The r-th Powers of the Rational Bernstein Polynomials

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Abstract

This paper is defined and studied a new r-th powers of rational Bernstein polynomials. The convergence theorem, the recurrence relation for the m-th order moment, and the Voronovskaya-type asymptotic formula for these polynomials in ordinary approximation are given. Also, a numerical example for these polynomials is applied to approximate the test function $\sin 10x \in C[0,1]$. The results obtained from this example are shown that these polynomials are given better than the corresponding numerical results for the classical Bernstein polynomials and the square rational Bernstein polynomials. The comparison is done by plot the graphs of the function and its approximations as well as the evaluation of the average absolute errors for these approximations.

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1. Introduction

For $f \in C[0,1]$, the well-known m-th order of Bernstein polynomials of the function f is defined as:

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x)\, f\left(\frac{k}{n}\right),$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $x \in [0,1]$.

In 2009, Pitul and Sablonniere are defined and studied a new family of univariate rational Bernstein polynomials defined as [6]

$$R_n(f\,;x) = \frac{\sum_{k=0}^{n} f\left(x_{n,k}\right).\, \overline{w}_{n,k}\, b_{n,k}(x)}{\sum_{j=0}^{n-1} w_{n,j}\, b_{n-1,j}(x)}, f \in C[0,1], n \geq 1,$$

the weights of the denominator are assumed to be strictly positive while the weights $\overline{w}_{n,k}$ and the abscissae $x_{n,k}$ of the numerator satisfy the following relations:

$$\overline{w}_{n,j} = \frac{j}{n} w_{n,j-1} + \left(1 - \frac{j}{n}\right) w_{n,j} \text{ for } 1 \leq j \leq n-1$$

$$x_{n,k} = \frac{k}{n} \frac{w_{n,k-1}}{\overline{w}_{n,k}} = \frac{k w_{n,k-1}}{k w_{n,k-1} + (n-k) w_{n,k}} \text{ for } 1 \leq k \leq n-1.$$

After that, many researchers have been interested in the sequences of linear positive operators of rational polynomials. Please see [8] and [5]. In 2014, Render has discussed some convergence properties and error estimates of rational Bernstein polynomials in the general case [7].In 2017, Gavrea and Ivan have defined the square Bernstein polynomials as [3]:

$$B_{n,2}(f;x) = \frac{\sum_{k=0}^{n} b_{n,k}^2(x) f\left(\frac{k}{n}\right)}{\sum_{k=0}^{n} b_{n,k}^2(x)}, n = 1,2, \ldots.$$

where $b_{n,k}^2(x) = \left(b_{n,k}(x)\right)^2$.

In 2019, Holhos proved the Voronovskaya-type asymptotic formula for squared Bernstein polynomials. [4]

This paper defines a new rational Bernstein polynomial of r-th power as follows:

For $f \in C[0,1]$ and $r \in \mathbb{N} := \{1,2, \ldots\}$, the r-th power $B_{n,r}(f;x)$ is defined

$$B_{n,r}(f;x) = \frac{\sum_{k=0}^{n} b_{n,k}^r(x) f\left(\frac{k}{n}\right)}{\sum_{k=0}^{n} b_{n,k}^r(x)}, b_{n,k}^r(x) = \left(b_{n,k}(x)\right)^r.$$

Clearly, $B_{n,1}(f;x) = B_n(f;x)$

Here, the convergence theorem, the m-th order moment, and the Voronovskaya-type asymptotic formula for the polynomials $B_{n,r}(f;x)$ are given. re, we assume that M is a constant not has the same value in different cases.

2. Preliminary Results

Some primary results relative to the r-th power of rational Bernstein polynomials are given in this section.

Lemma 2.1

For $r \in \mathbb{N}$, the function $b_{n,k}^{r}(x)$ has the following:

(i) $x(1-x)\left(b_{n,k}^{r}(x)\right)' = (rk - rnx)b_{n,k}^{r}(x);$

(ii) $\sum_{k=0}^{n} b_{n,k}^{r}(x) = (1-x)^{rn}\sum_{k=0}^{n}\frac{((-n)_k)^r}{(k!)^r}\frac{x^{rk}}{(-1+x)^{rk}}.$

Proof.

