-r)^2}{4r^2(n+1)^2}\right)$$
$$\mu^r_{n,1}(x)\simeq\frac{1-2x}{2r(n+1)}.$$

Then,

$$\omega_{n+1,m+2}(x)-\varphi_{n+1,m+2}(x)$$
$$=\left(\frac{x(1-x)}{r(n+1)}\left(\omega'_{n+1,m+1}(x)+(m+1)\omega_{n+1,m}(x)\right)+\omega_{n+1,m+1}(x)\omega_{n+1,1}(x)\right.$$
$$\left.-\frac{x(1-x)}{r(n+1)}\left(\varphi'_{n+1,m+1}(x)+(m+1)\varphi_{n+1,m}(x)\right)+\varphi_{n+1,m+1}(x)\varphi_{n+1,1}(x)\right)$$
$$=\frac{x(1-x)}{r(n+1)}\left\{\left(\omega'_{n+1,m+1}(x)+(m+1)\omega_{n+1,m}(x)\right)-\left(\varphi'_{n+1,m+1}(x)+(m+1)\varphi_{n+1,m}(x)\right)\right\}$$
$$+\omega_{n+1,m+1}(x)\omega_{n+1,1}(x)-\varphi_{n+1,m+1}(x)\varphi_{n+1,1}(x)$$

$$=\frac{x(1-x)}{r(n+1)}\left\{\omega'_{n+1,m+1}(x)-\varphi'_{n+1,m+1}(x)+(m+1)\left(\omega_{n+1,m}(x)-\varphi_{n+1,m}(x)\right)\right\}$$

$$+\omega_{n+1,m+1}(x)\omega_{n+1,1}(x)-\varphi_{n+1,m+1}(x)\varphi_{n+1,1}(x)$$

$$=\frac{x(1-x)}{r(n+1)}\left\{\omega'_{n+1,m+1}(x)-\varphi'_{n+1,m+1}(x)+(m+1)\left(\omega_{n+1,m}(x)-\varphi_{n+1,m}(x)\right)\right\}$$

$$+\frac{(1-2x)+r}{2r(n+1)}\omega_{n+1,m+1}(x)-\frac{(1-2x)-r}{2r(n+1)}\varphi_{n+1,m+1}(x)$$

$$=\frac{x(1-x)}{r(n+1)}\left\{\omega'_{n+1,m+1}(x)-\varphi'_{n+1,m+1}(x)+(m+1)\left(\omega_{n+1,m}(x)-\varphi_{n+1,m}(x)\right)\right\}$$

$$+\frac{(1-2x)}{2r(n+1)}\left(\omega_{n+1,m+1}(x)-\varphi_{n+1,m+1}(x)\right)+\frac{1}{2(m+1)}\left(\omega_{n+1,m+1}(x)+\varphi_{n+1,m+1}(x)\right).$$

In view of Lemma 2.3, one has

$$\frac{m+1}{n+1}\mu^r_{n,m}(x)=\left(\omega_{n+1,m+1}(x)-\varphi_{n+1,m+1}(x)\right)$$

$$\mu^r_{n,m+1}(x)=\frac{(m+1)x(1-x)}{r(m+2)(n+1)}\left(\left(\mu^r_{n,m}(x)\right)'+m\mu^r_{n,m-1}(x)\right)+\frac{(m+1)}{(m+2)}\frac{(1-2x)}{2r(n+1)}\mu^r_{n,m}(x)+$$

$$\frac{1}{2(m+1)}\left(\omega_{n+1,m+1}(x)+\varphi_{n+1,m+1}(x)\right) \blacksquare$$

Now, by the direct evaluations and apply the recurrence relation above, one gets

$$\mu^r_{n,2}(x)\simeq\frac{rnx(1-x)}{r^2(n+1)^2}-\frac{2x(1-x)}{r^2(n+1)^2}+\frac{rx(1-x)}{r^2(n+1)^2}+\frac{r^2+3}{12r^2(n+1)^2}.$$

3.Main Results.

