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Solution of Linear Fuzzy Fractional Differential Equations Using Fuzzy Natural Transform

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Abstract

The purpose of this paper is to apply the fuzzy natural transform (FNT) for solving linear fuzzy fractional ordinary differential equations (FFODEs) involving fuzzy Caputo's H-difference with Mittag-Leffler laws. It is followed by proposing new results on the property of FNT for fuzzy Caputo's H-difference. An algorithm was then applied to find the solutions of linear FFODEs as fuzzy real functions. More specifically, we first obtained four forms of solutions when the FFODEs is of order $\alpha \in (0,1]$, then eight systems of solutions when the FFODEs is of order $\alpha \in (1,2]$ and finally, all of these solutions are plotted using MATLAB. In fact, the proposed approach is an effective and practical to solve a wide range of fractional models.

1. Introduction

The subject of fractional calculus, a generality of the ordinary calculus, described when the order of the functions' derivative is non-integer. Fractional calculus has a long history, where, explored and studied by various researchers in different fields of mathematics, engineering and physics have been intensively described via fractional derivatives, and fractional differential equations have come up as powerful instrument for modeling many difficult types of complex problems. Many authors considered the terms such as Riemann-Liouville, Grünwald-Letnikov and Caputo fractional derivative when dealing with fractional differential equations ([3], [23], [29], [40]). As times moving on, some drawbacks seem in the fractional differential equations especially, when dealing

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with physical phenomena. One of which is that a model involves uncertainty quantities and thus it is difficult to determine the initial values of it, as seen in ([13], [39]). To handle uncertainty quantities, researchers suggest several concepts. What stands out among those concepts is fuzzy set theory [44]. This theory is able to deal with differential equations having uncertainties at initial values. The first contribution on handling fractional differential equations with uncertainty was studied in [3]. This has prompted many researchers to further explore the topic ([4]-[6], [34], [23], [41]). It is also possible to solve many types of equations using integral transforms. Where integral transforms were used in solving linear ordinary differential equations, as well as linear fractional differential equations. The integral transforms were preceded by Fourier transform. Following, several integral transforms have been proposed, namely, Laplace, Mellin, Hankel and Sumudu transforms ([17], [24], [27], [32]). In this paper, we go to apply the natural transform to solve linear fuzzy fractional ordinary differential equations FFODEs involving fuzzy Caputo's H-difference. The natural transform is a new integral transform was introduced by [10], and its properties were giving by [38] also they proved that the natural transform method converges to the Laplace and Sumudu transforms. This transform was applied to solve the ordinary differential equations ([20], [28]). Moreover, in [28], some certain fundamental properties of generalized natural transform in generalized spaces were established. In [37], this new transform was applied to find analytical solutions of fractional-order heat and wave equations. In the recent years and with the rapid development of linear and nonlinear science, many effectiveness numerical and analytical different methods for solving the fuzzy fractional differential equations, as using the fuzzy Laplace and fuzzy Sumudu transforms, see ([1], [14]). The objective of this paper is to discuss the numerically solutions for each form and system of FFODEs based on orders of fuzzy Caputo's H-difference by using FNT.

The paper is presented as follows: Section 2 gives the fundamental definitions and theorems of fuzzy calculus, fuzzy fractional derivatives and the types of Mittag-Leffler functions. The definition of fuzzy natural transform FNT for Caputo's H-difference is given in Section 3. Algorithm of the solution of linear fuzzy fractional ordinary differential equations FFODEs of order $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$ is presented in Section 4. The solutions of the FFODEs by four forms and eight systems are described in Section 5, respectively. Section 6 focuses on the results of the numerical applications are performed to test the validity and reliability of the present method in Section 4. Finally, the conclusions are drawn in Section 7.

2. Basics Concepts

Fuzzy Calculus 2.1. In this subsection, several essential definitions and theorems of fuzzy calculus will be introduced.

Definition 2.2. A fuzzy number u is a pair (\underline{u}, \bar{u}) of functions $\underline{u}(r), \bar{u}(r), 0 \leq r \leq 1$ which satisfy the following requests [36]:

i: $\underline{u}(r)$ is bounded monotonic increasing left continuous function.

ii: $\bar{u}(r)$ is bounded monotonic decreasing left continuous function.

iii: $\underline{u}(r) \leq \bar{u}(r), 0 \leq r \leq 1$.

Arithmetic operation of arbitrary fuzzy numbers $u = (\underline{u}(r), \bar{u}(r))$ and $v = (\underline{v}(r), \bar{v}(r)) \in \mathbb{R}$, can be defined as [1],

(a) $u = v$, if and only if $\underline{u}(r) = \underline{v}(r)$ and $\bar{u}(r) = \bar{v}(r)$.

(b) $u + v = (\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r))$.

(c) $u - v = (\underline{u}(r) - \bar{v}(r), \bar{u}(r) - \underline{v}(r))$.

(d) $\lambda u = \begin{cases} (\lambda \underline{u}(r), \lambda \bar{u}(r)) & , \lambda \geq 0, \\ (\lambda \underline{u}(r), \lambda \bar{u}(r)) & , \lambda < 0. \end{cases}$

The fuzzy numbers are classified into several types of fuzzy membership function. The definition of triangular fuzzy number will be mentioned, because it's the most used membership functions in our study.

Definition 2.3 [14]. In a triangular fuzzy number (TFN) is defined by notation $A = [a, b, c]$. Then the members help function for this TFN is defined as

$$A(t) = \begin{cases} 0 & , & t < a \\ \frac{t-a}{b-a} & , & a \leq t \leq b \\ \frac{c-t}{c-b} & , & b \leq t \leq c \end{cases}$$

The r -cuts of TFN is $A^r = [(b - a)r + a, (b - c)r + c], \forall r \in (0,1]$.

Theorem 2.4 [15]. Let $F(t): \mathbb{R} \rightarrow \mathcal{F}(\mathbb{R})$ and it is denoted by $[f_r(t), \bar{f}_r(t)]$. So for any fixed $r \in [0,1]$, imagine $f_r(t)$ and $\bar{f}_r(t)$ are Riemann-integrable on $[a, b]$ for every $b \geq a$, and two positive \underline{M}_r and \bar{M}_r such that, $\int_a^b |f_r(t)| dx \leq \underline{M}_r$ and $\int_a^b |\bar{f}_r(t)| dx \leq \bar{M}_r$ for every $b \geq a$. Then, $F(t)$ is inadvisable fuzzy Riemann-integrable on $[a, \infty)$ and it is a fuzzy number. Thereby,

$$\int_a^\infty F(t) dx = \left[\int_a^\infty f_r(t) dx, \int_a^\infty \bar{f}_r(t) dx \right].$$

Hukuhara (H-difference) of fuzzy number is defined as follows.