The proof of the consequence (i) is going as:

$$\left(b_{n,k}^{r}(x)\right)' = rb_{n,k}^{r-1}(x)\binom{n}{k}\left(kx^{k-1}(1-x)^{n-k} + x^k(n-k)(-1)(1-x)^{n-k-1}\right)$$

$$= rb_{n,k}^{r-1}(x)\binom{n}{k}x^{k-1}(1-x)^{n-k-1}(k-nx).$$

Hence, $x(1-x)\left(b_{n,k}^{r}(x)\right)' = (rk - rnx)b_{n,k}^{r}(x)$.

The proof of the consequence (ii) is doing by the direct evaluation as follows:

$$\sum_{k=0}^{n} b_{n,k}^{r}(x) = \sum_{k=0}^{n}\left(\binom{n}{k}x^k(1-x)^{n-k}\right)^r$$

$$= (1-x)^{rn}\left(1 + \sum_{k=1}^{n}\left(\frac{n(n-1)(n-2)\dots(n-k+1)}{k!}\right)^r\frac{x^{rk}}{(1-x)^{rk}}\right)$$

$$= (1-x)^{rn}\left[1 + n^r\frac{x^r}{(1-x)^r} + \frac{n^r(n-1)^r}{(2!)^r}\frac{x^{2r}}{(1-x)^{2r}} + \frac{n^r(n-1)^r(n-2)^r}{(3!)^r}\frac{x^{3r}}{(1-x)^{3r}} + \cdots + \frac{x^{rn}}{(1-x)^{rn}}\right]$$

$$= (1-x)^{rn}\left[1 + (-n)^r\frac{x^r}{(-1+x)^r} + \frac{(-n)^r(-n+1)^r}{(2!)^r}\frac{x^{2r}}{(-1+x)^{2r}} + \frac{(-n)^r(-n+1)^r(-n+2)^r}{(3!)^r}\frac{x^{3r}}{(-1+x)^{3r}} + \cdots + \frac{x^{rn}}{(1-x)^{rn}}\right].$$

Therefore,

$$\sum_{k=0}^{n} b_{n,k}^{r}(x) = (1-x)^{rn}\sum_{k=0}^{n}\frac{((-n)_k)^r}{(k!)^r}\frac{x^{rk}}{(-1+x)^{rk}}. \blacksquare$$

For $m \in \mathbb{N}^0$, we define $Q_{n,m}^r(x)$ of the polynomials $B_{n,r}(f;x)$ as follow:

$$Q_{n,m}^r(x) = \frac{\sum_{k=0}^{n} k^m b_{n.k}^r(x)}{\sum_{k=0}^{n} b_{n.k}^r(x)}.$$

Lemma 2.2

For $m \in \mathbb{N}^0$.The function $Q_{n,m}^r(x)$ has the following recurrence relation

$$Q_{n,m+1}^r(x) = \frac{x(1-x)}{r}\left(Q_{n,m}^r(x)\right)' + Q_{n,m}^r(x)Q_{n,1}^r(x).$$

Where $Q_{n,0}^r(x) = 1$ and $Q_{n,1}^r(x) \simeq \frac{((n+1)r-1)x}{rn} + \frac{1-r}{2rn}$

Proof.

By direct evaluation one has

$$Q_{n,0}^r(x) = 1, Q_{n,1}^r(x) \simeq \frac{((n+1)r-1)x}{rn} + \frac{1-r}{2rn}$$

Now,

$$\left(Q_{n,m}^r(x)\right)' = \left(\frac{\sum_{k=0}^{n} k^m b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)}\right)'$$

$$= \frac{r\sum_{k=0}^{n} b_{n,k}^r(x)\sum_{k=0}^{n} k^m b_{n,k}^{r-1}(x)\left(b_{n,k}(x)\right)'}{\left(\sum_{k=0}^{n} b_{n,k}^r(x)\right)^2} - \frac{r\sum_{k=0}^{n} k^m b_{n,k}^r(x)\sum_{k=0}^{n} b_{n,k}^{r-1}(x)\left(b_{n,k}(x)\right)'}{(\sum_{k=0}^{n} b_{n.k}^r(x))^2}$$

$$x(1-x)\left(Q_{n,m}^r(x)\right)'$$

$$= \frac{r\sum_{k=0}^{n} k^m b_{n,k}^{r-1}(x)x(1-x)\left(b_{n,k}(x)\right)'}{\sum_{k=0}^{n} b_{n,k}^r(x)}$$

