The Rational $r$- Powers of Bernstein-Kantorovich Sequence

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Abstract.

The rational $r$-powers of Bernstein-Kantorovich sequence $KB_{n,r}(f; x)$ is defined and studied in this paper. In the beginning the study shows that the sequence $KB_{n,r}(f; x)$ is converged to the function $f \in C[0,1]$ whenever $n \to \infty$. Next, for this sequence, the moments of order $m$ and the Voronovskaja-type asymptotic formula are given. Also, a numerical application for the sequence $KB_{n,r}(f; x)$ is given for the test function $f(t) = \sin 10t$ to explain the convergence properties and compared with the numerical results of the classical Bernstein-Kantorovich sequence $KB_n(f; x)$. It shows that, the sequence $KB_{n,r}(f; x)$ has better numerical approximation properties than the sequence $KB_n(f; x)$.

Keywords: Voronovskaja-type asymptotic formula, Ordinary approximation, Rational Bernstein sequence.

MSC 2010: 41A25, 41A36.

*This paper is a part of M.Sc. Thesis in Mathematics Department, College of Education for pure Sciences, University of Basrah.
2.1 Introduction.

The well-known Bernstein sequence of order \( n \):

\[
B_n(f; x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right),
\]

where \( f \in C[0,1] \), \( b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \) and \( x \in [0,1] \).

In 1930, Kantorovich [6], gave a modification to the sequence of Bernstein for a function \( f \in C[0,1] \) as:

\[
KB_n(f; x) = (n + 1) \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) \, dt,
\]

where \( k = 0,1,2, \ldots, n \).

After that, some other generalizations and modifications have been introduced by some researchers (see, [3], [4], [5]).

In 2000, Cal and Vall [2], introduced a class of sequences that are general to particular many cases.

In 2013, Mahmudov and Sabancigil [8], have introduced a generalization of q-type Bernstein-Kantorovich sequence as follows:

\[
B_{n,q}(f; x) = \sum_{k=0}^{n} b_{n,k}(q, x) \int_{0}^{1} f\left(\frac{[k] + q^{k} t}{[n + 1]}\right) \, d_{q} t.
\]

where \( f \in C[0,1] \), \( 0 < q < 1 \).

In 2018, Acu, Manav and Sofonea [1], have studied the approximation properties and asymptotic type results concerning the Kantorovich variant of \( \lambda \)-Bernstein sequences.

\[
KB_{n,\lambda}(f; x) = (n + 1) \sum_{k=0}^{n} \hat{b}_{n,k}(\lambda; x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) \, dt,
\]

where \( \lambda \in [-1,1] \) and \( \hat{b}_{n,k}(\lambda; x) = b_{n,k}(x) + \lambda \left( \frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x) \right) \).

In 2019, Karahan and Izgi [7], have studied the generalized of Bernstein-Kantorovich sequences for the function of two variables.

\[ KB_{n,m}(f; x, y) = \frac{n + 1}{2} \frac{m + 1}{2} \sum_{k=0}^{n} \sum_{j=0}^{m} \phi^{k,j}_{n,k}(x, y) \int_{2}^{k+1} \int_{2}^{j+1} f(t, u) \, dt \, du, \]

where \( f \in C(A), \quad A = [-1,1] \times [-1,1], \quad n, m \in \mathbb{N}, \quad \phi^{k,j}_{n,m}(x, y) = b_{n,k}(x)b_{m,j}(y), \) and 
\[ b_{n,k}(x) = \frac{1}{2^n} \binom{n}{k} (1 + x)^k (1 - x)^{n-k}. \]

In 2020, Mohiuddine and Özger [10], construct Stancu-type Bernstein-Kantorovich sequences based on parameter \( \alpha. \)

\[ S^{\theta,\beta}_{n,\alpha}(f; y) = (n + \beta + 1) \sum_{i=0}^{n} p_{n,i}^{(\alpha)}(y) \int_{i+\theta}^{i+\theta+1} f(s) \, ds, \]

where \( p_{n,i}^{(\alpha)}(y) = \left[ (1 - \alpha)y^\left(\frac{n-2}{j}\right) + (1 - \alpha)(1 - y)^\left(\frac{n-2}{j-2}\right) + \alpha y(1 - y)^\left(\frac{n}{j}\right) \right] y^{j-1}(1 - y)^{n-j-1} \quad n \geq 2 \quad \text{and} \quad \alpha, y \in [0,1]. \]

In 2021, Mohammad and Abdul Samad [9], have introduced and studied the rational \( r\)- powers of the Bernstein sequence \( B_{n,r}(f; x) \) for \( f \in C[0,1] \) and \( r \in \mathbb{N} := \{1, 2, \ldots\} \) as follows:

\[ B_{n,r}(f; x) = \frac{\sum_{k=0}^{n} b_{n,k}^r(x) f \left( \frac{k}{n} \right)}{\sum_{k=0}^{n} b_{n,k}^r(x)}. \]