The Korovkin theorem and the Voronovskaja theorem for the sequence $KB_{n,r}(f;x)$ are proved here.

Theorem 3.1.

If $x\in[0,1], f\in C[0,1]$, exists , then $\lim_{n\to\infty}KB_{n,r}(f(t);x)=f(x)$.

Proof.

The proof of this Theorem holds From Lemma 2.2. ■

Theorem 3.2.

Let $x\in(0,1)$ and $f\in C[0,1]$, if f'' exists, the sequence $KB_{n,r}(f,x)$ is satisfied the following $\lim_{n\to\infty}n\{KB_{n,r}(f,x)-f(x)\}=\frac{(1-2x)}{2r}f'(x)+\frac{x(1-x)}{2r}f''(x)$.

Proof.

Using Taylor's expansion, one has

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2!}(t-x)^2+\varepsilon(t,x)(t-x)^2$$

where $\varepsilon(t,x) \to 0 \;\; as \; t \to x$. So,

$$KB_{n,r}(f(t),x) = f(x)KB_{n,r}(1;x) + f'(x)KB_{n,r}\big((t-x);x\big)$$

$$+\frac{1}{2}f''(x)KB_{n,r}\left(\left(\frac{k}{n}-x\right)^2;x\right) + KB_{n,r}(\varepsilon(t,x)(t-x)^2;x).$$

Then,

$$n\{KB_{n,r}(f;x)-f(x)\} = n\mu_{n,1}^r(x)f'(x) + n\mu_{n,2}^r(x)\frac{f''(x)}{2!} + n\left(\, KB_{n,r}(\varepsilon(t,x)(t-x)^2;x\right)$$

$$\lim_{n\to\infty} n\{KB_{n,r}(f;x)-f(x)\} = \lim_{n\to\infty} n\left(\frac{1-2x}{2r(n+1)}\right)f'(x)$$

$$+\lim_{n\to\infty} n\left(\frac{rnx(1-x)}{r^2(n+1)^2} - \frac{2x(1-x)}{r^2(n+1)^2} + \frac{rx(1-x)}{r^2(n+1)^2} + \frac{r^2+3}{6r^2(n+1)^2}\right)\frac{f''(x)}{2!}$$

$$+\lim_{n\to\infty} n\left(\, KB_{n,r}(\varepsilon(t,x)(t-x)^2;x\right)$$

$$= \frac{(1-2x)}{2r}f'(x) + \frac{x(1-x)}{2r}\, f''(x) + \lim_{n\to\infty} n\left(\, B_{n,r}(\varepsilon(t,x)(t-x)^2;x\right).$$

Now,

$$n\left|\, B_{n,r}(\varepsilon(t,x)(t-x)^2;x)\right| = \frac{n(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|\varepsilon(t,x)(t-x)^2|dx}{\sum_{k=0}^{n} b_{n,k}^r(x)}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

Now, for $\varepsilon > 0 \exists \delta > 0$ such that either $0 < |t-x| < \delta \to \;|\varepsilon(t,x)| < \varepsilon$ or $|t-x| \geq \delta \to$ $|\varepsilon(t,x)(t-x)^2| \leq Mt^\alpha$

$$n\left|\, KB_{n,r}(\varepsilon(t,x)(t-x)^2;x)\right|$$

$$\leq \frac{n(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)\int_{|t-x|<\delta}|\varepsilon(t,x)(t-x)^2|dt}{\sum_{k=0}^{n} b_{n,k}^r(x)}$$

$$+\frac{n(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)\int_{|t-x|\geq\delta}|\varepsilon(t,x)(t-x)^2|dt}{\sum_{k=0}^{n} b_{n,k}^r(x)}.$$

$$\leq n\varepsilon\mu_{n,2}^r(x) + \frac{n(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)\int_{|t-x|\geq\delta}|\varepsilon(t,x)(t-x)^2|dt}{\sum_{k=0}^{n} b_{n,k}^r(x)}$$