$$- \frac{r\sum_{k=0}^{n} k^m b_{n,k}^r(x)\sum_{k=0}^{n} b_{n,k}^{r-1}(x)x(1-x)\left(b_{n,k}(x)\right)'}{\left(\sum_{k=0}^{n} b_{n,k}^r(x)\right)^2}$$

we have $x(1-x)\left(b_{n,k}(x)\right)' = (k-nx)b_{n,k}(x)$

$$x(1-x)\left(Q_{n,m}^r(x)\right)' = \frac{r\sum_{k=0}^{n} k^m(k-nx)b_{n,k}^r(x)}{\sum_{k=0}^{n} b_{n,k}^r(x)} - \frac{r\sum_{k=0}^{n} k^m b_{n.k}^r(x)\sum_{k=0}^{n}(k-nx)b_{n.k}^r(x)}{(\sum_{k=0}^{n} b_{n.k}^r(x))^2}$$

$$= \frac{r\sum_{k=0}^{n} k^{m+1} b_{n,k}^{r}(x)}{\sum_{k=0}^{n} b_{n,k}^{r}(x)} - rnx\frac{\sum_{k=0}^{n} k^{m} b_{n,k}^{r}(x)}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$$-\frac{r\sum_{k=0}^{n} k^{m} b_{n,k}^{r}(x)}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}\left[\frac{\sum_{k=0}^{n} k b_{n,k}^{r}(x)}{\sum_{k=0}^{n} b_{n,k}^{r}(x)} - \frac{nx\sum_{k=0}^{n} b_{n,k}^{r}(x)}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}\right]$$

$$x(1-x)\left(Q_{n,m}^{r}(x)\right)' = rQ_{n,m+1}^{r}(x) - rQ_{n,m}^{r}(x)Q_{n,1}^{r}(x).$$

Hence,

$Q_{n,m+1}^{r}(x) = \frac{x(1-x)}{r}\left(Q_{n,m}^{r}(x)\right)' + Q_{n,m}^{r}(x)Q_{n,1}^{r}(x).$ ■

Lemma 2.3

(i) $B_{n,r}(1;x) = 1;$

(ii) $B_{n,r}(t;x) = \frac{((n+1)r-1)}{rn}x + \frac{1-r}{2rn} + o(1);$

(iii) $B_{n,r}(t^2;x) = \frac{((n+1)^2r^2-3(n+1)r+2)}{r^2n^2}x^2 - \frac{\left((n+1)r^2-(2n+3)r+2\right)}{r^2n^2}x + \frac{(1-r)^2}{4r^2n^2} + o(1);$

(iv) $B_{n,r}(t^m;x) = \frac{((n+1)r-1)!}{r^m n^m((n+1)r-m-1)!}x^m + \frac{m(m-r)}{2}\frac{((n+1)r-1)!}{r^m n^m((n+1)r-m)!}x^{m-1} + TLP(x).$

Proof.

The consequence (i) holds immediately by the direct computations. The consequence (ii) claims by using Maple software, from Lemma 2.2, the consequences (iii) hold, Finally, (iv) using indction on m, the general relation gets immediate. ■

Lemma 2. 4

For, $x \in [0,1]$ and $s \in \mathbb{N}$, there exist the polynomials $q_{i,j,s}(x)$independent of n and k such that

$$x^s(1-x)^s\left(b_{n,k}^{r}(x)\right)^{(s)} = \sum_{\substack{2i+j\le s\\ i,j\ge 0}} n^i(rk-rnx)^j\, q_{i,j,s}(x)b_{n,k}^{r}(x).$$

Proof.

If $x = 0$ or $x = 1$, then the relation is true.

For $x \in (0,1)$ the proof is doing by induction on s. The relation is true for $s = 1$. Suppose that the relation is true for s. Then,

$$\left(b_{n,k}^{r}(x)\right)^{(s+1)} = \frac{d^{s+1}}{dx^{s+1}}\{b_{n,k}^{r}(x)\} = \frac{d}{dx}\left\{\left(b_{n,k}^{r}(x)\right)^{(s)}\right\}$$

$$= \binom{n}{k}^{r}\left\{\sum_{\substack{2i+j\le s\\ i,j\ge 0}} n^{i+1}(rk-rnx)^{j-1}(-jr\, q_{i,j,s}(x))x^{rk-s}(1-x)^{rn-rk-s}\right\}$$