In this paper, the rational \( r\)-power of the Bernstein-Kantorovich sequence \( KB_{n,r}(f(t); x) \) is defined as:

\[ KB_{n,r}(f(t); x) = \frac{(n + 1) \sum_{k=0}^{n} b_{n,k}^r(x) \int_{\frac{n+1}{k}}^{\frac{n+1}{k+1}} f(t) \, dt}{\sum_{k=0}^{n} b_{n,k}^r(x)}, \]

where \( f \in C[0,1], \quad r \in \mathbb{N} \) and 
\[ b_{n,k}^r(x) = \left( \binom{n}{k} x^k (1 - x)^{n-k} \right)^r. \]

For the sequence \( KB_{n,r}(f(t); x) \), the moment of order \( m \), Korovkin theorem, and the Voronovskaja-type asymptotic formula is studied.  

2. Preliminary Results.

Some preliminaries relative to the sequence \( r\)-th powers of the rational Bernstein-Kantorovich polynomials are introduced here.
Lemma 2.1. [9]

(i) \( B_{n,r}(1; x) = 1; \)

(ii) \( B_{n,r}(t; x) \approx \frac{(r(n+1)-1)x}{r(n+1)} + \frac{1-r}{2r(n+1)}; \)

(iii) \( B_{n,r}(t^2; x) \approx \frac{(r(n+1)-1)(r(n+1)-2)}{r^2n^2} x^2 + \frac{2(1-r)(r(n+1)-1)}{r^4n^2} x + \frac{(1-r)^2}{4r^2n^2}; \)

(iv) \( B_{n,r}(t^m; x) = \frac{(r(n+1)-1)!}{r^m n^m(r(n+1)-m)!} x^m + \frac{(r(n+1)-1)}{2r^m n^m(r(n+1)-m)!} x^{m-1} + TLP(x). \)

Lemma 2.2.

For \( x \in [0,1], m \in \mathbb{N}^0; = \{0,1,2, \ldots \}, \) the following conditions are satisfied

(i) \( KB_{n,r}(1; x) = 1; \)

(ii) \( KB_{n,r}(t; x) = \frac{(r(n+1)-1)x}{r(n+1)} + \frac{1}{2r(n+1)}; \)

(iii) \( KB_{n,r}(t^2; x) = \frac{(r(n+1)-1)(r(n+1)-2)}{r^2(n+1)^2} x^2 + \frac{2(r(n+1)-1)}{r^3(n+1)^2} x + \frac{r^2+3}{12r^2(n+1)^2}; \)

(iv) \( KB_{n,r}(t^m; x) = \frac{(r(n+1)-1)!}{r^m(n+1)^m(r(n+1)-m)!} x^m + \frac{m(r(n+1)-1)!}{2r^m(n+1)^m(n+1-r)!} x^{m-1} + TLP(x). \)

Where \( TLP(x) \) means terms in lower powers of \( x. \)

Proof.

The proof of the above polynomials is going as:

By direct evaluation, one has

\( KB_{n,r}(1; x) = 1. \)

To prove (ii)

\[
KB_{n,r}(t; x) = \frac{(n+1) \sum_{k=0}^{n} b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t \, dt
\]

\[
= \frac{(n+1) \sum_{k=0}^{n} b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} \left[ \frac{k+1}{n+1} \right]^2 - \left( \frac{k}{n+1} \right)^2
\]

\[
= \frac{2(n+1) \sum_{k=0}^{n} b_{n,k}^r(x)}{2(n+1)^2} + \frac{(n+1) \sum_{k=0}^{n} b_{n,k}^r(x)}{2(n+1)^2} k
\]

Applying Lemma 2.1, one has

\[
KB_{n,r}(t; x) \approx \frac{(r(n+1)-1)x}{r(n+1)} + \frac{1-r}{2r(n+1)} + \frac{1}{2(n+1)}
\]

\[
= \frac{(r(n+1)-1)x}{r(n+1)} + \frac{1}{2r(n+1)}.
\]

To prove (iii)

\[
KB_{n,r}(t^2; x) = \frac{(n+1) \sum_{k=0}^{n} b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t^2 \, dt
\]
Applying Lemma 2.1, one has

\[
\sum_{k=0}^{n} x^k = \frac{(r(n+1) - 1)(r(n+1) - 2)}{r^2(n+1)^2} x^2 + \frac{(2 - r)(r(n+1) - 1)}{r^2(n+1)^2} x + \frac{(1-r)^2}{4r^2(n+1)^2} + \frac{r(n+1) - 1}{2r(n+1)^2} x + \frac{1}{3(n+1)^2}.
\]