$$\leq \varepsilon O(1) + \frac{n(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)\int_{|t-x|\geq\delta} Mt^\alpha dt}{\sum_{k=0}^{n} b_{n,k}^r(x)}$$

since ε arbitrary then $\varepsilon O(1) \to 0$

$$n\left|\, KB_{n,r}(\varepsilon(t,x)(t-x)^2;x)\right| = \frac{n(n+1)\sum_{k=0}^{n} b_{n,k}^r(x)\int_{|t-x|\geq\delta} Mt^\alpha dt}{\sum_{k=0}^{n} b_{n,k}^r(x)}$$

$$= nM \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x) \int_{|t-x|\geq\delta} \sum_{i=0}^{\infty} \frac{(\alpha)_i x^{\alpha-i}}{i!} (t-x)^i \, dt}{\sum_{k=0}^{n} b_{n,k}^{r}(x)},$$

where $(\alpha)_i = \alpha(\alpha-1)\ldots(\alpha-i+1)$

$$\leq \sup_{x\in[0,1]} nM \sum_{i=0}^{\infty} \frac{(\alpha)_i \, x^{\alpha-i}}{i!} \; \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x) \int_{|t-x|\geq\delta} |t-x|^i dt}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$\leq nM \sum_{i=0}^{\infty} \frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x) \int_{|t-x|\geq\delta} |t-x|^i dt}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}.$$

Now, applying Cauchy-Schwarz inequality for integration and then for summation, one gets

$$\leq nM \sum_{i=0}^{\infty} \frac{(n+1)^{\frac{1}{2}+\frac{1}{2}} \sum_{k=0}^{n} b_{n,k}^{\frac{1}{2}r+\frac{1}{2}r}(x) \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right)^{1/2} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^{2i} dt\right)^{1/2}}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)^{\frac{1}{2}+\frac{1}{2}}}$$

$$\leq nM \sum_{i=0}^{\infty} \left(\frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)}\right)^{1/2} \left(\frac{(n+1)\sum_{k=0}^{n} b_{n,k}^{r}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^{2i} dt}{\left(\sum_{k=0}^{n} b_{n,k}^{r}(x)\right)}\right)^{1/2}$$

$\leq nM \sum_{i=0}^{\infty} \left(O\left(n^{-2i}\right)\right)^{\frac{1}{2}} \leq O(n^{-s})$ for any $s > 0$, for $i > 1$

$= o(1)$. Hence,

$\lim_{n\to\infty}(n+1)\ KB_{n,r}\left(\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^2;\ x\right) \to 0$ as $n \to \infty$. This completes the proof. ■

4. Numerical Example.

This example is a graph comparison among the convergence of the sequence, $KB_n(f;x) = KB_{n,1}(f;x)$ (black color), $KB_{n,2}(f;x)$ (red color), $KB_{n,3}(f;x)$ (green color), $KB_{n,5}(f;x)$ (blue color) and the test function $f(t) = \sin 10t \in C[0,1]$ (brawn color) (Fig4.1). Also, it is giving the graphs of the error functions $E(x) = \left(KB_{n,r}(f;x) - f(x)\right)$, $r = 1,2,3,5$ for these polynomials (in same colors above) for the values of $n = 25$ and 50 (Fig4.2).