Definition 2.5 [36]. If $u, v \in \mathcal{F}(\mathbb{R})$ and exists a fuzzy subset $w \in \mathcal{F}(\mathbb{R})$ such as $w + u = v$, thereby w is unique. In this case, w is called H-difference, of u and v denoted by $v -_H u$ and also note that $-_H u \neq u + (-1)v$.

The concept of strongly generalized differentiable is a popularization of H-difference for fuzzy functions, it is defined as follows.

Definition 2.6 [43]. Let $F(t) : (a, b) \rightarrow \mathcal{F}(\mathbb{R})$ be a fuzzy function and $t_0 \in (a, b)$. That $F(t)$ is strongly generalized differentiable at t_0 if there exists an element $F'(t_0) \in \mathcal{F}(\mathbb{R})$, then:

For all $h > 0$ sufficiently small, $\exists F(t_0 + h) -_H F(t_0)$, $\exists F(t_0) -_H F(t_0 - h)$, and the limits (in the metric D)

$$\lim_{h \rightarrow 0} \frac{F(t_0 + h) -_H F(t_0)}{h} = \lim_{h \rightarrow 0} \frac{F(t_0) -_H F(t_0 - h)}{h} = F'(t_0).$$

Or,

For every $h > 0$ enough small, $\exists F(t_0) -_H F(t_0 + h)$, $\exists F(t_0 - h) -_H F(t_0)$ and so the limits (in the metric D)

$$\lim_{h \rightarrow 0} \frac{F(t_0) -_H F(t_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{F(t_0 - h) -_H F(t_0)}{-h} = F'(t_0).$$

The symbols $\mathcal{C}_{\mathcal{F}}[a, b]$ and $\mathcal{L}_{\mathcal{F}}[a, b]$ are used to refer for the spaces of all continuous fuzzy functions and Lebesgue integrable fuzzy functions, respectively on $[a, b] \subseteq \mathbb{R}$.

Fuzzy Fractional Derivatives 2.7. In this subsection, some of these definitions and theorems are presented on the subject of fuzzy Caputo's H-difference. This subject extended of crisp case for fuzzy discussed in [35].

Lemma 2.8. Let $f(t)$ be a crisp continuous function and $([\alpha])$ –time differentiable in the independent variable t over the interval of differentiation (integration) $[0, t]$. Then the relation,

$${}^C D^\alpha f(t) = {}^{RL} D^\alpha \left(f(t) - \sum_{p=0}^{[\alpha]} \frac{t^p}{p!} f_0^{(p)} \right), \quad \alpha \in (n - 1, n], \quad n \in \mathbb{N},$$

holds, where $f_0^{(p)} = \left. \frac{d^p f(t)}{dt^p} \right|_{t=0}$ and ${}^C D^\alpha$ means Caputo derivative operator, while $[\alpha]$ and $\lfloor \alpha \rfloor$ are the value α rounded up and down to the closet integer number, respectively. ${}^{RL} D^\alpha$ is the common Riemann–Liouville fractional derivative operator which is defined as follows:

$${}^{RL} D^\alpha f(t) = \frac{1}{\Gamma([\alpha] - \alpha)} \frac{d^{[\alpha]}}{dt^{[\alpha]}} \int_0^t \frac{f(s)}{(t - s)^{1 - [\alpha] + \alpha}} ds.$$

Definition 2.9. Let $F(t) \in \mathcal{C}_{\mathcal{F}}[0, b] \cap L_{\mathcal{F}}[0, b]$, $K(t) = \frac{1}{\Gamma([\alpha] - \alpha)} \int_0^t \frac{f(s) - \sum_{p=0}^{[\alpha]} \frac{s^p}{p!} f_0^{(p)}}{(t - s)^{1 - [\alpha] + \alpha}} ds$, and

$$\lim_{h \rightarrow 0^+} \frac{K(t_0 + h) - {}_H K(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{K(t_0) - {}_H K(t_0 - h)}{h} = Q(t_0),$$

and

$$\lim_{h \rightarrow 0^+} \frac{K(t_0) - {}_H K(t_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{K(t_0 - h) - {}_H K(t_0)}{-h} = L(t_0)$$

$F(t)$ is Caputo’s H–difference function of order $0 < \alpha < 2$, $\alpha \neq 1$ at $t_0 \in (0, b)$, if there exists an element ${}^C D^\alpha F(t_0) \in \mathcal{C}_{\mathcal{F}}$ such that for every $0 \leq r \leq 1$ and for $h > 0$ adequately near zero, either,

(a) $\lim_{h \rightarrow 0^+} \frac{K(t_0 + h) - {}_H K(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{K(t_0) - {}_H K(t_0 - h)}{h} = {}^C D^\alpha F(t_0).$

(b) $\lim_{h \rightarrow 0^+} \frac{K(t_0) - {}_H K(t_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{K(t_0 - h) - {}_H K(t_0)}{-h} = {}^C D^\alpha F(t_0),$

for $0 < \alpha < 1$.

(c) $\lim_{h \rightarrow 0^+} \frac{Q(t_0 + h) - {}_H Q(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{Q(t_0) - {}_H Q(t_0 - h)}{h} = {}^C D^\alpha F(t_0).$

(d) $\lim_{h \rightarrow 0^+} \frac{Q(t_0) - {}_H Q(t_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{Q(t_0 - h) - {}_H Q(t_0)}{-h} = {}^C D^\alpha F(t_0).$

$$(e) \lim_{h \rightarrow 0^+} \frac{L(t_0+h) - {}_H L(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{L(t_0) - {}_H L(t_0-h)}{h} = {}^c D^\alpha F(t_0).$$

$$(f) \lim_{h \rightarrow 0^+} \frac{L(t_0) - {}_H L(t_0+h)}{-h} = \lim_{h \rightarrow 0^+} \frac{L(t_0-h) - {}_H L(t_0)}{-h} = {}^c D^\alpha F(t_0),$$

for $1 < \alpha < 2$.