$$+\binom{n}{k}^r\left\{\sum_{\substack{2i+j\le s\\ i,j\ge 0}} n^i(rk-rnx)^j(\,q'_{i,j,s}(x))x^{rk-s}(1-x)^{rn-rk-s}\right\}$$

$$+\binom{n}{k}^r\left\{\sum_{\substack{2i+j\le s\\ i,j\ge 0}} n^i(rk-rnx)^j(\,q_{i,j,s}(x))[(rk-s)x^{rk-s-1}(1-x)^{rn-rk-s}\right.$$

$$\left.+\,x^{rk-s}(rn-rk-s)(-1)(1-x)^{rn-rk-s-1}]\right\}$$

$$=\binom{n}{k}^r\left\{\sum_{\substack{2i+j\le s\\ i,j\ge 0}} n^{i+1}(rk-rnx)^{j-1}(-jr\,q_{i,j,s}(x))x^{rk-s}(1-x)^{rn-rk-s}\right\}$$

$$+\binom{n}{k}^r\left\{\sum_{\substack{2i+j\le s\\ i,j\ge 0}} n^i(rk-rnx)^j(\,q'_{i,j,s}(x))x^{rk-s}(1-x)^{rn-rk-s}\right\}$$

$$+\binom{n}{k}^r\left\{\sum_{\substack{2i+j\le s\\ i,j\ge 0}} n^i(rk-rnx)^{j+1}(\,q_{i,j,s}(x))x^{rk-s-1}(1-x)^{rn-rk-s-1}\right\}.$$

Hence,

$$x^{(s+1)}(1-x)^{(s+1)}\left(b^r_{n,k}(x)\right)^{(s+1)}=\left\{\sum_{\substack{2i+j\le s+1\\ i,j\ge 0}} n^i(rk-rnx)^j\,q_{i,j,s+1}(x)b^r_{n,k}(x)\right\}.$$

The relation is true for $s+1$. ■

For $m\in\mathbb{N}^0$, the m-th order moment $T^r_{n,m}(x)$ for the polynomials $B_{n,r}$ is defined as

$$T^r_{n,m}(x)=B_{n,r}((t-x)^m;x)=\frac{\sum_{k=0}^n b^r_{n,k}(x)\left(\frac{k}{n}-x\right)^m}{\sum_{k=0}^n b^r_{n,k}(x)}.$$

Lemma 2.5

The function $T^r_{n,m}(x)$ has the following recurrence relation

$$T^r_{n,m+1}(x)=\frac{x(1-x)}{rn}\left(\left(T^r_{n,m}(x)\right)'+mT^r_{n,m-1}(x)\right)+T^r_{n,m}(x)T^r_{n,1}(x), m\in\mathbb{N}$$

where $T_{n,0}^{r}(x)=1$, and $T_{n,1}^{r}(x)\simeq\frac{(1-r)(1-2x)}{2rn}$.

Proof.

Clearly, $T_{n,0}^{r}(x)=1$, and $T_{n,1}^{r}(x)\simeq\frac{(1-r)(1-2x)}{2rn}$. Then,

$$\left(T_{n,m}^{r}(x)\right)'$$

$$=\frac{\sum_{k=0}^{n}b_{n,k}^{r}(x)\sum_{k=0}^{n}b_{n,k}^{r}(x)\left((-1)m\left(\frac{k}{n}-x\right)^{m-1}\right)+\sum_{k=0}^{n}b_{n,k}^{r}(x)\sum_{k=0}^{n}\left(b_{n,k}^{r}(x)\right)'\left(\frac{k}{n}-x\right)^{m}}{\left(\sum_{k=0}^{n}b_{n,k}^{r}(x)\right)^{2}}$$

$$-\frac{\sum_{k=0}^{n}b_{n,k}^{r}(x)\left(\frac{k}{n}-x\right)^{m}\sum_{k=0}^{n}\left(b_{n,k}^{r}(x)\right)'}{\left(\sum_{k=0}^{n}b_{n,k}^{r}(x)\right)^{2}}$$