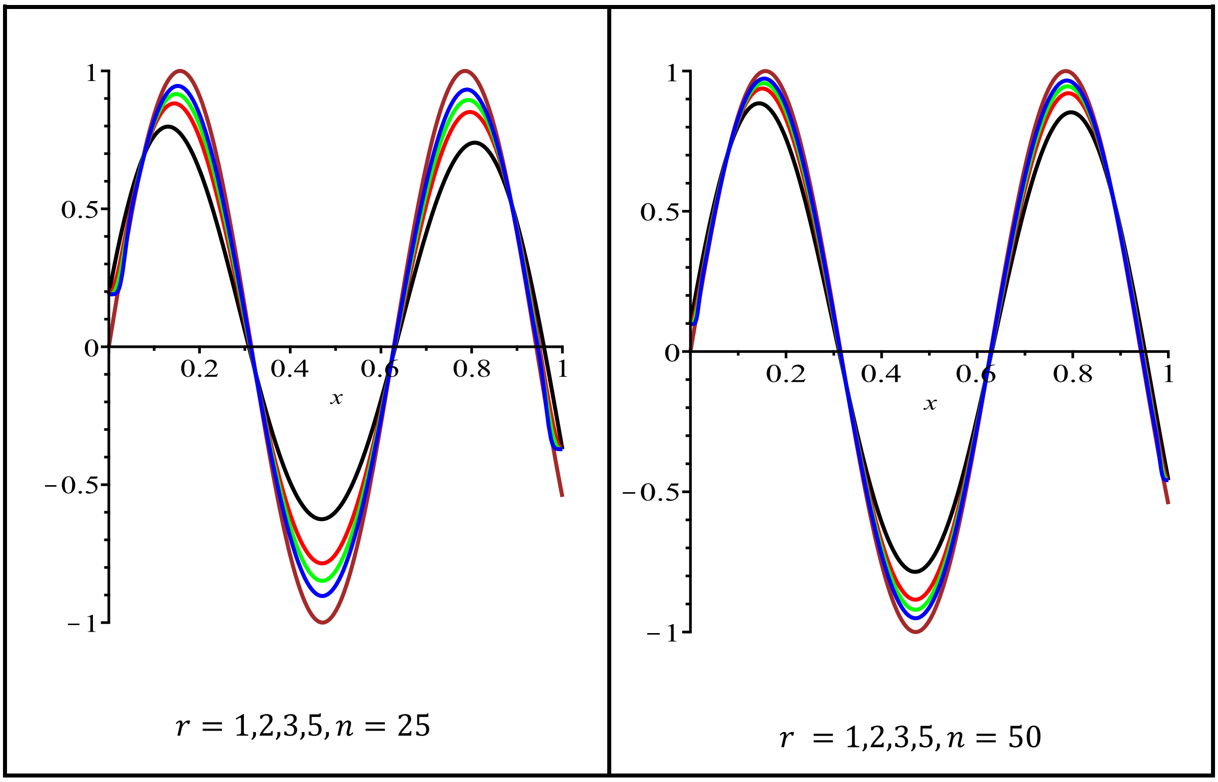

Fig 4.1: The convergence of $KB_{n,r}$ to the function $f(x)$ whenever $n = 25,50, r = 1,2,3,5$.

