MAXIMUM A POSTERIORI ESTIMATION (MAP) FOR NESTED-FACTORIAL EXPERIMENTS OF THREE PHASES WITH REPEATED MEASUREMENTS

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Abstract

Agronomic experiments are often complex and difficult to interpret, and the proper use of appropriate statistical methodology is essential for an efficient and reliable analysis. In this paper, the basics of the statistical analysis of nested factorial experiments for three phases with repeated measurements are discussed using real examples from agricultural field trials. The presented experiment consists of three phases factors. A third factor represents the experimental units (subjects) taken as measurements or repeated treatments of the experimental units. While, the fourth factor represents the treatments (repeated measurements), this factor will in turn interact with the other factors that resulted from the nested factorial experiments. As the maximum a posteriori estimators (MAP) is a common method of point estimation in Bayesian Statistics, this study used a MAP approach to provide conclusions about the treatment of nested factor experiments for three phases. A complete block design was used to apply the suggested model to the real world based on data from cold storage unit of Horticulture Department, Agriculture College, Baghdad University during the spring growth seasons of 2011- 2012 years.

Keywords: Maximum a posteriori estimation (MAP), Nested factorial experiment, Repeated measurements**.**

1. Introduction

A repeated measures design element refers to the practice of measuring the outcome on each study unit multiple times. Most frequently the multiple measurements occur over time, although other factors can be studied such as repeated exposure of individuals to changing levels such as sound or light [1]. Inclusion of a repeated measurements design element can be desirable for a variety of reasons. Repeated measurement of outcomes is often included if the impact of the factor, time, or changes in the effect of treatment over time are of explicit interest to the investigator [2, 3]. Studying multiple outcomes for each subject might also allow investigators to reduce subject-to-subject and within-subject variation in the investigation of the relative effects of different treatments. Reduced variability might increase the study's power, and this is another rationale for implementing repeated measurements [4] Inclusion of a repeated measurements design element can be desirable for a variety of reasons. Repeated measurement of outcomes is often included if the impact of the factor, time, or changes in the effect of treatment over time are of explicit interest to the investigator. Studying multiple outcomes for each subject might also allow investigators to reduce subject-to-subject and within-subject variation in the investigation of the relative effects of different treatments. Reduced variability might increase the study's power, and this is another rationale for implementing repeated measurements [5, 6].

Repeated measurement designs with nested rows and columns have received much attention in the literature. Recently, Bailey and Lacka (2015) pioneered work on such designs when the treatments consist of treatment combinations of two factors, plus a control. They called these nested row-column designs for near factorial experiments and discussed how these designs can have wide ranging applications, notably in agriculture, plant protection experiments and clinical trials. On the other hand, Bose & Mukerjee (2018) proposed a method for constructing a nested row-column design, involving a control treatment and test treatments, starting from a Latin square and an incomplete block design.

This study suggested an intertwined experience of two factors. The first factor (β) has (n) levels. The second factor (γ) has (q) levels, as the levels of the second factor (γ) are intertwined within the levels of the first factor (β), and then this relationship is denoted by the symbol γ(β) the second factor (γ) is called the interfering factor and the first factor (β) is a factor Interference (nested factor) then the experimental units are taken for each level of the interfering factor (γ) where these experimental units are considered as a third factor (η) and has (p) levels that are interconnected within both factors (β) and (γ) and symbolize this relationship η(βγ) In this case, we obtain an interconnected global experiment in three stages. When taking more than one response for each experimental unit at different periods where repeated measurements are formed which can be counted as a fourth factor (π) and has (s) levels and be crossed with levels factor (β) as well as intersecting with factor levels (γ) within factor levels (β) as well as intersecting with factor levels (η) within the levels of both factors (β) and (γ) Thus we will have an interfering factorial experiment with the repeated measurements. A maximum posteriori estimation (MAP) approach [9, 10] was employed to making inferences on the nested factorial experiments for three phases with repeated measurements. This paper is organized as follows. Section 2 reviews Prior Distribution for Nested-Factorial Experiments of three phase with Repeated Measurements. Section 3 presents the Maximum a posteriori estimation for Nested-Factorial Experiments of three phase with repeated measurements. Section 4 presents the numerical experiments Results and conclusions discussed in Section 5.

2. Prior Distribution for Nested-Factorial Experiments of three phases with Repeated Measurements.

In this section a maximum posteriori estimation (MAP) approach is formulated to determine the inferences on the treatment of nested factorial experiments for three phases with repeated measurements. Table 2.1 below summarizes the symbols that are used.