$$=-mT_{n,m-1}^{r}(x)+\frac{\sum_{k=0}^{n}\left(b_{n,k}^{r}(x)\right)'\left(\frac{k}{n}-x\right)^{m}}{\sum_{k=0}^{n}b_{n,k,}^{r}(x)}-\frac{\sum_{k=0}^{n}b_{n,k}^{r}(x)\left(\frac{k}{n}-x\right)^{m}\sum_{k=0}^{n}\left(b_{n,k}^{r}(x)\right)'}{\left(\sum_{k=0}^{n}b_{n,k}^{r}(x)\right)^{2}}.$$

$$x(1-x)\left(\left(T_{n,m}^{r}(x)\right)'+mT_{n,m-1}^{r}(x)\right)$$

$$=\frac{\sum_{k=0}^{n}x(1-x)\left(b_{n,k}^{r}(x)\right)'\left(\frac{k}{n}-x\right)^{m}}{\sum_{k=0}^{n}b_{n,k}^{r}(x)}$$

$$-\frac{\sum_{k=0}^{n}b_{n,k}^{r}(x)\left(\frac{k}{n}-x\right)^{m}\sum_{k=0}^{n}x(1-x)\left(b_{n,k}^{r}(x)\right)'}{\left(\sum_{k=0}^{n}b_{n,k}^{r}(x)\right)^{2}}$$

$$=\frac{rn\sum_{k=0}^{n}b_{n,k}^{r}(x)\left(\frac{k}{n}-x\right)^{m+1}}{\sum_{k=0}^{n}b_{n,k}^{r}(x)}-\frac{rn\sum_{k=0}^{n}b_{n,k}^{r}(x)\left(\frac{k}{n}-x\right)^{m}\sum_{k=0}^{n}b_{n,k}^{r}(x)\left(\frac{k}{n}-x\right)}{\left(\sum_{k=0}^{n}b_{n,k}^{r}(x)\right)^{2}}$$

$$x(1-x)\left(\left(T_{n,m}^{r}(x)\right)'+mT_{n,m-1}^{r}(x)\right)=rnT_{n,m+1}^{r}(x)-rnT_{n,m}^{r}(x)T_{n,1}^{r}(x)$$

$$T_{n,m+1}^{r}(x)=\frac{x(1-x)}{rn}\left(\left(T_{n,m}^{r}(x)\right)'+mT_{n,m-1}^{r}(x)\right)+T_{n,m}^{r}(x)T_{n,1}^{r}(x).\ \blacksquare$$

Using the recurrence relation above and the direct evaluations, one has

$$T_{n,2}^{r}(x)\simeq\frac{x(1-x)}{rn}+\frac{(r-1)x(1-x)}{r^{2}n^{2}}+\frac{(1-2x)^{2}(1-r)^{2}}{4r^{2}n^{2}};$$

Lemma 2.6

The order of the above moment is $T^{r}_{n,m}(x) = O\left((rn)^{-\left[\frac{m+1}{2}\right]}\right), \forall x \in [0,1]$, where $\left[\frac{m+1}{2}\right]$ the integer part of $\frac{m+1}{2}$.

Proof.

From the values of $T^{r}_{n,0}(x) = 1 = O\left((rn)^{-\left[\frac{1}{2}\right]}\right) = O(1)$

$T^{r}_{n,1}(x) = \frac{(1-r)(1-2x)}{2rn} \simeq O((rn)^{-1})$. The relation above holds.

Suppose that the relation be true for m, show it true for $m+1$

$$rnT^{r}_{n,m+1}(x) = O\left((rn)^{-\left[\frac{m+1}{2}\right]}\right) + O\left((rn)^{-\left[\frac{m}{2}\right]}\right) + O\left((rn)^{-\left[\frac{m+1}{2}\right]}\right)$$

$$rnT^{r}_{n,m+1}(x) = \begin{cases} O\left((rn)^{-\left[\frac{m+1}{2}\right]}\right) & \text{if } m \text{ is odd} \\ O\left((rn)^{-\left[\frac{m}{2}\right]}\right), & \text{if } m \text{ is even} \end{cases}$$

$$T^{r}_{n,m+1}(x) = \begin{cases} (rn)^{-1}O\left((rn)^{-\left[\frac{m+1}{2}\right]}\right) & \text{if } m \text{ is odd} \\ (rn)^{-1}O\left((rn)^{-\left[\frac{m}{2}\right]}\right), & \text{if } m \text{ is even} \end{cases} = \begin{cases} O\left((rn)^{\left[\frac{-m-3}{2}\right]}\right) & \text{if } m \text{ is odd} \\ O\left((rn)^{\left[\frac{-m-2}{2}\right]}\right), & \text{if } m \text{ is even} \end{cases}$$

$$= O\left((rn)^{-\left[\frac{m+2}{2}\right]}\right) = O\left((rn)^{-\left[\frac{(m+1)+1}{2}\right]}\right) \text{ is hold for } m+1. \blacksquare$$

3. Main Results

The convergence theorem and the Voronovskaya-type asymptotic formula for the polynomials $B_{n,r}(f;x)$ are given in this section.