In general,

\[
KB_{n,r}(t^2; x) \approx \frac{(r(n+1) - 1)(r(n+1) - 2)}{r^2(n+1)^2} x^2 + \frac{2(r(n+1) - 1)}{r^2(n+1)^2} x + \frac{r^2 + 3}{12r^2(n+1)^2}.
\]

\[
\int_0^\infty t^m dt = \frac{k^{m+1}}{m+1} = \frac{(n+1)^{m+1}}{m+1} \sum_{k=0}^n b^r_{n,k}(x) \left[ \left( \frac{k+1}{n+1} \right)^{m+1} - \left( \frac{k}{n+1} \right)^{m+1} \right]
\]

\[
= \sum_{k=0}^n b^r_{n,k}(x) \int_0^\infty t^m dt = \frac{(n+1)^m}{(n+1)^{m+1}} \sum_{k=0}^n b^r_{n,k}(x) \left[ (k+1)^{m+1} - k^{m+1} \right]
\]

\[
= \frac{n}{(n+1)^{m+1}} \sum_{k=0}^n b^r_{n,k}(x) \left( k^{m+1} + (m+1)k^m + \frac{m(m+1)}{2} k^{m-1} + \cdots + (m+1)k + 1 - k^{m+1} \right)
\]

\[
= \frac{n}{(n+1)^{m+1}} \sum_{k=0}^n b^r_{n,k}(x) \left( (m+1)k^m + \frac{m(m+1)}{2} k^{m-1} + \cdots + (m+1)k + 1 \right) + \frac{1}{(n+1)^m}(m+1)
\]

\[
= \frac{(m+1) \sum_{k=0}^n k^m b^r_{n,k}(x)}{(n+1)^m} + \frac{m(m+1) \sum_{k=0}^n k^{m-1} b^r_{n,k}(x)}{2(n+1)^m} + TLP(x)
\]
For, we define the following:

The $m$-th order moment $\mu_{n,m}^r(x)$ of the polynomials $KB_{n,r}(f(t); x)$

$$\mu_{n,m}^r(x) = \frac{(n+1) \sum_{k=0}^n k^m b_{n,k}^r(x)}{\sum_{k=0}^n b_{n,k}^r(x)} \int_{\frac{r}{n+1}}^{\frac{k+1}{n+1}} (t-x)^m dt.$$ 

The two functions $\omega_{n+1,m+1}(x)$ and $\varphi_{n+1,m+1}(x)$

$$\omega_{n+1,m+1}(x) = \frac{\sum_{k=0}^n b_{n,k}^r(x) \left( \frac{k+1}{n+1} - x \right)^{m+1}}{\sum_{k=0}^n b_{n,k}^r(x)}$$

$$\varphi_{n+1,m+1}(x) = \frac{\sum_{k=0}^n b_{n,k}^r(x) \left( \frac{k}{n+1} - x \right)^{m+1}}{\sum_{k=0}^n b_{n,k}^r(x)}.$$ 

The next lemma shows the relation between the two functions above and the function $\mu_{n,m}^r(x)$. 

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Lemma 2.3.

For \( m \in \mathbb{N}^0 \), the functions \( \mu_{n,m}^r(x) \) have

\[
\mu_{n,m}^r(x) = \frac{n+1}{m+1} \left( \omega_{n+1,m+1}^x(x) - \varphi_{n+1,m+1}^x(x) \right).
\]

Proof.

\[
\mu_{n,m}^r(x) = \frac{(n+1) \sum_{k=0}^{n} b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} \int \frac{(t-x)^m \, dt}{k+1} = \frac{(n+1) \sum_{k=0}^{n} b_{n,k}^r(x)}{(m+1) \sum_{k=0}^{n} b_{n,k}^r(x)} \left[ \left( \frac{k+1}{n+1} - x \right)^{m+1} - \left( \frac{k}{n+1} - x \right)^{m+1} \right]
\]

\[
\mu_{n,m}^r(x) = \frac{n+1}{m+1} \left[ \sum_{k=0}^{n} b_{n,k}^r(x) \left( \frac{k+1}{n+1} - x \right)^{m+1} - \sum_{k=0}^{n} b_{n,k}^r(x) \left( \frac{k}{n+1} - x \right)^{m+1} \right]
\]

\[
\mu_{n,m}^r(x) = \frac{n+1}{m+1} \left( \omega_{n+1,m+1}(x) - \varphi_{n+1,m+1}(x) \right).
\]

Lemma 2.4.

The functions \( \omega_{n+1,m+1}(x) \) and \( \varphi_{n+1,m+1}(x) \) have the following recurrence relations

(i) \( \omega_{n+1,m+2}(x) = \frac{x(1-x)}{r(n+1)} \left( \omega_{n+1,m+1}(x) + (m+1) \omega_{n+1,m}(x) \right) + \omega_{n+1,m+1}(x) \omega_{n+1,1}(x) \),

where \( \omega_{n+1,1}(x) \approx \frac{(1-2x)+r}{2r(n+1)} \), \( \omega_{n+1,2}(x) \approx \frac{(2-r(n+1))x^2}{r^2(n+1)^2} + \frac{1}{4r^2(n+1)^2} \).