Remark 2.10. The fuzzy function $F(t)$ is Caputo's H-difference and denoted by ${}^c D_1^\alpha F(t)$, ${}^c D_2^\alpha F(t)$ respectively if it is (n) –differentiable for $n = 1, 2$ as in Definition 2.9 cases (a,b). Also, the fuzzy function $F(t)$ is Caputo's H-difference and denoted by ${}^c D_{1,1}^\alpha F(t)$, ${}^c D_{1,2}^\alpha F(t)$, ${}^c D_{2,1}^\alpha F(t)$, ${}^c D_{2,2}^\alpha F(t)$ respectively if it is (n, m) – differentiable for $n, m = 1, 2$ as in Definition 2.9 cases(c,d,e,f).

Theorem 2.11. Let $F(t) \in \mathbb{C}_{\mathcal{F}}[0, b] \cap L_{\mathcal{F}}[0, b]$ be a fuzzy valued function and where $[F(t)]^r = [\underline{f}_r(t), \overline{f}_r(t)]$, for $r \in [0, 1]$ and $t_0 \in (0, b)$. Then:

i- If $F(t)$ is Caputo's H-difference function in (1)-differentiable form, then for $0 < \alpha < 1$

$$[{}^c D_1^\alpha F(t_0)]^r = [{}^c D^\alpha \underline{f}_r(t_0), {}^c D^\alpha \overline{f}_r(t_0)].$$

ii- If $F(t)$ is Caputo's H-difference function in (2)-differentiable form, then for $0 < \alpha < 1$

$$[{}^c D_2^\alpha F(t_0)]^r = [{}^c D^\alpha \overline{f}_r(t_0), {}^c D^\alpha \underline{f}_r(t_0)].$$

iii- If ${}^c D_1^\alpha F(t)$ is Caputo's H-difference function in (1,1)-differentiable form, then for $1 < \alpha < 2$

$$[{}^c D_{1,1}^\alpha F(t_0)]^r = [{}^c D^\alpha \underline{f}_r(t_0), {}^c D^\alpha \overline{f}_r(t_0)].$$

iv- If ${}^c D_1^\alpha F(t)$ is Caputo's H-difference function in (1,2)-differentiable form, then for $1 < \alpha < 2$

$$[{}^c D_{1,2}^\alpha F(t_0)]^r = [{}^c D^\alpha \overline{f}_r(t_0), {}^c D^\alpha \underline{f}_r(t_0)].$$

v- If ${}^c D_2^\alpha F(t)$ is Caputo's H-difference function in (2,1)-differentiable form, then for $1 < \alpha < 2$

$$[{}^c D_{2,1}^\alpha F(t_0)]^r = [{}^c D^\alpha \overline{f}_r(t_0), {}^c D^\alpha \underline{f}_r(t_0)].$$

vi- If ${}^c D_{2,2}^\alpha F(t)$ is Caputo's H -difference function in $(2,2)$ -differentiable form, then for $1 < \alpha < 2$

$$[{}^c D_{2,2}^\alpha F(t_0)]^r = [{}^c D^\alpha \underline{f}_r(t_0), {}^c D^\alpha \overline{f}_r(t_0)].$$

where

$${}^c D^\alpha \underline{f}_r(t_0) = \left[\frac{1}{\Gamma(|\alpha|-\alpha)} \int_0^t \frac{D^{|\alpha|} \underline{f}_r(t)}{(t-s)^{1-|\alpha|+\alpha}} ds \right]_{t=t_0},$$

$${}^c D^\alpha \overline{f}_r(t_0) = \left[\frac{1}{\Gamma(|\alpha|-\alpha)} \int_0^t \frac{D^{|\alpha|} \overline{f}_r(t)}{(t-s)^{1-|\alpha|+\alpha}} ds \right]_{t=t_0}.$$

$$D^k f(t) = \frac{d^k f(t)}{dt^k}.$$

Mittag–Leffler Function 2.12. This function is a direct generalization of the exponential function, and has an affinity for fractional calculus. In this subsection, one and two parameter representations of Mittag–Leffler function are presented by the following definitions [22].

Definition 2.13. A one-parameter function of Mittag-Leffler type defined by the series expansion

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}.$$

Definition 2.14. A two-parameter function of Mittag–Leffler type defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0, \quad z \in \mathbb{C}.$$

Consequently, the definition for the classical natural transform is given, when dealing with Caputo's fractional derivative of crisp type and then some properties.

Theorem 2.15 [26]. Let $n \in \mathbb{N}$ and $\alpha > 0$ be such that $n - 1 < \alpha \leq n$ and $R^+(s, u)$ be a natural transform of the function $f(t)$. Then the natural transform denoted by $R_\alpha^+(s, u)$ of Caputo fractional derivative of the function $f(t)$ of order α , is given by

$$N^+[{}^c D^\alpha f(t)] = R_\alpha^+(s, u) = \frac{s^\alpha}{u^\alpha} R^+(s, u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} [D^k f(t)]_{t=0}.$$

Since in this paper, the orders $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$ are considered, Theorem 2.16 can be simplified as

$$N^+[{}^c D^\alpha f(t)] = R_\alpha^+(s, u) = \frac{s^\alpha}{u^\alpha} R^+(s, u) - \frac{s^{\alpha-1}}{u^\alpha} D^0 f(0), \quad \alpha \in (0, 1],$$

$$N^+[{}^c D^\alpha f(t)] = R_\alpha^+(s, u) = \frac{s^\alpha}{u^\alpha} R^+(s, u) - \frac{s^{\alpha-1}}{u^\alpha} D^0 f(0) - \frac{s^{\alpha-2}}{u^{\alpha-1}} D^1 f(0), \quad \alpha \in (1, 2].$$

Note that when $\alpha = 1$ and $\alpha = 2$, the definition is like to the definition of natural transform for first and second orders derivative.

Property 2.16 [26]. If $\alpha > -1$, then the natural transform of t^α is given by

$$N^+[t^\alpha] = \frac{\Gamma(\alpha + 1)u^\alpha}{s^{\alpha+1}}.$$

Property 2.17 [18]. If $\alpha, \beta > 0$, $\alpha \in \mathbb{R}$ and $\frac{s^\alpha}{u^\alpha} > |a|$, then the inverse natural transform formula,

$$N^{-1} \left[\frac{u^{\beta-1} s^{\alpha-\beta}}{s^\alpha + au^\alpha} \right] = t^{\beta-1} E_{\alpha, \beta}(-at^\alpha).$$

3. Fuzzy Natural transforms for Caputo's H-difference

The definition of FNT for Caputo's H-difference will be introduced as follows ([26], [33]).