3.1 Theorem (Korovkin Theorem)

If $f \in C[0,1]$, exists and continuous and $x \in [0,1]$, then

$$\lim_{n\to\infty} B_{n,r}(f(t);x) = f(x).$$

Proof.

From Lemma 2.3, the proof of this theorem holds. ■

3.2 Theorem (Voronovskaya-type asymptotic formula)

Let $f \in C[0,1]$ and $x \in (0,1)$, if f'' exists, the polynomials $B_{n,r}(f,x)$ satisfy the asymptotic relation:

$$\lim_{n\to\infty} n\{B_{n,r}(f,x)-f(x)\} = \frac{(1-\mathrm{r})(1-2\mathrm{x})}{2r} f'(x)+\frac{x(1-x)}{2r}f''(x).$$

Proof.

By Taylor's expansion, one has

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2!}(t-x)^2+\varepsilon(t,x)(t-x)^2$$

where $\varepsilon(t,x)\to 0\ \ as\ t\to x$. Therefore,

$$B_{n,r}(f(t),x)$$

$$= f(x)B_{n,r}(1;x)+f'(x)B_{n,r}\left(\left(\frac{k}{n}-x\right);x\right)+\frac{1}{2}f''(x)B_{n,r}\left(\left(\frac{k}{n}-x\right)^2;x\right)$$

$$+B_{n,r}\left(\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^2;x\right).$$

Then,

$$n\{B_{n,r}(f;x)-f(x)\}=nT_{n,1}^{r}(x)f'(x)+nT_{n,2}^{r}(x)\frac{f''(x)}{2!}+n\left(B_{n,r}(\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^2;x\right)$$

$$\lim_{n\to\infty} n\{B_{n,r}(f;x)-f(x)\}=\lim_{n\to\infty} n\left(\frac{(1-\mathrm{r})(1-2\mathrm{x})}{2rn}\right)f'(x)$$

$$+\lim_{n\to\infty} n\left(\frac{(1-r)^2(1-2x)^2}{4r^2n^2}+\frac{(r-1)x(1-x)}{r^2n^2}+\frac{x(1-x)}{rn}\right)\frac{f''(x)}{2!}$$

$$+\lim_{n\to\infty} n\left(B_{n,r}(\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^2;x\right)$$

$$=\frac{(1-\mathrm{r})(1-2\mathrm{x})}{2r} f'(x)+\frac{x(1-x)}{r}\frac{f''(x)}{2!}+\lim_{n\to\infty} n\left(B_{n,r}(\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^2;x\right).$$

Now,

$$n\left|B_{n,r}(\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^2;x)\right|=\frac{n\sum_{k=0}^{n}b_{n,k}^{r}(x)\left(\left|\varepsilon\left(\frac{k}{n},x\right)\right|\left(\frac{k}{n}-x\right)^2\right)}{\sum_{k=0}^{n}b_{n,k}^{r}(x)}$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

Now, given $\varepsilon>0\ \exists\delta>0$ such that either $0<\left|\frac{k}{n}-x\right|<\delta\to\left|\varepsilon\left(\frac{k}{n},x\right)\right|<\varepsilon$

Or $\left|\frac{k}{n}-x\right|\geq\delta\to\left|\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^2\right|<Mt^{\alpha}$

$$n\left|B_{n,r}(\varepsilon(t,x)(t-x)^2;x)\right|$$

$$\leq n \frac{\sum_{\left|\frac{k}{n}-x\right|<\delta} b_{n,k}^{r}(x) \left|\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^{2}\right|}{\sum_{k=0}^{n} b_{n,k}^{r}(x)} + n \frac{\sum_{\left|\frac{k}{n}-x\right|\geq\delta} b_{n,k}^{r}(x) \left|\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^{2}\right|}{\sum_{k=0}^{n} b_{n,k}^{r}(x)}$$

$\leq n\varepsilon T_{n,2}^{r}(x) + nO((nr)^{-\gamma}) \forall \gamma > 0$

$= o(1)$ since ε is arbitrary.