(ii) \( \varphi_{n+1,m+2}(x) = \frac{x(1-x)}{r(n+1)} \left( \varphi_{n+1,m+1}(x) + (m+1) \varphi_{n+1,m}(x) \right) + \varphi_{n+1,m+1}(x) \varphi_{n+1,1}(x) \),

where \( \varphi_{n+1,1}(x) \approx \frac{(1-2x)-r}{2r(n+1)} \), \( \varphi_{n+1,2}(x) \approx \frac{(2-r(n+1))x^2}{r^2(n+1)^2} - \frac{1}{4r^2(n+1)^2} + \frac{(1-r)^2}{4r^2(n+1)^2} \).

Proof.

The proof of the consequence (i) is going as:

\[
\omega_{n+1,1}(x) = \frac{\sum_{k=0}^{n} b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} = \frac{\sum_{k=0}^{n} kb_{n,k}^r(x)}{(n+1) \sum_{k=0}^{n} b_{n,k}^r(x)} + \frac{1}{n+1} - x
\]

\[
\approx \frac{(r(n+1)-1)x}{r(n+1)} + \frac{1-r}{2r(n+1)} + \frac{1}{n+1} - x
\]

\[
\approx \frac{2rx + 2rx - 2x + 1 - r + 2r - 2rx - 2rx}{2r(n+1)} = \frac{(1-2x)+r}{2r(n+1)}.
\]
\[
\omega_{n+1,2}(x) = \frac{\sum_{k=0}^{n} b_{n,k}^r(x) \left(\frac{k+1}{n+1} - x\right)^2}{\sum_{k=0}^{n} b_{n,k}^r(x)} \\
\approx \frac{\sum_{k=0}^{n} k^2 b_{n,k}^r(x)}{(n+1)^2 \sum_{k=0}^{n} b_{n,k}^r(x)} + \frac{2}{(n+1)^2} \frac{\sum_{k=0}^{n} k b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} + \frac{1}{(n+1)^2} - 2x \frac{\sum_{k=0}^{n} k b_{n,k}^r(x)}{(n+1) \sum_{k=0}^{n} b_{n,k}^r(x)} \\
- \frac{2x}{(n+1)} + x^2 \\
\approx \frac{(r(n+1) - 1)(r(n+1) - 2)}{r^2(n+1)^2} x^2 + \frac{(2 - r)(r(n+1) - 1)}{r^2(n+1)^2} x + \frac{(1 - r)^2}{4r^2(n+1)^2} \\
+ 2 \left(\frac{r(n+1) - 1}{r(n+1)^2} + \frac{1 - r}{2r(n+1)^2}\right) + \frac{1}{(n+1)^2} \\
- 2x \left(\frac{r(n+1) - 1}{r(n+1)} + \frac{1 - r}{2r(n+1)^2}\right) - \frac{2x}{(n+1)} + x^2 \\
\approx \left(\frac{r^2(n+1)^2 - 3r(n+1) + 2}{r^2(n+1)^2}\right) x^2 \\
+ \left(\frac{2r(n+1) - 2}{r(n+1)^2} - \frac{2r(n+1) - 1}{r(n+1) - 1}\right) x \\
+ \frac{(1 - r)^2}{4r^2(n+1)^2} + \frac{1 - r}{r(n+1)^2} + \frac{1}{(n+1)^2} \\
\omega_{n+1,2}(x) \approx \frac{(2 - r(n+1)) x^2}{r^2(n+1)^2} - \frac{(2 - r) r x}{r^2(n+1)^2} + \frac{(1 + r)^2}{4r^2(n+1)^2}. \\
\]

Now,
\[
\omega_{n+1,m+1}(x) = \frac{\sum_{k=0}^{n} b_{n,k}^r(x) \left(\frac{k+1}{n+1} - x\right)^m}{\sum_{k=0}^{n} b_{n,k}^r(x)} \\
\omega_{n+1,m+1}'(x) = \frac{\sum_{k=0}^{n} b_{n,k}^r(x) \sum_{k=0}^{n} b_{n,k}^r(x) \left(-1\right)(m+1) \left(\frac{k+1}{n+1} - x\right)^m}{\left(\sum_{k=0}^{n} b_{n,k}^r(x)\right)^2} \\
+ \frac{\sum_{k=0}^{n} b_{n,k}^r(x) \sum_{k=0}^{n} \left(b_{n,k}^r(x)\right)' \left(\frac{k+1}{n+1} - x\right)^m}{\left(\sum_{k=0}^{n} b_{n,k}^r(x)\right)^2} \\
- \frac{\sum_{k=0}^{n} b_{n,k}^r(x) \left(\frac{k+1}{n+1} - x\right)^m}{\left(\sum_{k=0}^{n} b_{n,k}^r(x)\right)^2} \sum_{k=0}^{n} \left(b_{n,k}^r(x)'\right) \\
= -(m+1) \omega_{n+1,m}(x) + \frac{\sum_{k=0}^{n} \left(b_{n,k}^r(x)'\right) \left(\frac{k+1}{n+1} - x\right)^m}{\sum_{k=0}^{n} b_{n,k}^r(x)}
\]