Definition 3.1. Let $F(t): \mathbb{R} \rightarrow \mathcal{F}(\mathbb{R})$ be a continuous fuzzy function. Assume $F(ut)e^{-st}$ is improper fuzzy Riemann-integrable on $[0, \infty)$, then $\int_0^\infty F(ut)e^{-st} dt$ is called fuzzy Natural transform and is denoted by,

$$R(s, u) = N^+[F(t)](u, s) = \frac{1}{u} \int_0^\infty F(ut)e^{-st} dt, \quad s, u \in [-\vartheta_1, \vartheta_2]$$

where, the variables u and s are used to factor the variable t in the argument of the fuzzy and $\vartheta_1, \vartheta_2 > 0$. Also, the FNT can be written in the following form:

$$N^+[F(t)](u, s) = \left[N^+ \left[\underline{f}_r(t_0) \right] (u, s), \underline{f}_r(t_0), N^+ \left[\overline{f}_r(t_0) \right] (u, s) \right].$$

4. Algorithm of the Solution of Linear Fuzzy Fractional Ordinary Differential Equations (FFODEs)

The FFODEs is given in this form:

$$[{}^c D^\alpha \tilde{y}(t)]^r = a[\tilde{y}(t)]^r + b, \tag{1}$$

where, ${}^c D^\alpha$ is Caputo's H-difference of order, $0 < \alpha \leq 1$ or $1 < \alpha \leq 2$, $r \in [0,1]$ and a, b are constants. The fuzzy initial condition FIC is the triangular fuzzy number:

$$[\tilde{y}(0)]^r = [S_1, S_2, S_3], \tag{2}$$

where, $[\tilde{y}(t)]^r$ is a fuzzy function of the crisp function $y(t)$. Then Eq. (1) and Eq. (2) are equivalent the following forms, respectively:

$$\left[{}^c D^\alpha \underline{y}_r(t), {}^c D^\alpha \bar{y}_r(t) \right] = a \left[\underline{y}_r(t), \bar{y}_r(t) \right] + b, \tag{3}$$

$$\left[\underline{y}_r(0), \bar{y}_r(0) \right] = [S_1 + r(S_2 - S_1), S_3 - r(S_3 - S_2)]. \tag{4}$$

Next, the algorithm for solving of FFODEs can be provided through the following steps:

Step 1: Eq. (3) is written in the form of a system consisting of two equations. The same procedure which applied to Eq.(4) as follows:

$${}^c D^\alpha \underline{y}_r(t) = a\underline{y}_r(t) + b, \tag{5}$$

$${}^c D^\alpha \bar{y}_r(t) = a\bar{y}_r(t) + b, \tag{6}$$

and

$$\underline{y}_r(0) = S_1 + r(S_2 - S_1), \tag{7}$$

$$\bar{y}_r(0) = S_3 - r(S_3 - S_2). \tag{8}$$

Step 2: Add the equations (5), (6) as well as equations (7), (8) to obtain the following equations, respectively.

$${}^c D^\alpha \psi_r(t) = a\psi_r(t) + 2b, \tag{9}$$

where,

$$\psi_r(t) = \underline{y}_r(t) + \bar{y}_r(t). \tag{10}$$

So,

$$\psi_r(0) = S_1 + S_3 + r(2S_2 - S_1 - S_3). \tag{11}$$

Step 3: Finding a solution to the Eq. (9) by applying FNT and the properties of Mittag-Leffler function.

Step 4: Subtract the equations (5), (6) as well as equations (7), (8) to obtain the following equations, respectively

$${}^C D^\alpha \phi_r(t) = a\phi_r(t), \quad (12)$$

where,

$$\phi_r(t) = \underline{y}_r(t) - \bar{y}_r(t). \quad (13)$$

So,

$$\phi_r(0) = S_1 - S_3 + r(S_3 - S_1). \quad (14)$$

Step 5: Getting the solution to the Eq.(12) by applying FNT and the properties of Mittag-Leffler function.

Step 6: Return to **step 1** and **step 2**, such that Firstly: adding the equations (10), (13) to get the lower bound of solution $\underline{y}_r(t)$. Secondly: subtracting the same equations to get the upper bound of solution $\bar{y}_r(t)$.

5. The Solutions of FFODEs

In this section, the solutions of FFODEs (3)-(4) will be based on the method steps mentioned above in the algorithm through two subsections depending on the orders of Caputo's H-difference ${}^C D^\alpha$. **First:** includes the solutions when the order is $0 < \alpha \leq 1$ and **Second:** includes other solutions when the order is $1 < \alpha \leq 2$.

5.1. The solutions of FFODEs of the order $0 < \alpha \leq 1$

Now, the FFODEs (3)-(4) of order $\alpha \in (0,1]$ can be solved through two cases depending on the signal of the constant a .

Case i: Let $a > 0$, then the solutions of (3)-(4) can be found by applying the (i) and (ii) forms of Theorem 2.11, as follows:

First Form: Assume that $[\tilde{y}(t)]^r$ is Caputo's H-difference function in (1) –differentiable, from Theorem 2.11 (i), then the solution is

$$\begin{aligned} \underline{y}_r(t) &= \{S_1 + r(S_2 - S_1)\}E_\alpha(at^\alpha) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha), \\ \bar{y}_r(t) &= \{S_3 + r(S_2 - S_3)\}E_\alpha(at^\alpha) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha). \end{aligned}$$

Second Form: Assume that $[\tilde{y}(t)]^r$ is Caputo's H-difference function in (2) –differentiable, then from Theorem 2.11 (ii), then the solution is

$$\underline{y}_r(t) = \frac{1}{2}\{S_1 + S_3 + r(2S_2 - S_1 - S_3)\}E_\alpha(at^\alpha) + \frac{1}{2}\{S_1 - S_3 + r(S_3 - S_1)\}E_\alpha(-at^\alpha) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha),$$

$$\bar{y}_r(t) = \frac{1}{2}\{S_1 + S_3 + r(2S_2 - S_1 - S_3)\}E_\alpha(at^\alpha) - \frac{1}{2}\{S_1 - S_3 + r(S_3 - S_1)\}E_\alpha(-at^\alpha) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha).$$

Case ii: Let $a < 0$, then (3)-(4) becomes

$$\begin{aligned} \left[{}^c D^\alpha \underline{y}_r(t), {}^c D^\alpha \bar{y}_r(t) \right] &= a \left[\bar{y}_r(t), \underline{y}_r(t) \right] + b, \\ \left[\underline{y}_r(0), \bar{y}_r(0) \right] &= [S_1 + r(S_2 - S_1), S_3 - r(S_3 - S_2)]. \end{aligned}$$

In order to get the solutions of the above equations. By applying the (i) and (ii) forms of Theorem 2.11, as follows:

First form: Consider that $[\tilde{y}(t)]^r$ is Caputo’s H-difference function in (1) –differentiable, from Theorem 2.11 (i). Note that getting the same lower and upper bound for a second form of Case i.