Hence $\lim_{n\to\infty} n\ B_{n,r}\left(\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^{2};\ x\right) \to 0$ as $n \to \infty$.■

4. Numerical Example.

This example compares among the polynomials, $B_n(g;x) = B_{n,1}(g;x)$ (brown color), $B_{n,2}(g;x)$ (blue color), $B_{n,3}(g;x)$ (red color), $B_{n,5}(g;x)$ (green color) and the test function is $g(t) = \sin 10t \in C[0,1]$ (black color). Also, evaluates the average absolute errors $\frac{\sum_{i=0}^{m}|B_{n,r}(g;x_i)-g(x_i)|}{m}$, for some values of $n = 50, 100$ and $x_i \in [0,1], i = 0,1,\ldots m$ where $r = 1,2,3,5$. It turns out that, the numerical results become more accurate whenever r increases.

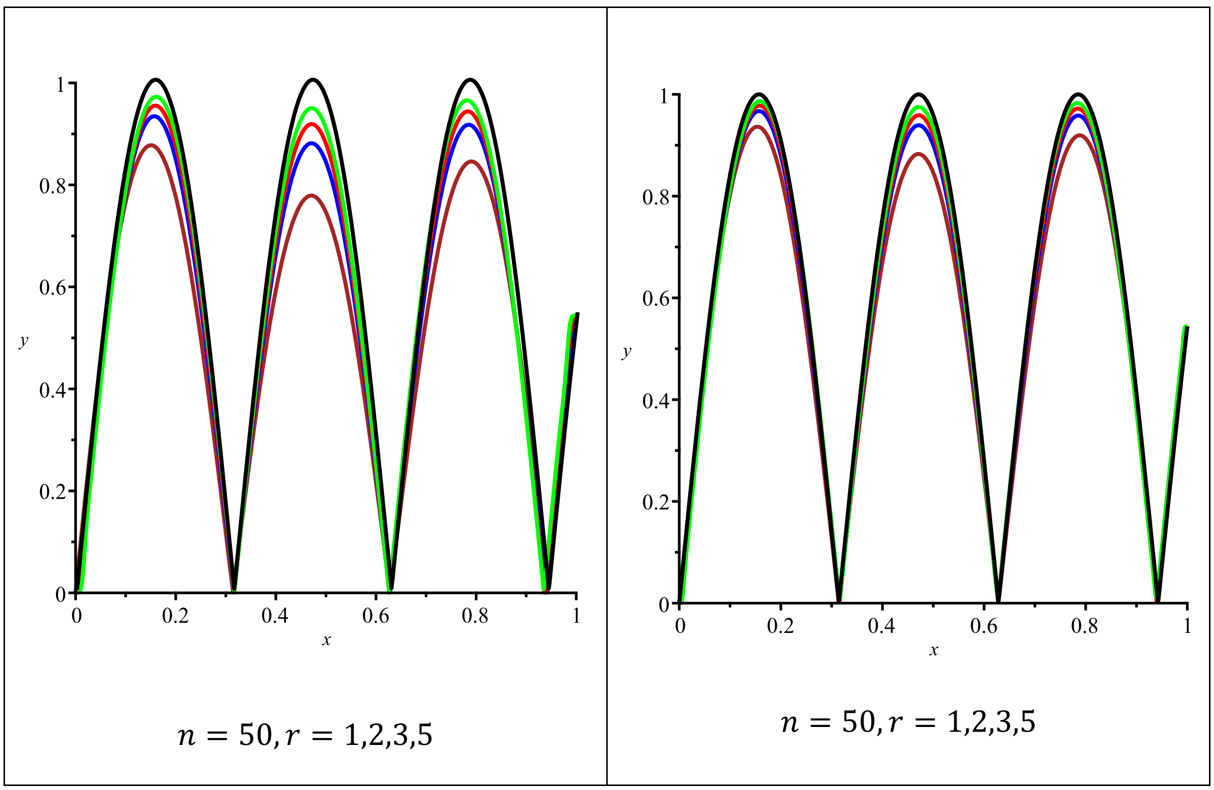
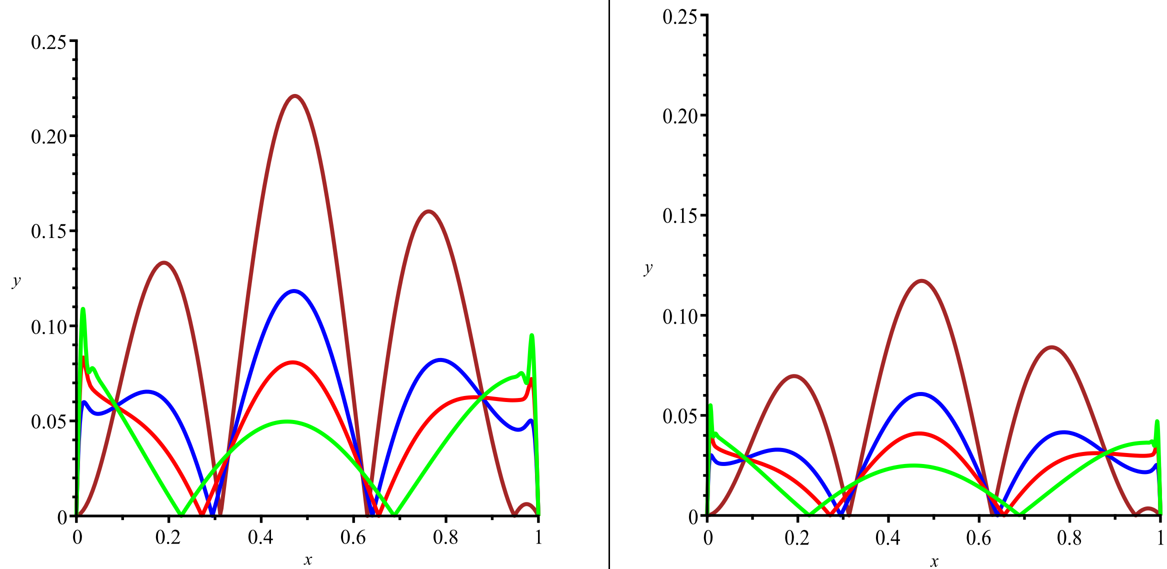