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\[ \sum_{k=0}^{n} b_{n;k}^r(x) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} \left(b_{n;k}^r(x)\right) \]

\[ x(1-x)\left(\omega_{n+1,m+1}(x) + (m+1)\omega_{n+1,m}(x)\right) \]

\[ \sum_{k=0}^{n} x(1-x)\left(b_{n;k}^r(x)\right) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} b_{n;k}^r(x) \]

\[ - \sum_{k=0}^{n} b_{n;k}^r(x) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} x(1-x)\left(b_{n;k}^r(x)\right) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} b_{n;k}^r(x) \]

by using the fact \( x(1-x)\left(b_{n;k}^r(x)\right)' = (kr - nrx)b_{n;k}^r(x) \), one has

\[ r \sum_{k=0}^{n} b_{n;k}^r(x)(k-nx+x-x+1-1) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} b_{n;k}^r(x) \]

\[ \sum_{k=0}^{n} b_{n;k}^r(x) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} b_{n;k}^r(x)(k-nx+x-x+1-1) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} b_{n;k}^r(x) \]

\[ r(n+1) \sum_{k=0}^{n} b_{n;k}^r(x) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} b_{n;k}^r(x) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} b_{n;k}^r(x) \]

\[ r(n+1) \sum_{k=0}^{n} b_{n;k}^r(x) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} b_{n;k}^r(x) \left(\frac{k+1}{n+1} - x\right)^{m+1} \sum_{k=0}^{n} b_{n;k}^r(x) \]

\[ r(n+1) \left(\omega_{n+1,m+2}(x)\right) + r(x-1)\omega_{n+1,m+1}(x) - r(n+1)\omega_{n+1,m+1}(x)\omega_{n+1,1}(x) \]

\[ -r(x-1)\omega_{n+1,m+1}(x) \]

\[ r(n+1)\omega_{n+1,m+2}(x) - r(n+1)\omega_{n+1,m+1}(x)\omega_{n+1,1}(x), \]

hence,

\[ \omega_{n+1,m+2}(x) = \frac{x(1-x)}{r(n+1)}\left(\omega_{n+1,m+1}(x) + (m+1)\omega_{n+1,m}(x)\right) + \omega_{n+1,m+1}(x)\omega_{n+1,1}(x). \]

Using similar steps, we have

\[ \varphi_{n+1,1}(x) = \frac{(1-2x) - r}{2r(n+1)}; \]

\[ \varphi_{n+1,2}(x) = \frac{2-r(n+1)}{r^2(n+1)^2}x^2 - \frac{2-r(n+2)}{r^2(n+1)^2}x + \frac{(1-r)^2}{4r^2(n+1)^2}; \]
Lemma 2.5.

The moment \( \mu_{n,m}^r(x) \) of order \( m \) has the recurrence relation

\[
\mu_{n,m+1}^r(x) = \frac{(m+1)(1-x)}{r(m+2)} \left( \mu_{n,m}^r(x) \right)' + \frac{(m+1)(1-2x)}{r(m+2)} \mu_{n,m}^r(x) + \frac{1}{2(m+2)} (\omega_{n+1,m+1}(x) + \varphi_{n+1,m+1}(x)),
\]

where \( \mu_{n,0}^r(x) = 1 \), and \( \mu_{n,1}^r(x) \approx \frac{(1-2x)}{2r(n+1)} \).

Proof.

From Lemma 2.3, one gets

\[
\frac{1}{n+1} \mu_{n,0}^r(x) = \omega_{n+1,1}(x) - \varphi_{n+1,1}(x)
\]

\[
\frac{1}{n+1} \mu_{n,0}^r(x) \approx \frac{1-2x + r}{2r(n+1)} - \frac{1-2x - r}{2r(n+1)} \approx \frac{2r}{2r(n+1)} \approx \frac{1}{n+1}
\]