Table 2.1: Summary of symbols that are used in the model

Symbol	Description
n	Index levels of the first factor β (indexed by *i*).
q	Index levels of the second factor γ (indexed by *j*).
p	Index levels of the third factor η (indexed by *k*).
s	Index levels of the fourth factor Π (indexed by *l*).
μ	over all mean.
β_i	Added effect of the i^{th} level of the first factor β.
$\gamma_{j(i)}$	Added effect of the j^{th} level of the second factor γ that nested within the i^{th} level of the first factor β.
$\eta_{k(ij)}$	Added effect of the k^{th} level of the third factor η that nested within the two levels (i,j) of the first factor β and the second factor γ, respectively, which is a random variable, i.e.: $\eta_{k(ij)} \sim \text{i.i.d} \quad N\left(0, \sigma_\eta^2\right)$.
Π_1	Added effect of the first level of the fourth factor Π that represents a repeated measurement.
$(\beta\Pi)_{i1}$	Added effect of the interaction between level i of the first factor β and the first level of the fourth factor Π.
$(\Pi\gamma)_{1j(i)}$	Added effect of the interaction between first level of the fourth factor Π and the j^{th} level of the second factor γ nested within the i^{th} level of the first factor β.
	Average service time of the ALS and BLS ambulances in the system, respectively.
$\varepsilon_{k1(ij)}$	Added effect of random error resulting of the interaction effect of the l^{th} of the fourth

factor π and the k^{th} from the third factor η intervening within the levels (i, j) of the first factors β and the second factors γ, respectively.

Also define the following set of conditions:

$\sum_{i=1}^{n} \beta_i = 0, \quad \sum_{j=1}^{q} \gamma_{j(i)} = 0 \qquad ; \forall \ i = 1, \dots, n$

$\sum_{l=1}^{s} \pi_l = 0, \quad \sum_{i=1}^{n} (\beta\pi)_{il} = 0 \qquad ; \forall \ l = 1, \dots, s$

$\sum_{l=1}^{s} (\beta\pi)_{il} = 0 \quad ; \forall \ i = 1, \dots, n$

$\sum_{j=1}^{q} (\pi\gamma)_{lj(i)} = 0 \quad ; \forall \ i = 1, \dots, n \ \text{ and } \ \forall \ l = 1, \dots, s$

$\sum_{l=1}^{s} (\pi\gamma)_{lj(i)} = 0 \quad ; \forall \ j = 1, \dots, q \ \text{ and } \ \forall \ i = 1, \dots, n$

$\sum_{i=1}^{n} \sum_{j=1}^{q} \gamma_{j(i)} = 0, \quad \sum_{j=1}^{q} \sum_{l=1}^{s} (\pi\gamma)_{lj(i)} = 0 \qquad ; \forall \ i = 1, \dots, n$

$\sum_{i=1}^{n} \sum_{j=1}^{q} (\pi\gamma)_{lj(i)} = 0 \quad ; \forall \ l = 1, \dots, s$

With these definitions, the proposed model can be formulated as:

$$y_{ijkl} = \mu + \beta_i + \gamma_{j(i)} + \eta_{k(ij)} + \pi_l + (\beta\pi)_{il} + (\pi\gamma)_{lj(i)} + \varepsilon_{kl(ij)} \tag{1}$$

This model assumed each of random error factor $\varepsilon_{k1(ij)}$ and the third factor $\eta_{k(ij)}$ are independent, such that:

$$\varepsilon_{kl(ij)} \sim \ \text{i.i.d} \ \ N\left(0, \sigma^2_{(\pi\eta)}\right) \ \text{ and } \ \eta_{k(ij)} \ \sim \ \text{i.i.d} \ \ N\left(0, \sigma^2_{\eta}\right) \tag{2}$$

Thus, the above model parameters can be estimated using the least squares method as:

Let $Q = \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{p} \sum_{l=1}^{s} \left(y_{ijkl} - E\left(y_{ijkl}\right)\right)^2 = \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{p} \sum_{l=1}^{s} (\varepsilon_{kl(ij)} + \eta_{k(ij)})^2$

$$\therefore Q = \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{p} \sum_{l=1}^{s} (y_{ijkl} - \mu - \beta_i - \gamma_{j(i)} - \pi_l - (\beta\pi)_{il} - (\pi\gamma)_{lj(i)})^2 \tag{3}$$

By derivation the equation (3) above and setting it equal to zero, get:

$\rightarrow \frac{\partial Q}{\partial \mu} = -2 \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{p} \sum_{l=1}^{s} (y_{ijkl} - \mu - \beta_i - \gamma_{j(i)} - \pi_l - (\beta\pi)_{il} - (\pi\gamma)_{lj(i)})^2 = 0$

$\sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{p} \sum_{l=1}^{s} y_{ijkl} - nqps\mu = 0 \ \rightarrow \ \hat{\mu} = \bar{y}_{....}$,

Derive the equation (3) for β_i and setting it equal to zero, get:

$\frac{\partial Q}{\partial \beta_i} = -2 \sum_{j=1}^{q} \sum_{k=1}^{p} \sum_{l=1}^{s} (y_{ijkl} - \mu - \beta_i - \gamma_{j(i)} - \pi_l - (\beta\pi)_{il} - (\pi\gamma)_{lj(i)})$

$\rightarrow \ \sum_{j=1}^{q} \sum_{k=1}^{p} \sum_{l=1}^{s} y_{ijkl} - nps\hat{\mu} - qps\beta_i = 0 \ \ \rightarrow \frac{y_{i...}}{qps} - \hat{\mu} - \beta_i = 0$

$\rightarrow \ \bar{y}_{i...} - \bar{y}_{....} - \beta_i = 0 \rightarrow \widehat{\beta_i} = \bar{y}_{i...} - \bar{y}_{....}$.

By using the least squares method, obtained an estimate of the all parameters as follows:

$$\widehat{\gamma_{j(i)}} = \bar{y}_{ij..} - \bar{y}_{i...}\,, \qquad \widehat{\eta_{k(ij)}} = \bar{y}_{ijk.} - \bar{y}_{ij..}\,, \qquad \widehat{\pi_l} = \bar{y}_{...l} - \bar{y}_{....}$$

$$\widehat{(\beta\pi)}_{il} = \bar{y}_{i..l} - \bar{y}_{i...} - \bar{y}_{...l} + \bar{y}_{....}\,, \qquad \widehat{(\pi\gamma)}_{lj(i)} = \bar{y}_{ij.l} - \bar{y}_{ij..} - \bar{y}_{i..l} + \bar{y}_{i...}$$

The sum of the squares will be total

$$SS_{\beta} = qps\sum_{i=1}^{n}\left(\bar{y}_{i...} - \bar{y}_{....}\right)^2$$

$$SS_{\gamma(\beta)} = ps\sum_{i=1}^{n}\sum_{j=1}^{q}\left(\bar{y}_{ij..} - \bar{y}_{i...}\right)^2$$

$$SS_{\eta(\beta\gamma)} = s\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\left(\bar{y}_{ijk.} - \bar{y}_{ij..}\right)^2$$

$$SS_{\pi} = nqp\sum_{l=1}^{s}\left(\bar{y}_{...l} - \bar{y}_{....}\right)^2$$

$$SS_{\beta\pi} = qp\sum_{i=1}^{n}\sum_{l=1}^{s}(\bar{y}_{i..l} - \bar{y}_{i...} - \bar{y}_{...l} + \bar{y}_{....})^2$$

$$SS_{\pi\gamma(\beta)} = p\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{l=1}^{s}(\bar{y}_{ij.l} - \bar{y}_{ij..} - \bar{y}_{i..l} + \bar{y}_{i...})^2$$

$$SS_{E} = \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{s}(y_{ijkl} - \bar{y}_{ijk.} - \bar{y}_{ij.l} + \bar{y}_{ij..})^2$$

where,

$$\bar{y}_{....} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{s}y_{ijkl}}{nqps}\,, \qquad \bar{y}_{i...} = \frac{\sum_{j=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{s}y_{ijkl}}{qps}$$

$$\bar{y}_{.j..} = \frac{\sum_{i=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{s}y_{ijkl}}{nps}\,, \qquad \bar{y}_{..k.} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{l=1}^{s}y_{ijkl}}{nqs}$$

$$\bar{y}_{...l} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}y_{ijkl}}{nqp}\,, \qquad \bar{y}_{.jk.} = \frac{\sum_{i=1}^{n}\sum_{l=1}^{s}y_{ijkl}}{ns}$$

$$\bar{y}_{.j.l} = \frac{\sum_{i=1}^{n}\sum_{k=1}^{p}y_{ijkl}}{np}\,, \qquad \bar{y}_{..kl} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}y_{ijkl}}{nq}$$