Second form: Assume that $[\tilde{y}(t)]^r$ is Caputo’s H-difference function in (2) –differentiable, then from Theorem 2.11 (ii). Note that the same lower and upper bound are getting to the first form of Case i.

5.2. The solutions of FFODEs of the order $1 < \alpha \leq 2$

If the FFODEs (3) - (4) are of the order $\alpha \in (1,2]$, then they are also solved using the same algorithm of the method and these equations take the form:

$$\left[{}^c D^\alpha \underline{y}_r(t), {}^c D^\alpha \bar{y}_r(t) \right] = a \left[\underline{y}_r(t), \bar{y}_r(t) \right] + b, \tag{15}$$

subject to the (FIC) $[\tilde{y}(0)]^r = \delta^r, [\tilde{y}_t(0)]^r = \varepsilon^r$. Put $\delta^r = \varepsilon^r$, then,

$$\begin{aligned} [\tilde{y}_t(0)]^r &= [\tilde{y}(0)]^r = [S_1, S_2, S_3], \\ &= [S_1 + r(S_2 - S_1), S_3 - r(S_3 - S_2)]. \end{aligned} \tag{16}$$

The above equation will be solved in two cases depending on the signal of the constant a .

Case j: Let $a > 0$, finding the solutions of (15)-(16) by applying the (iii), (iv), (v) and (vi) forms of Theorem 2.11, respectively as in the following the systems.

(1-1) System: Assume that $[\tilde{y}(t)]^r$ is Caputo's H-difference function (1,1) –differentiable, from Theorem 2.11 (iii), then the solution of (15)-(16) is

$$\underline{y}_r(t) = \{S_1 + r(S_2 - S_1)\}(E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha),$$

$$\bar{y}_r(t) = \{S_3 + r(S_2 - S_3)\}(E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha).$$

(1-2) System: Suppose that $[\tilde{y}(t)]^r$ is Caputo's H-difference function in (1,2) –differentiable, from Theorem 2.11 (iv), note that (15)-(16) becomes

$$\left[{}^c D^\alpha \bar{y}_r(t), {}^c D^\alpha \underline{y}_r(t) \right] = a \left[\underline{y}_r(t), \bar{y}_r(t) \right] + b,$$

and FIC is

$$\underline{y}_r(0) = \underline{\delta}^r, \quad \bar{y}_r(0) = \bar{\delta}^r,$$

$$\underline{y}_{tr}(0) = \underline{\varepsilon}^r, \quad \bar{y}_{tr}(0) = \bar{\varepsilon}^r.$$

Then the solution is

$$\underline{y}_r(t) = \frac{1}{2} \{S_1 + S_3 + r(2S_2 - S_1 - S_3)\}(E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) + \frac{1}{2} \{S_1 - S_3 + r(S_3 - S_1)\}(E_\alpha(-at^\alpha) + tE_{\alpha,2}(-at^\alpha)) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha),$$

$$\bar{y}_r(t) = \frac{1}{2} \{S_1 + S_3 + r(2S_2 - S_1 - S_3)\}(E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) + \frac{1}{2} \{S_1 - S_3 + r(S_3 - S_1)\}(E_\alpha(-at^\alpha) + tE_{\alpha,2}(-at^\alpha)) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha).$$

(2-1) System: Consider that $[\tilde{y}(t)]^r$ is Caputo's H-difference function in(2,1) –differentiable, from Theorem 2.11 (v),

Note that (15)-(16) becomes

$$\left[{}^c D^\alpha \bar{y}_r(t), {}^c D^\alpha \underline{y}_r(t) \right] = a \left[\underline{y}_r(t), \bar{y}_r(t) \right] + b,$$

and FIC is

$$\underline{y}_r(0) = \underline{\delta}^r, \quad \bar{y}_r(0) = \bar{\delta}^r,$$

$$\underline{y}_{tr}(0) = \underline{\varepsilon}^r, \quad \bar{y}_{tr}(0) = \bar{\varepsilon}^r.$$

Then the solution is

$$\underline{y}_r(t) = \frac{1}{2} \{S_1 + S_3 + r(2S_2 - S_1 - S_3)\}(E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) + \frac{1}{2} \{S_1 - S_3 + r(S_3 - S_1)\}(E_\alpha(-at^\alpha) - tE_{\alpha,2}(-at^\alpha)) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha),$$

$$\bar{y}_r(t) = \frac{1}{2}\{S_1 + S_3 + r(2S_2 - S_1 - S_3)\}(E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) - \frac{1}{2}\{S_1 - S_3 + r(S_3 - S_1)\}(E_\alpha(-at^\alpha) - tE_{\alpha,2}(-at^\alpha)) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha).$$

(2-2) System: Assume that $[\tilde{y}(t)]^r$ is Caputo's H-difference function in(2,2) –differentiable, from Theorem 2.11 (vi),

Note that (15)-(16) becomes

$$\left[{}^c D^\alpha \underline{y}_r(t), {}^c D^\alpha \bar{y}_r(t) \right] = a \left[\underline{y}_r(t), \bar{y}_r(t) \right] + b,$$

and FIC is

$$\underline{y}_r(0) = \underline{\delta}^r, \quad \bar{y}_r(0) = \bar{\delta}^r,$$

$$\underline{y}_{tr}(0) = \underline{\varepsilon}^r, \quad \bar{y}_{tr}(0) = \bar{\varepsilon}^r.$$

Then the solution is

$$\underline{y}_r(t) = \{S_1 + r(S_2 - S_1)\}E_\alpha(at^\alpha) + \{S_3 + r(S_2 - S_3)\}tE_{\alpha,2}(at^\alpha) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha),$$

$$\bar{y}_r(t) = \{S_3 + r(S_2 - S_3)\}E_\alpha(at^\alpha) + \{S_1 + r(S_2 - S_1)\}tE_{\alpha,2}(at^\alpha) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha).$$