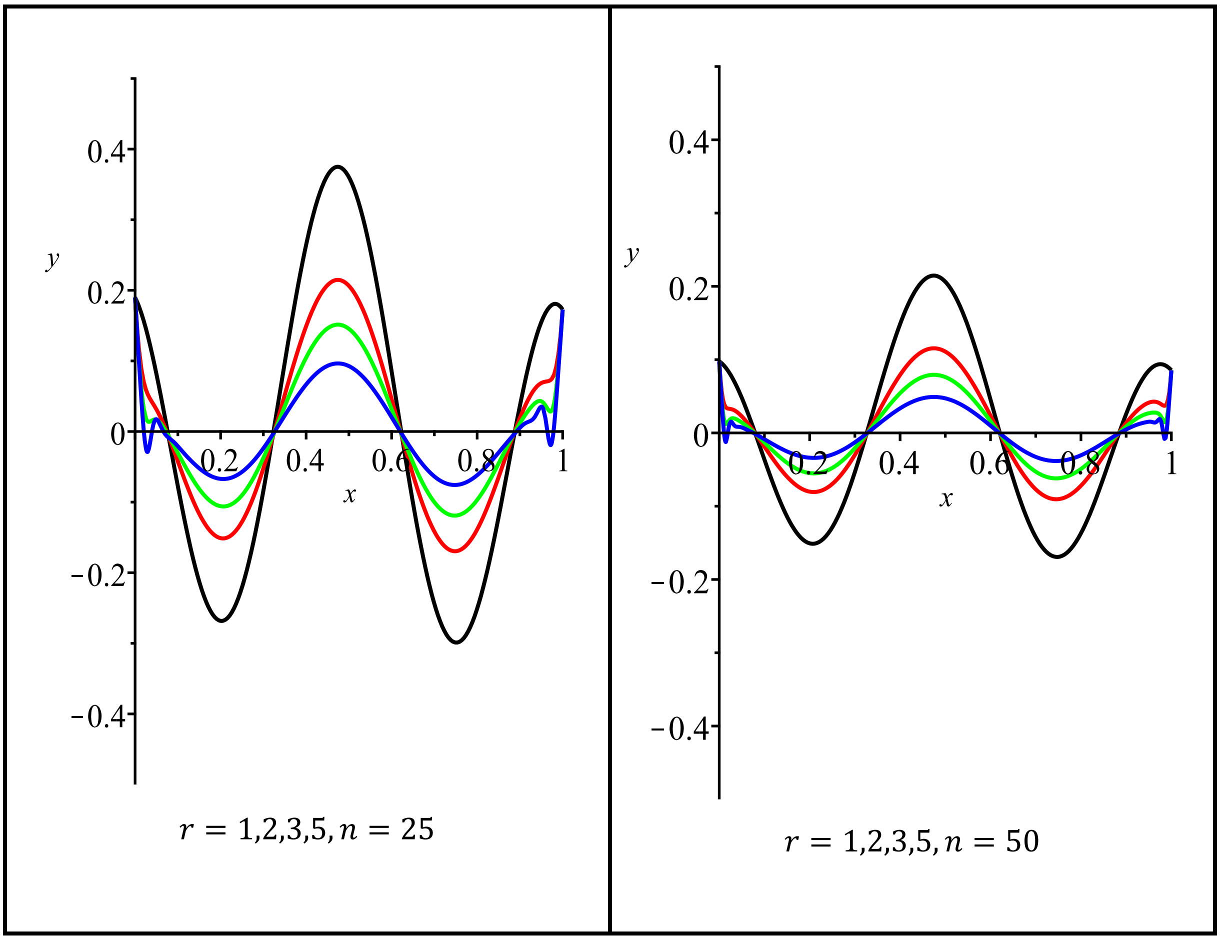

Fig 4.2: The functions $E(x)$.

Conclusion

This study is a generalization of well-known sequences of linear positive operators which are deduced as a special case from the r-th powers of the rational Bernstein polynomials. Also, the study gives a numerical example which are showed the numerical convergence of the polynomials $B_{n,r}(f;x)$ to the test function. This numerical convergence shows by the graphs of the $B_{n,r}(f;x)$ with the function $f(x)$. The numerical results appeared that numerical results became more accurate whenever r increase.

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متتابعة برنشتاين-كانتروفج الكسرية من القوىr

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المستخلص .

متتابعة برنشتاين- كانتروفج الكسرية من القوى (f; x) (n,r)_ 〖KB〗 r عرفت ودرست في هذا البحث. في البداية تظهر الدراسة ان المتتابعة KB_(n,r) (f;x) تتقارب الى دالة الاختبار f∈[0,1] عندما nتقترب الى اللانهاية. بعد ذلك اعطي العزم من الرتبة m وصيغة فرونفسكيا للتقارب. ايضاً اعطي تطبيق عددي للمتتابعة KB_(n,r) (f;x) ولدالة الاختبار f(x)=sin⁡10x لشرح خصائص التقارب ومقارنة النتائج العددية لهذا التقريب مع متتابعة برنشتاين- كانتروفج الكلاسيكية .(f;x) n_ 〖KB〗 اتضح ان خصائص التقريب العددي للمتتابعة (f;x) (n,r)_ 〖KB〗 يكون أفضل من التقريب العددي لـ 〖KB〗 _n (f;x).