$$\bar{y}_{.jkl} = \frac{\sum_{i=1}^{n}y_{ijkl}}{n}\,, \qquad \bar{y}_{ij..} = \frac{\sum_{i=1}^{p}\sum_{l=1}^{s}y_{ijkl}}{ps}$$

$$\bar{y}_{i..l} = \frac{\sum_{i=1}^{q}\sum_{l=1}^{p}y_{ijkl}}{qp}\,, \qquad \bar{y}_{ijk.} = \frac{\sum_{i=1}^{s}y_{ijkl}}{s}\,, \qquad \bar{y}_{ij.l} = \frac{\sum_{i=1}^{p}y_{ijkl}}{p}$$

Table 2.2 below summarizes the ANOVA table for nested-factorial experiments of three phases with repeated measurements.

Table 2.1: ANOVA table for nested-factorial experiments of three phases with repeated measurements.

Source of variation	**d.f**	**S.S**	**M.S**	**(E.M.S)**	**F_{table}**
Between (sub.)	nqp-1	SS_B			
β	n-1	SS_β	$\frac{SS_\beta}{n-1}$	$\frac{qps}{(n-1)}\sum_{i=1}^{n}\beta_i^2 + s\sigma_\eta^2$	$\frac{MS_\beta}{MS_{E(B)}}$
$\gamma(\beta)$	n(q-1)	$SS_{\gamma(\beta)}$	$\frac{SS_{\gamma(\beta)}}{n(q-1)}$	$\frac{ps}{n(q-1)}\sum_{i=1}^{n}\sum_{j=1}^{q}(\gamma(\beta))_{j(i)}^2 + s\sigma_\eta^2$	$\frac{MS_{\gamma(\beta)}}{MS_{E(B)}}$
Error (Bet.) $=\eta(\beta\gamma)$	nq(p-1)	$SS_{E(B)}$	$\frac{SS_{E(B)}}{nq(p-1)}$	$S\sigma_\eta^2$	
Within (sub.)	nap (s-1)	SS_W			
π	s-1	SS_π	$\frac{SS_\pi}{(s-1)}$	$\frac{nqp}{(s-1)}\sum_{l=1}^{s}\pi_l^2 + \sigma_{(\pi\eta)}^2$	$\frac{MS_\pi}{MS_{E(W)}}$
$\beta\pi$	(n-1) (s-1)	$SS_{\beta\pi}$	$\frac{SS_{\beta\pi}}{(n-1)(s-1)}$	$\frac{qp}{n(q-1)}\sum_{i=1}^{n}\sum_{l=1}^{s}(\beta\pi)_{il}^2 + \sigma_{(\pi\eta)}^2$	$\frac{MS_{\beta\pi}}{MS_{E(W)}}$
$\pi\gamma(\beta)$	n(s-1) (q-1)	$SS_{\pi\gamma(\beta)}$	$\frac{SS_{\pi\gamma(\beta)}}{n(s-1)(q-1)}$	$\frac{p}{n(s-1)(q-1)}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{l=1}^{s}(\pi\gamma(\beta))_{lj(i)}^2 + \sigma_{(\pi\eta)}^2$	$\frac{MS_{\pi\gamma(\beta)}}{MS_{E(W)}}$
Error(Within)	nq(s-1) (p-1)	$SS_{E(W)}$	$\frac{SS_{E(W)}}{nq(s-1)(p-1)}$	$\sigma_{(\pi\eta)}^2$	
TOTAL	nqps-1	SS_T			

3. Maximum a Posteriori Estimation for Nested-Factorial Experiments of Three Phases with Repeated Measurements

Suppose that the prior distribution of the coefficients for the suggested model is given by:

$$\mu \sim N(\mu_\mu, \sigma_\mu^2), \quad \beta_i \sim N(\mu_\beta, \sigma_\beta^2), \quad \gamma_{j(i)} \sim N(\mu_\gamma, \sigma_\gamma^2), \quad \pi_l \sim N(\mu_\pi, \sigma_\pi^2),$$

$$(\beta\pi)_{il} \sim N(\mu_{\beta\pi}, \sigma_{(\beta\pi)}^2), \quad (\pi\gamma)_{lj(i)} \sim N(\mu_{\pi\gamma}, \sigma_{(\pi\gamma)}^2),$$

$$\sigma_\eta^2 \sim IG(\alpha_\eta, \beta_\eta) \text{ and } \sigma_{(\pi\eta)}^2 \sim IG(\alpha_{(\pi\eta)}, \beta_{(\pi\eta)}) \qquad (4)$$

In this section, the maximum a posteriori estimation of each coefficients $\left(\mu, \beta_i, \gamma_{j(i)}, \pi_l, (\beta\pi)_{il}, (\pi\gamma)_{lj(i)}, \sigma_\eta^2, \sigma_{(\pi\eta)}^2\right)$ for the suggested model has been formulated.