Fig. 1: The convergence of the polynomials $B_{n,1}, B_{n,2}, B_{n,3}, B_{n,5}$ to the test function g



$n = 50, r = 1,2,3,5$

$n = 100, r = 1,2,3,5$

Fig. 2: The graph of average error functions $|B_{n,r} - g|, r = 1,2,3,5$.

Table 1: The average absolute errors of $\frac{\sum_{i=0}^{m}|B_{n,r}(g;x_i)-g(x_i)|}{m}$, $x_i \in [0,1]$ and $|x_i - x_{i+1}| = 0.1$.

r	$n = 50$	$n = 100$
1	0.08860203574	0.04643920283
2	0.05094378977	0.02585223416
3	0.03897204886	0.01956340062
5	0.02910692553	0.01445737662

Conclusions

This study is a generalization of some well-known sequences of linear positive operators which are deduced as a special case from the r-th powers of the rational Bernstein polynomials. Also, the study gives a numerical example which is shown the numerical convergence of the polynomials $B_{n,r}(f;x)$ to the test function. This numerical convergence shows by the graphs of the $B_{n,r}(f;x)$ with the function $f(x)$. The numerical results appeared that numerical results became more accurate whenever r increase.

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القوى الرائية لمتعددة حدود برنشتاين الكسرية

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المستخلص

في هذا البحث تم تعريف متعددات حدود جديدة لبرنشتاين من القوى r ودراسة تقاربها والعزم من الرتبة m وصيغة فرونوفسكيا للتقارب لها. تم تطبيق مثال عددي لتقريب دالة اختبار $\sin 10x$. النتائج العددية التي تم الحصول عليها من تطبيق ذلك المثال تظهر بان متعددات حدود برنشتاين الكسرية من النمط r تعطي نتائج أفضل من متعددات حدود برنشتاين الكسرية التربيعية ومتعددات حدود برنشتاين الاصلية. تم اجراء المقارنة عن طريق رسم المخططات البيانية للدالة وتقريباتها وحساب معدل الأخطاء المطلقة التي حدثت بين هذه التقريبات.