Case jj: Let $a < 0$, then (15)-(16) take this form

$$\left[{}^c D^\alpha \underline{y}_r(t), {}^c D^\alpha \bar{y}_r(t) \right] = a \left[\bar{y}_r(t), \underline{y}_r(t) \right] + b, \tag{17}$$

subject to the FIC $[\tilde{y}(0)]^r = \delta^r, [\tilde{y}_t(0)]^r = \varepsilon^r$. Put $\delta^r = \varepsilon^r$, then,

$$\begin{aligned} [\tilde{y}_t(0)]^r &= [\tilde{y}(0)]^r = [S_1, S_2, S_3], \\ &= [S_1 + r(S_2 - S_1), S_3 - r(S_3 - S_2)]. \end{aligned} \tag{18}$$

Now, finding the solutions of (17)-(18). By applying the **(iii)**, **(iv)**, **(v)** and **(vi)** forms of Theorem 2.11, respectively as in the following systems:

(1-1) System: Consider that $[\tilde{y}(t)]^r$ is Caputo's H-difference function in (1,1) –differentiable, from Theorem 2.11 (iii), then the solution of (17)-(18) is

$$\underline{y}_r(t) = \frac{1}{2}\{S_1 + S_3 + r(2S_2 - S_1 - S_3)\}(E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) + \frac{1}{2}\{S_1 - S_3 + r(S_3 - S_1)\}(E_\alpha(-at^\alpha) + tE_{\alpha,2}(-at^\alpha)) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha),$$

$$\bar{y}_r(t) = \frac{1}{2} \{S_1 + S_3 + r(2S_2 - S_1 - S_3)\} (E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) - \frac{1}{2} \{S_1 - S_3 + r(S_3 - S_1)\} (E_\alpha(-at^\alpha) + tE_{\alpha,2}(-at^\alpha)) + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha).$$

(1-2) System: Suppose that $[\tilde{y}(t)]^r$ is Caputo's H-difference function in (1,2) –differentiable, from Theorem 2.11 (iv),

Note that (17)-(18) becomes

$$\left[{}^c D^\alpha \bar{y}_r(t), {}^c D^\alpha \underline{y}_r(t) \right] = a \left[\bar{y}_r(t), \underline{y}_r(t) \right] + b,$$

and FIC is

$$\begin{aligned} \underline{y}_r(0) &= \underline{\delta}^r, & \bar{y}_r(0) &= \bar{\delta}^r, \\ \underline{y}_{tr}(0) &= \underline{\varepsilon}^r, & \bar{y}_{tr}(0) &= \bar{\varepsilon}^r. \end{aligned}$$

After solving this system, obtaining the same results in (1-1) system for the Case j.

(2-1) System: Assume that $[\tilde{y}(t)]^r$ is Caputo's H-difference function in (2,1) –differentiable, from Theorem 2.11 (v),

Note that (17)-(18) becomes

$$\left[{}^c D^\alpha \bar{y}_r(t), {}^c D^\alpha \underline{y}_r(t) \right] = a \left[\bar{y}_r(t), \underline{y}_r(t) \right] + b,$$

and (FIC) is

$$\begin{aligned} \underline{y}_r(0) &= \underline{\delta}^r, & \bar{y}_r(0) &= \bar{\delta}^r, \\ \underline{y}_{tr}(0) &= \bar{\varepsilon}^r, & \bar{y}_{tr}(0) &= \underline{\varepsilon}^r. \end{aligned}$$

After solving this system obtaining the same results in (2-2) system for the Case j.

(2-2) System: Assume that $[\tilde{y}(t)]^r$ is Caputo's H-difference function in (2,2) –differentiable, from Theorem 2.11 (vi),

Note that (17)-(18) becomes

$$\left[{}^c D^\alpha \underline{y}_r(t), {}^c D^\alpha \bar{y}_r(t) \right] = a \left[\bar{y}_r(t), \underline{y}_r(t) \right] + b,$$

and (FIC) is

$$\underline{y}_r(0) = \underline{\delta}^r, \quad \bar{y}_r(0) = \bar{\delta}^r,$$

$$\underline{y}_{tr}(0) = \underline{\varepsilon}^r, \quad \bar{y}_{tr}(0) = \bar{\varepsilon}^r.$$

Then the solution is

$$\begin{aligned} \underline{y}_r(t) &= \frac{1}{2}\{S_1 + S_3 + r(2S_2 - S_1 - S_3)\}(E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) \\ &\quad + \frac{1}{2}\{S_1 - S_3 + r(S_3 - S_1)\}(E_\alpha(-at^\alpha) - tE_{\alpha,2}(-at^\alpha)) \\ &\quad + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha), \\ \bar{y}_r(t) &= \frac{1}{2}\{S_1 + S_3 + r(2S_2 - S_1 - S_3)\}(E_\alpha(at^\alpha) + tE_{\alpha,2}(at^\alpha)) \\ &\quad - \frac{1}{2}\{S_1 - S_3 + r(S_3 - S_1)\}(E_\alpha(-at^\alpha) - tE_{\alpha,2}(-at^\alpha)) \\ &\quad + bt^\alpha E_{\alpha,\alpha+1}(at^\alpha). \end{aligned}$$

6. Main Results and Discussion

In this section, assume that the FIC is $[\hat{y}_t(0)]^r = [\hat{y}(0)]^r = [1,2,3]$. Taking $a = b = 1$ for Cases i and j so, $a = -1, b = 1$ for Cases ii and jj. Firstly, the numerical solutions of (3)-(4) for Cases i and ii are listed in Tables 1 and 2, respectively. For graphical results, please see in Figures 1 and 2.