\( \mu_{n,0}^r(x) = 1 \), and

\[
\frac{2}{n+1} \mu_{n,1}^r(x) = \omega_{n+1,2}(x) - \varphi_{n+1,2}(x)
\]

\[
\frac{2}{n+1} \mu_{n,1}^r(x) \approx \left( \frac{2-r(n+1)}{r^2(n+1)^2} x^2 - \frac{2-rn}{r^2(n+1)^2} x + \frac{(1+r)^2}{4r^2(n+1)^2} \right)
\]

\[
- \left( \frac{2-r(n+1)}{r^2(n+1)^2} x^2 - \frac{2-r(n+2)}{r^2(n+1)^2} x + \frac{(1-r)^2}{4r^2(n+1)^2} \right)
\]

\[
\approx \left( \frac{2-rn}{r^2(n+1)^2} x + \frac{(1+r)^2}{4r^2(n+1)^2} \right) - \left( \frac{2-r(n+2)}{r^2(n+1)^2} x + \frac{(1-r)^2}{4r^2(n+1)^2} \right)
\]

\[
\mu_{n,1}^r(x) \approx \frac{1-2x}{2r(n+1)}.
\]

Then,

\[
\omega_{n+1,m+2}(x) - \varphi_{n+1,m+2}(x)
\]

\[
= \frac{x(1-x)}{r(n+1)} \left( \omega_{n+1,m+1}'(x) + (m+1)\omega_{n+1,m}(x) \right) + \omega_{n+1,m+1}(x)\omega_{n+1,1}(x)
\]

\[
- \frac{x(1-x)}{r(n+1)} \left( \varphi_{n+1,m+1}'(x) + (m+1)\varphi_{n+1,m}(x) \right) + \varphi_{n+1,m+1}(x)\varphi_{n+1,1}(x)
\]

\[
= \frac{x(1-x)}{r(n+1)} \left( \omega_{n+1,m+1}'(x) + (m+1)\omega_{n+1,m}(x) \right) - \left( \varphi_{n+1,m+1}'(x) + (m+1)\varphi_{n+1,m}(x) \right)
\]

\[
+ \omega_{n+1,m+1}(x)\omega_{n+1,1}(x) - \varphi_{n+1,m+1}(x)\varphi_{n+1,1}(x)
\]

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In view of Lemma 2.3, one has

\begin{align*}
&= \frac{x(1-x)}{r(n+1)}\left\{\omega_{n+1,m+1}(x) - \varphi_{n+1,m+1}(x) + (m+1)\left(\omega_{n+1,m}(x) - \varphi_{n+1,m}(x)\right)\right\} \\
&+ \omega_{n+1,m+1}(x)\omega_{n+1,1}(x) - \varphi_{n+1,m+1}(x)\varphi_{n+1,1}(x) \\
&= \frac{x(1-x)}{r(n+1)}\left\{\omega_{n+1,m+1}(x) - \varphi_{n+1,m+1}(x) + (m+1)\left(\omega_{n+1,m}(x) - \varphi_{n+1,m}(x)\right)\right\} \\
&+ \frac{(1-2x)+r}{2r(n+1)}\omega_{n+1,m+1}(x) - \frac{(1-2x)-r}{2r(n+1)}\varphi_{n+1,m+1}(x) \\
&= \frac{x(1-x)}{r(n+1)}\left\{\omega_{n+1,m+1}(x) - \varphi_{n+1,m+1}(x) + (m+1)\left(\omega_{n+1,m}(x) - \varphi_{n+1,m}(x)\right)\right\} \\
&+ \frac{(1-2x)}{2r(n+1)}\left(\omega_{n+1,m+1}(x) - \varphi_{n+1,m+1}(x)\right) + \frac{1}{2(m+1)}\left(\omega_{n+1,m+1}(x) + \varphi_{n+1,m+1}(x)\right).
\end{align*}

In view of Lemma 2.3, one has

\[ m + 1 = \mu_{n+1,m}(x) = \left(\omega_{n+1,m+1}(x) - \varphi_{n+1,m+1}(x)\right) \]

\[ \mu_{n+1,m+1}(x) = \frac{(m+1)x(1-x)}{r(n+1)}\left(\mu_{n,m}(x)\right) + m\mu_{n,m-1}(x) \]

\[ + \frac{(m+1)}{2(m+1)}\mu_{n,m}(x) + \frac{1}{2(m+1)}\left(\omega_{n+1,m+1}(x) + \varphi_{n+1,m+1}(x)\right) \]

Now, by the direct evaluations and apply the recurrence relation above, one gets

\[ \mu_{n,2}(x) = \frac{rnx(1-x)}{r^2(n+1)^2} - \frac{2x(1-x)}{r^2(n+1)^2} + \frac{r^2+3}{12r^2(n+1)^2}. \]

3. Main Results.

The Korovkin theorem and the Voronovskaja theorem for the sequence KB_{n,r}(f;x) are proved here.

**Theorem 3.1.**

If \( x \in [0,1], f \in C[0,1], \) exists, then \( \lim_{n \to \infty} KB_{n,r}(f(t);x) = f(x). \)

**Proof.**

The proof of this Theorem holds From Lemma 2.2.