The likelihood function for the equation (1) is given by:

$$f\left(y_{ijkl}/\mu, \beta_i, \gamma_{j(i)}, \pi_l, (\beta\pi)_{il}, (\pi\gamma)_{lj(i)}, \sigma_\eta^2, \sigma_{(\pi\eta)}^2\right) = \prod_{i=1}^{n}\prod_{j=1}^{q}\prod_{k=1}^{p}\prod_{l=1}^{s}\frac{1}{\sqrt{2\pi(\sigma_\eta^2+\sigma_{(\pi\eta)}^2)}} \exp\left[\frac{-(y_{ijkl}-\mu-\beta_i-\gamma_{j(i)}-\pi_l-(\beta\pi)_{il}-(\pi\gamma)_{lj(i)})^2}{2(\sigma_\eta^2+\sigma_{(\pi\eta)}^2)}\right]$$

$$= \frac{1}{\sqrt{2\pi(\sigma_\eta^2+\sigma_{(\pi\eta)}^2)}} \exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{s}(y_{ijkl}-\mu-\beta_i-\gamma_{j(i)}-\pi_l-(\beta\pi)_{il}-(\pi\gamma)_{lj(i)})^2}{2(\sigma_\eta^2+\sigma_{(\pi\eta)}^2)}\right] \quad (5)$$

So, the maximum a posteriori estimation (MAP) of each coefficient for the suggested model and the variances ($\sigma_{(\pi\eta)}^2$) and (σ_η^2) can be computing as follows:

The function to be maximized is then given by

$$f(\mu)f\big(y_{ijkl}/\mu,\beta_i,\gamma_{j(i)},\pi_l,(\beta\pi)_{il},(\pi\gamma)_{lj(i)},\sigma_\eta^2,\sigma_{(\pi\eta)}^2\big) = \pi(\mu)L\big(\mu,\beta_i,\gamma_{j(i)},\pi_l,(\beta\pi)_{il},(\pi\gamma)_{lj(i)},\sigma_\eta^2,\sigma_{(\pi\eta)}^2\big) =$$
$$(2\pi\sigma_\mu^2)^{-\frac{1}{2}} \exp\left[\frac{-(\mu-\mu_\mu)^2}{2\sigma_\mu^2}\right] \times \frac{1}{\sqrt{2\pi(\sigma_\eta^2+\sigma_{(\pi\eta)}^2)}} \exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{s}(y_{ijkl}-\mu-\beta_i-\gamma_{j(i)}-\pi_l-(\beta\pi)_{il}-(\pi\gamma)_{lj(i)})^2}{2(\sigma_\eta^2+\sigma_{(\pi\eta)}^2)}\right]$$

Which is equivalent to minimizing the following function of μ:

$$\left[\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{s}(y_{ijkl}-\mu-\beta_i-\gamma_{j(i)}-\pi_l-(\beta\pi)_{il}-(\pi\gamma)_{lj(i)})^2}{(\sigma_\eta^2+\sigma_{(\pi\eta)}^2)}\right] + \left[\frac{(\mu-\mu_\mu)^2}{\sigma_\mu^2}\right]$$

Thus, the MAP estimator for μ is given by:

$$\hat{\mu}_{MAP} = \left[\frac{(nqps)(\sigma_\mu^2)(\bar{y}_{....})}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+nqps\sigma_\mu^2}\right] + \left[\frac{(\sigma_\eta^2+\sigma_{(\pi\eta)}^2)\mu_\mu}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+nqps\sigma_\mu^2}\right] = \frac{(nqps)(\sigma_\mu^2)(\bar{y}_{....})+\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)\mu_\mu}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+nqps\sigma_\mu^2} \quad (6)$$

The MAP estimator for β_i is given by

$$\hat{\beta}_{(i)_{\mu_{MAP}}} = \left[\frac{(qps)\left(\sigma_\beta^2\right)(\bar{y}_{i...}-\bar{y}_{....})}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+qps\sigma_\beta^2}\right] + \left[\frac{(\sigma_\eta^2+\sigma_{(\pi\eta)}^2)\mu_\beta}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+qps\sigma_\beta^2}\right] = \frac{(qps)(\sigma_\beta^2)(\bar{y}_{i...}-\bar{y}_{....})+\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)\mu_\beta}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+qps\sigma_\beta^2} \quad (7)$$