Table 1: Numerical solutions at Case i; First and Second forms of FFODEs (3)-(4) for $\alpha = 0.5$.

r	Case i; First form $\alpha = 0.5$		Case i; Second form $\alpha = 0.5$	
	\underline{y}	\bar{y}	\underline{y}	\bar{y}
0	27.3517	55.5461	41.0128	41.8850
0.1	28.7614	54.1364	41.0564	41.8414
0.2	30.1711	52.7267	41.1000	41.7978
0.3	31.5808	51.3170	41.1436	41.7542
0.4	32.9906	49.9072	41.1872	41.7106
0.5	34.4003	48.4975	41.2308	41.6669
0.6	35.8100	47.0878	41.2745	41.6233
0.7	37.2197	45.6781	41.3181	41.5797
0.8	38.6295	44.2683	41.3617	41.5361
0.9	40.0392	42.8586	41.4053	41.4925
1.0	41.4489	41.4489	41.4489	41.4489

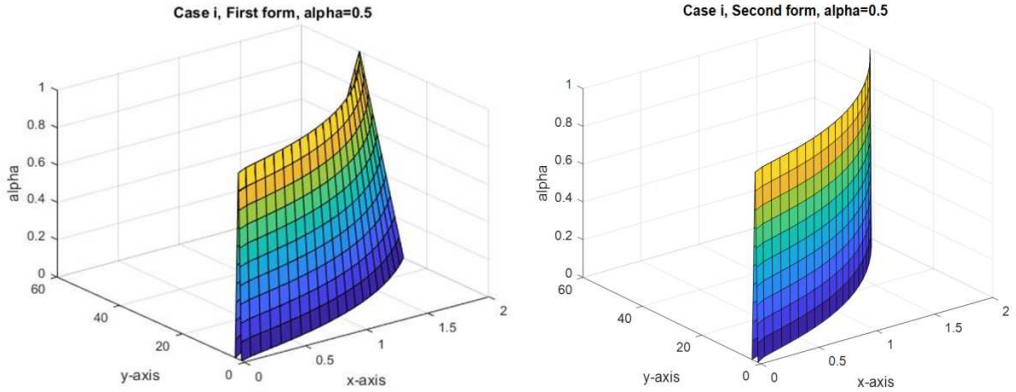


Figure 1: The numerical solutions at Case i; First and Second forms of FFODEs (3)-(4) for $\alpha = 0.5$.

Table 2: Numerical solutions at Case ii; First and Second forms of FFODEs (3)-(4) for $\alpha = 0.5$.

	Case i; First form $\alpha = 0.5$		Case i; Second form $\alpha = 0.5$	
r	\underline{y}	\bar{y}	\underline{y}	\bar{y}
0	-12.5039	15.6905	1.1572	2.0294
0.1	-11.0942	14.2808	1.2008	1.9858
0.2	-9.6845	12.8711	1.2444	1.9422
0.3	-8.2748	11.4614	1.2880	1.8986
0.4	-6.8650	10.0516	1.3316	1.8550
0.5	-5.4553	8.6419	1.3752	1.8113
0.6	-4.0456	7.2322	1.4189	1.7677
0.7	-2.6359	5.8225	1.4625	1.7241
0.8	-1.2261	4.4127	1.5061	1.6805
0.9	0.1836	3.0030	1.5497	1.6369
1.0	1.5933	1.5933	1.5933	1.5933

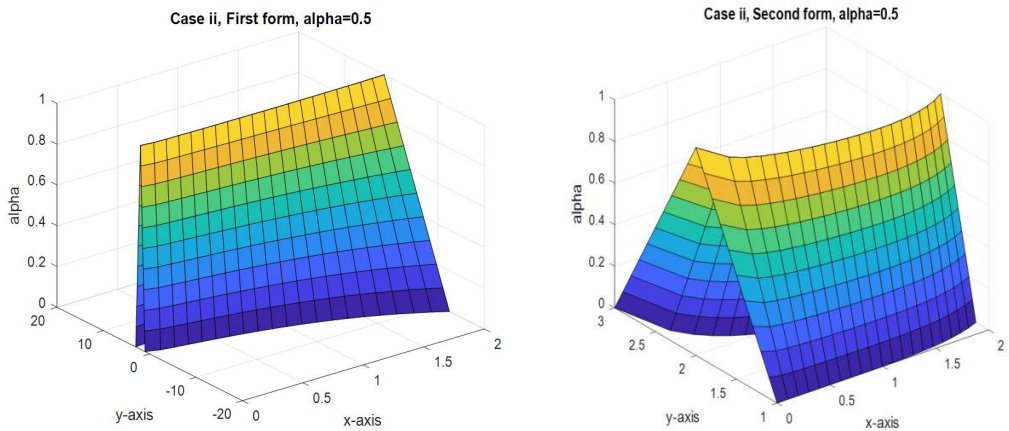


Figure 2: The numerical solutions at Case ii; First and Second forms of FFODEs (3)-(4) for $\alpha = 0.5$.

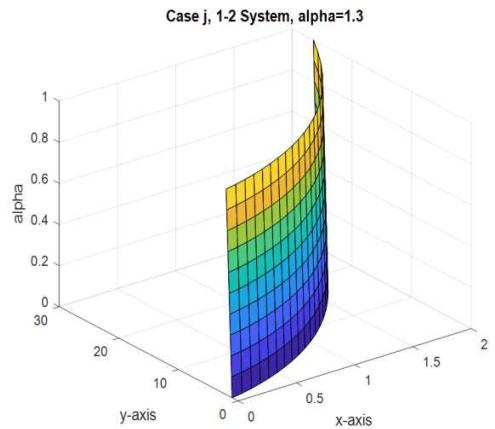
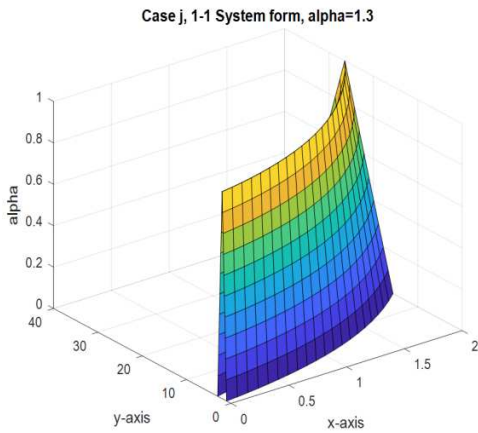
Secondly, the numerical solutions of (13)-(14) for Cases j and jj are listed in Tables 3, 4, 5 and 6, respectively. For graphical results, please see in Figures 3 and 4. Numerical solutions for all cases are got by extending Mittag-Leffler functions up to 10 terms.