**Theorem 3.2.**

Let \( x \in (0,1) \) and \( f \in C[0,1], \) if \( f'' \) exists, the sequence \( KB_{n,r}(f,x) \) is satisfied the following \( \lim_{n \to \infty} n\{KB_{n,r}(f,x) - f(x)\} = \frac{(1-2x)}{2r}f'(x) + \frac{x(1-x)}{2r}f''(x). \)

**Proof.**

Using Taylor's expansion, one has

\[ f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2!}(t-x)^2 + \varepsilon(t,x)(t-x)^2 \]

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where $\varepsilon(t, x) \to 0$ as $t \to x$. So,

$$KB_{n,r}(f(t), x) = f(x)KB_{n,r}(1; x) + f'(x)KB_{n,r}((t - x); x) + \frac{1}{2}f''(x)KB_{n,r}\left(\left(\frac{n}{r} - x\right)^2; x\right) + KB_{n,r}(\varepsilon(t, x)(t - x)^2; x).$$

Then,

$$n\{KB_{n,r}(f; x) - f(x)\} = n\mu_{n,1}(x)f'(x) + n\mu_{n,2}(x)f''(x)2! + n\{KB_{n,r}(\varepsilon(t, x)(t - x)^2; x)\}$$

$$\lim_{n \to \infty} n\{KB_{n,r}(f; x) - f(x)\} = \lim_{n \to \infty} n\left(\frac{1 - 2x}{2r(n + 1)}\right)f'(x)$$

$$+ \lim_{n \to \infty} n\left(\frac{rnx(1 - x)}{r^2(n + 1)^2} - \frac{2x(1 - x)}{r^2(n + 1)^2} + \frac{r(1 - x)}{r^2(n + 1)^2} + \frac{r^2 + 3}{6r^2(n + 1)^2}\right)f''(x)2!$$

$$+ \lim_{n \to \infty} n\{KB_{n,r}(\varepsilon(t, x)(t - x)^2; x)\}$$

$$= \frac{(1 - 2x)}{2r}f'(x) + \frac{x(1 - x)}{2r}f''(x) + \lim_{n \to \infty} n\{B_{n,r}(\varepsilon(t, x)(t - x)^2; x)\}.$$ 

Now,

$$n\mid B_{n,r}(\varepsilon(t, x)(t - x)^2; x)\mid = \frac{n(n + 1)\sum_{k=0}^{n} b_{n,k}(x)\int_{t-x}^{t+x}{\varepsilon(t, x)(t - x)^2}dt}{\sum_{k=0}^{n} b_{n,k}(x)}$$

where $\varepsilon(t, x) \to 0$ as $t \to x$.

Now, for $\varepsilon > 0 \exists \delta > 0$ such that either $0 < |t - x| < \delta \to |\varepsilon(t, x)| < \varepsilon$ or $|t - x| \geq \delta \to |\varepsilon(t, x)|(t - x)^2 \leq Mt^\alpha$

$$n\mid KB_{n,r}(\varepsilon(t, x)(t - x)^2; x)\mid$$

$$\leq \frac{n(n + 1)\sum_{k=0}^{n} b_{n,k}(x)\int_{|t-x|<\delta}|\varepsilon(t, x)(t - x)^2|dt}{\sum_{k=0}^{n} b_{n,k}(x)}$$

$$+ \frac{n(n + 1)\sum_{k=0}^{n} b_{n,k}(x)\int_{|t-x|\geq\delta}|\varepsilon(t, x)(t - x)^2|dt}{\sum_{k=0}^{n} b_{n,k}(x)}.$$ 

$$\leq n\mu_{n,2}(x) + \frac{n(n + 1)\sum_{k=0}^{n} b_{n,k}(x)\int_{|t-x|<\delta}\varepsilon(t, x)(t - x)^2|dt}{\sum_{k=0}^{n} b_{n,k}(x)}$$

$$\leq \varepsilon O(1) + \frac{n(n + 1)\sum_{k=0}^{n} b_{n,k}(x)\int_{|t-x|\geq\delta}M t^\alpha dt}{\sum_{k=0}^{n} b_{n,k}(x)}$$

since $\varepsilon$ arbitrary then $\varepsilon O(1) \to 0$

$$n\mid KB_{n,r}(\varepsilon(t, x)(t - x)^2; x)\mid = \frac{n(n + 1)\sum_{k=0}^{n} b_{n,k}(x)\int_{|t-x|\geq\delta}M t^\alpha dt}{\sum_{k=0}^{n} b_{n,k}(x)}$$
\[
= nM \left( \sum_{k=0}^{n} \frac{b_{n,k}^{r}(x) \int_{|t-x| \geq \delta} \sum_{i=0}^{\infty} \frac{(\alpha)^{i} x^{\alpha-i}}{i!} (t-x)^{i} \, dt}{\sum_{k=0}^{n} b_{n,k}^{r}(x)} \right),
\]
where \((\alpha)_{i} = \alpha(\alpha - 1) \ldots (\alpha - i + 1)\)

\[
\leq \sup_{x \in [0,1]} nM \sum_{i=0}^{\infty} \frac{(\alpha)_{i} x^{\alpha-i}}{i!} \frac{(n + 1) \sum_{k=0}^{n} b_{n,k}^{r}(x) \int_{|t-x| \geq \delta} |t-x|^{i} \, dt}{\sum_{k=0}^{n} b_{n,k}^{r}(x)} \leq nM \sum_{i=0}^{\infty} \frac{(n + 1) \sum_{k=0}^{n} b_{n,k}^{r}(x) \int_{|t-x| \geq \delta} |t-x|^{i} \, dt}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}.
\]