The MAP estimator for $\gamma_{j(i)}$ is

$$\hat{\gamma}_{j(i)_{MAP}} = \frac{(ps)(\sigma_\gamma^2)(\bar{y}_{ij..}-\bar{y}_{i...})+\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)\mu_\gamma}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+ps\sigma_\gamma^2} \quad (8)$$

The MAP estimator for π_l is

$$\hat{\pi}_{(l)_{MAP}} = \frac{(nqp)(\sigma_\pi^2)(\bar{y}_{...l}-\bar{y}_{....})+\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)\mu_\pi}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+nqp\sigma_\pi^2} \quad (9)$$

The MAP estimator for $(\beta\pi)_{il}$ is

$$\widehat{\beta\pi}_{(il)_{MAP}} = \frac{(qp)(\sigma_{\beta\pi}^2)(\bar{y}_{i..l}-\bar{y}_{i...}-\bar{y}_{...l}+\bar{y}_{....})+\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)\mu_{\beta\pi}}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+qp\sigma_{\beta\pi}^2} \quad (10)$$

The MAP estimator for $(\pi\gamma)_{lj(i)}$ is

$$\widehat{\pi\gamma}_{(lj(i))_{MAP}} = \frac{(p)(\sigma_{\pi\gamma}^2)(\bar{y}_{ij.l}-\bar{y}_{ij..}-\bar{y}_{i..l}+\bar{y}_{i...})+\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)\mu_{\pi\gamma}}{\left(\sigma_\eta^2+\sigma_{(\pi\eta)}^2\right)+p\sigma_{(\pi\gamma)}^2} \quad (11)$$

where,

$$\mathrm{RSS} = \begin{bmatrix} SS_{(\pi\eta)} + SS_{\eta} + nqps\left(\bar{y}_{....} - \mu\right)^2 + qps\sum_{i=1}^{n}\left((\bar{y}_{i...} - \bar{y}_{....}) - \beta_i\right)^2 \\ +ps\sum_{i=1}^{n}\sum_{j=1}^{q}\left((\bar{y}_{ij..} - \bar{y}_{i...}) - \gamma_{j(i)}\right)^2 + nqp\sum_{l=1}^{s}((\bar{y}_{...l} - \bar{y}_{....}) - \pi_l)^2 \\ +qp\sum_{i=1}^{n}\sum_{l=1}^{s}\left((\bar{y}_{i..l} - \bar{y}_{i...} - \bar{y}_{...l} + \bar{y}_{....}) - (\beta\pi)_{il}\right)^2 \\ +p\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{l=1}^{s}((\bar{y}_{ij.l} - \bar{y}_{ij..} - \bar{y}_{i..l} + \bar{y}_{i...}) - (\pi\gamma)_{lj(i)})^2 \end{bmatrix}$$

The MAP estimator for σ_{η}^2 is

$$\widehat{\sigma^2_{(\pi\eta)}}_{MAP} \propto \left(\sigma^2_{(\pi\eta)}\right)^{-(\alpha_{(\pi\eta)}+\frac{nqp}{2}+1)} \frac{\beta_{(\pi\eta)}^{\alpha_{(\pi\eta)}}}{\Gamma\left(\alpha_{(\pi\eta)}\right)} \exp\left[\frac{-\left(\frac{RSS}{2}\right)-\beta_{(\pi\eta)}}{\sigma^2_{(\pi\eta)}}\right] \tag{12}$$

The MAP estimator for σ_{η}^2 is given by

$$\widehat{\sigma^2_{\eta}}_{MAP} \propto \left(\sigma^2_{\eta}\right)^{-(\alpha_{\eta}+\frac{nqp}{2}+1)} \frac{\beta_{\eta}^{\alpha_{\eta}}}{\Gamma\left(\alpha_{\eta}\right)} \exp\left[\frac{-\left(\frac{RSS}{2}\right)-\beta_{\eta}}{\sigma^2_{\eta}}\right] \tag{13}$$