Now, applying Cauchy-Schwarz inequality for integration and then for summation, one gets

\[
\leq nM \sum_{i=0}^{\infty} \left( \frac{1}{2} \left( \sum_{k=0}^{n} b_{n,k}^{r}(x) \right)^{1/2} \right)^{1/2} \left( \sum_{k=0}^{n} \frac{1}{2} \left( \sum_{k=0}^{n} b_{n,k}^{r}(x) \right)^{k+1} \frac{1}{n+1} \int_{|t-x| \geq \delta} \frac{k+1}{n+1} |t-x|^{2i} \, dt \right)^{1/2}
\]

\[
\leq nM \sum_{i=0}^{\infty} \left( \frac{1}{2} \left( \sum_{k=0}^{n} b_{n,k}^{r}(x) \right)^{1/2} \right)^{1/2} \left( \sum_{k=0}^{n} \frac{1}{2} \left( \sum_{k=0}^{n} b_{n,k}^{r}(x) \right)^{k+1} \frac{1}{n+1} \int_{|t-x| \geq \delta} \frac{k+1}{n+1} |t-x|^{2i} \, dt \right)^{1/2}
\]

\[
\leq nM \sum_{i=0}^{\infty} \left( O \left( n^{-2i} \right) \right)^{1/2} \leq O \left( n^{-s} \right) \text{ for any } s > 0, \text{ for } i > 1
\]

\[
= o(1). \text{ Hence,}
\]

\[
\lim_{n \to \infty} (n + 1) KB_{n,r} \left( \varepsilon \left( \frac{k}{n}, x \right) \left( \frac{k}{n} - x \right)^{2} ; x \right) \to 0 \text{ as } n \to \infty. \text{ This completes the proof. } \]


This example is a graph comparison among the convergence of the sequence, \(KB_{n}(f; x) = KB_{n,1}(f; x)\) (black color), \(KB_{n,2}(f; x)\) (red color), \(KB_{n,3}(f; x)\) (green color), \(KB_{n,5}(f; x)\) (blue color) and the test function \(f(t) = \sin 10t \in C[0,1]\) (brown color) (Fig4.1). Also, it is giving the graphs of the error functions \(E(x) = \left( KB_{n,r}(f; x) - f(x) \right)\), \(r = 1,2,3,5\) for these polynomials (in same colors above) for the values of \(n = 25\) and \(50\) (Fig4.2).
Fig 4.1: The convergence of $KB_{n,r}$ to the function $f(x)$ whenever $n = 25, 50, r = 1, 2, 3, 5$. 

$r = 1, 2, 3, 5, n = 25$  

$r = 1, 2, 3, 5, n = 50$
**Conclusion**

This study is a generalization of well-known sequences of linear positive operators which are deduced as a special case from the $r$-th powers of the rational Bernstein polynomials. Also, the study gives a numerical example which are showed the numerical convergence of the polynomials $B_{n,r}(f;x)$ to the test function. This numerical convergence shows by the graphs of the $B_{n,r}(f;x)$ with the function $f(x)$. The numerical results appeared that numerical results became more accurate whenever $r$ increase.

**Reference.**


متباعدة برشتانيا-كانتروفج الكرسية من القوى

علي جاسم伸び

العراق / جامعة البصرة / كلية التربية للعلوم الصرفة / قسم الرياضيات.

المتخصص.

متباعدة برشتانيا-كانتروفج الكرسية من القوى

ر

عوفت ودرست في

هزا الثحث. فٍ الثذاَح ذظهش الذساسح اى الورراتعح

تقترب الى دالة الاختبار [0,1] عندما

تقترب الى اللانهاية. بعد ذلك اعتي العزم من الرتبة

تشرح خصائص التقارب لـ(،،) ودالة الاختبار KB_(n,r) (f;x) عددية للمتتابعة

تضح اى خصائص الرمشَة العذدٌ للورراتعح KB_n (f;x)

النتائج العددية لهذا التقرب مع متتابعة برشتانيا-كانتروفج الكلاسيكية.

ينصح ان هزا الرمشَة للورراتعح KB_n (f;x)

كون أفضل من التقرب العددي لـ KB_(n,r) (f;x).