4. Experimental Results

This section considers the application of the suggested model to the real world based on data from cold storage unit of Horticulture Department, Agriculture College, Baghdad University during the spring growth seasons of 2011 and 2012 years. The field work was carried out on potato tuber Solanum tuberosum L. Desiree cv., Tubers were lifted at three time i.e. early morning, mid-day and in the evening. Tubers were stored either immediately after lifting or left for one, two or three hours in the field under a shade or without it. Curing was done at $15 \pm 1\ C^{\circ}$ and 80-85% RH for ten days, tuber then stored at $4 \pm C^{\circ}$ and 85-90% RH for three months. Reconditioning was done at room temperature (28-32 C°) and 45-52% RH for ten days. Randomized complete Block Design (RCBD) with four replicates for each treatment were adapted and the comparison was done using L.S.D at 5% level of significance. The design of the experiment was done according to the suggested model in equation (1). Table 4.1 below shows the experimental results for the analysis of variance for suggested model. On the other hand, Table 4.2 below shows the estimation results of parameters for suggested model based on ANOVA table. From Table 4.1 can see that the calculated F-values is greater than the tabulated F-values at (0.05) level that is means there is significant effect for a shade on storage capability for potato under different temperatures.

Table 4.1: Experimental results of ANOVA table for nested-factorial experiments of three phases with repeated measurements.

Source of variation	d.f	S.S	M.S	F_C	F_{table}
β	2	4.48	2.24	5.296	$F_{(2,12)}=3.89$
$\gamma(\beta)$	9	23.049	2.561	6.0543	$F_{(9,12)}=2.8$
Error (Bet.) $= \eta(\beta\gamma)$	12	5.071	0.423		
π	3	3.1198	1.0399	13.506	$F_{(3,24)}=3.01$
$\beta\pi$	6	0.724	0.121	1.571	$F_{(6,24)}=2.51$
$\pi\gamma(\beta)$	27	21.251	0.787	10.221	$F_{(27,24)}=1.95$
Error (Within)	24	1.859	0.077		
TOTAL	83				

Table 4.2: The estimation results of parameters of suggested model based on ANOVA table.

$\hat{\mu}$	$\hat{\beta}_i$	$\hat{\pi}_l$	$\widehat{(\beta\pi)}_{il}$	$\widehat{(\pi\gamma)}_{lj(i)}$	$\hat{\gamma}_{j(i)}$	$\widehat{\sigma^2_\eta}$	$\widehat{\sigma^2_{(\pi\eta)}}$
3.865	7.73	11.595	23.189	44.811	89.265	0.0811	0.077

By applied the maximum a posteriori estimation method to the storage experiment data, Table 4.3 represents the results of the parameters ($\mu, \beta_i, \gamma_{j(i)}, \pi_l, (\beta\pi)_{il}, (\pi\gamma)_{lj(i)}, \sigma^2_\eta, \sigma^2_{(\pi\eta)}$) based on maximum a posteriori estimation method.

Table 4.3: The results of the parameters of suggested model based on maximum a posteriori estimation method.

$\hat{\mu}$	$\hat{\beta}_i$	$\hat{\pi}_l$	$\widehat{(\beta\pi)}_{il}$	$\widehat{(\pi\gamma)}_{lj(i)}$	$\hat{\gamma}_{j(i)}$	$\widehat{\sigma^2_\eta}$	$\widehat{\sigma^2_{(\pi\eta)}}$
3.043	8.582	11.2	22.63	44.14	88.8	0.121	0.099

From Tables 4.2 and 4.3, one can see that the values of parameters obtained in both ANOVA and maximum a posteriori estimation are nearly alike and encouraging. On the other hand, the results of estimates are close to each other, which means that the MAP method is a good compared with the classic methods. Figures 4.1 and 4.2 below represent the maximum a posteriori density of all parameters for the suggested model and the maximum a posteriori density of the variances $\sigma^2_{(\pi\eta)}$ and σ^2_η.

Figure 4.1: The maximum a posteriori density of all parameters for the suggested model.

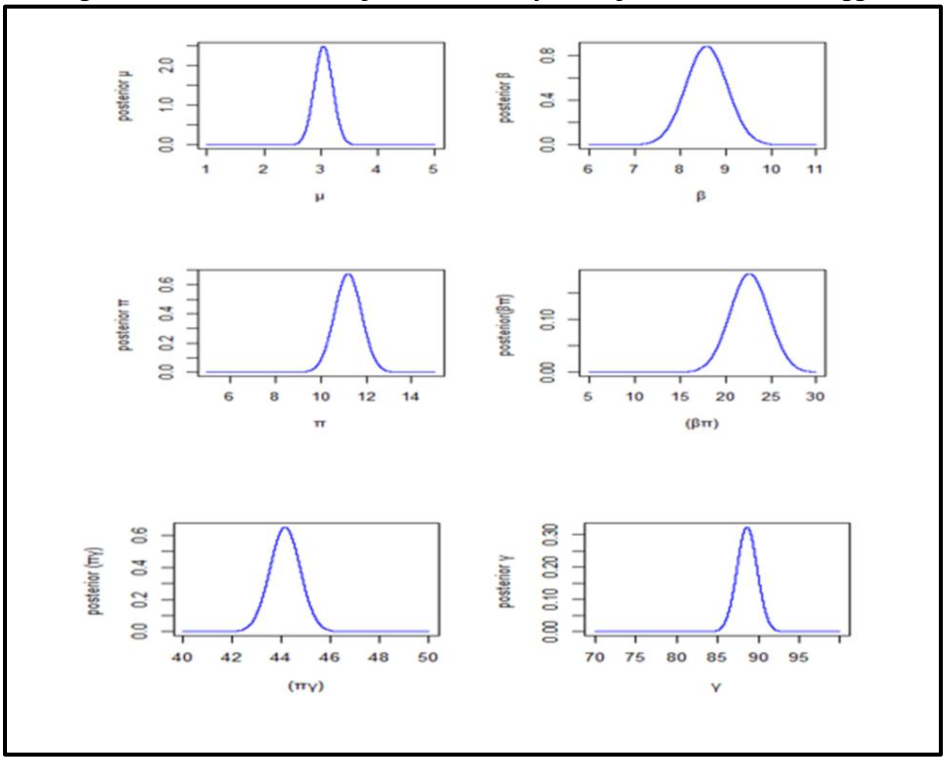
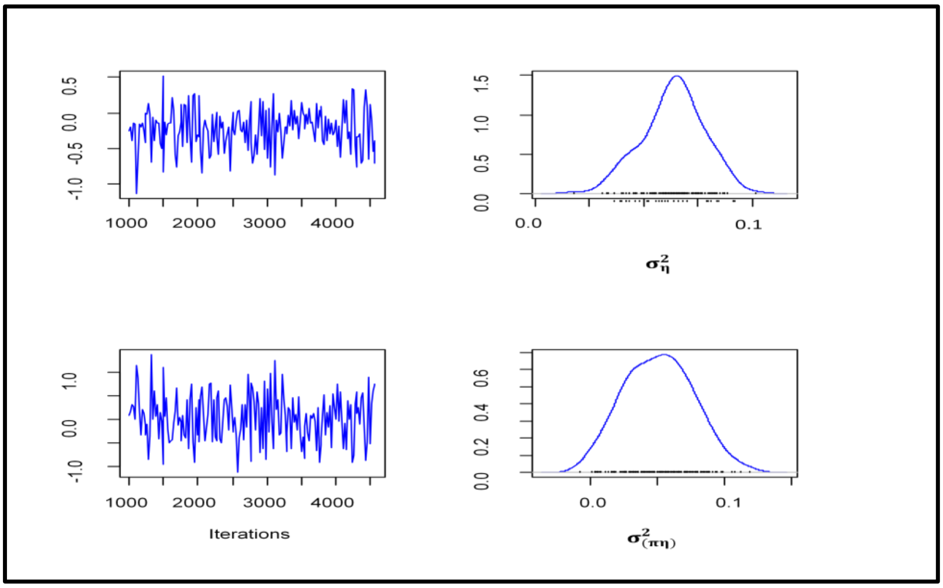


Figure 4.2: The maximum a posteriori density of the variances $\sigma^2_{(\pi\eta)}$ and σ^2_{η}.



5. Conclusion

This paper provides a method for discussing the basics of the statistical analysis of nested factorial experiments for three phases in repeated measurements. The factor that represents an experimental units called the subject, while, the factor that represents the treatments called repeated measurements. Factors that were considered to be repeated measurements interact in turn with the rest of the factors that resulted from the nested factorial experiments. A maximum a posteriori estimators (MAP) was utilized to provide conclusions about the treatment of nested factor experiments for three phases. For more validity, the suggested model was applied by using a real examples from agricultural field trials based on a potato tuber data that storage in unit of Horticulture Department, Agriculture College, Baghdad University. Randomized complete Block Design (RCBD) with four replicates for each treatment were adapted and the comparison was done using L.S.D at 5% level of significance. The experimental results show there are a significant effect for a shade on storage capability for potato under different temperatures.

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