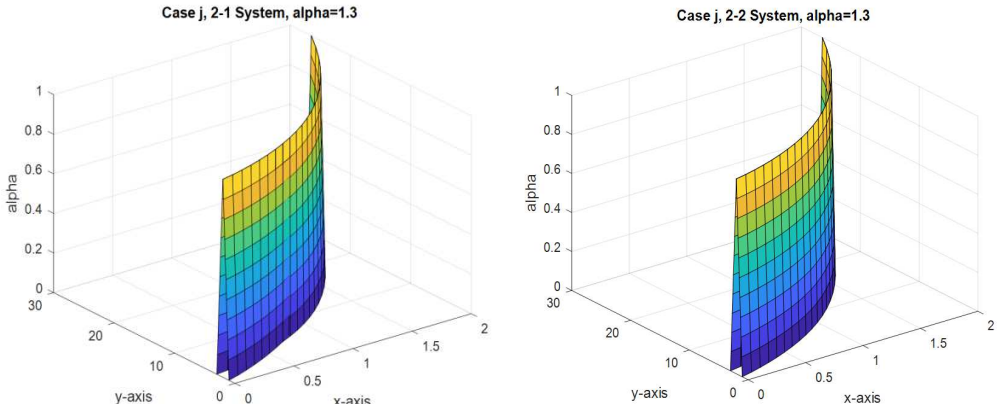
Table 3: Numerical solutions at Case j; (1-1) and (1-2) systems of FFODEs (13)-(14) for $\alpha = 1.3$.

	Case j; (1-1) system, $\alpha = 1.3$		Case j; (1-2) system, $\alpha = 1.3$	
r	\underline{y}	\bar{y}	\underline{y}	\bar{y}
0	15.5994	37.3200	25.6504	27.3203
0.1	16.6854	36.2340	25.7313	27.2343
0.2	17.7714	35.1479	25.8122	27.1482
0.3	18.8575	34.0619	25.8932	27.0621
0.4	19.9435	32.9759	25.9741	26.9761
0.5	21.0295	31.8899	26.0550	26.8900
0.6	22.1156	30.8038	26.1360	26.8040
0.7	23.2016	29.7178	26.2169	26.7179
0.8	24.2876	28.6318	26.2978	26.6318
0.9	25.3737	27.5457	26.3788	26.5458
1.0	26.4597	26.4597	26.4597	26.4597

Table 4: numerical solutions at Case j; (2-1) and (2-2) systems of FFODEs (13)-(14) for $\alpha = 1.3$.

	Case j; (2-1) system, $\alpha = 1.3$		Case j; (2-2) system, $\alpha = 1.3$	
r	\underline{y}	\bar{y}	\underline{y}	\bar{y}
0	25.5991	27.3203	25.8419	27.0775
0.1	25.6851	27.2343	25.9037	27.0157
0.2	25.7712	27.1482	25.9654	26.9539
0.3	25.8572	27.0621	26.0272	26.8922
0.4	25.9433	26.9761	26.0890	26.8304
0.5	26.0294	26.8900	26.1508	26.7686
0.6	26.1154	26.8040	26.2126	26.7068
0.7	26.2015	26.7179	26.2744	26.6450
0.8	26.2876	26.6318	26.3361	26.5833
0.9	26.3736	26.5458	26.3979	26.5215
1.0	26.4597	26.4597	26.4597	26.4597





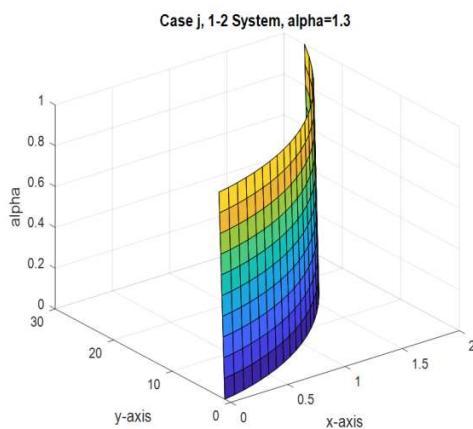
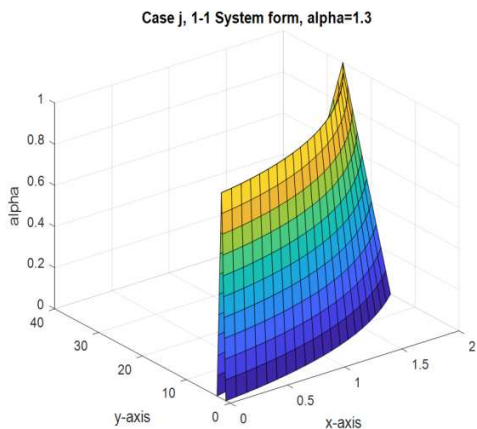
Figures 3: The numerical solutions at Case **j**; (1-1),(1-2),(2-1) and (2-2) systems of FFODEs (13)-(14) for $\alpha = 1.3$.

Table 5: Numerical solutions at Case **jj**; (1-1) and (1-2) systems of FFODEs (13)-(14) for $\alpha = 1.3$.

	Case jj ; (1-1) system, $\alpha = 1.3$		Case jj ; (1-2) system, $\alpha = 1.3$	
r	\underline{y}	\bar{y}	\underline{y}	\bar{y}
0	-8.2160	3.2621	1.8350	3.4536
0.1	-7.1300	3.2003	1.9159	3.3727
0.2	-6.0440	3.1385	1.9968	3.2917
0.3	-4.9579	3.0767	2.0778	3.2108
0.4	-3.8719	3.0150	2.1587	3.1299
0.5	-2.7859	2.9532	2.2396	3.0489
0.6	-1.6998	2.8914	2.3206	2.9680
0.7	-0.6138	2.8296	2.4015	2.8871
0.8	0.4722	2.7678	2.4824	2.8061
0.9	1.5582	2.7061	2.5634	2.7252
1.0	2.6443	2.6443	2.6443	2.6443

Table 6: Numerical solutions at Case **jj**; (2-1) and (2-2) systems of FFODEs (13)-(14) for $\alpha = 1.3$.

Case jj ; (2-1) system, $\alpha = 1.3$			Case jj ; (2-2) system, $\alpha = 1.3$	
r	\underline{y}	\bar{y}	\underline{y}	\bar{y}
0	3.5049	1.7836	2.0265	3.2621
0.1	3.4189	1.8697	2.0883	3.2003
0.2	3.3328	1.9558	2.1500	3.1385
0.3	3.2467	2.0418	2.2118	3.0767
0.4	3.1607	2.1279	2.2736	3.0150
0.5	3.0746	2.2140	2.3354	2.9532
0.6	2.9885	2.3000	2.3972	2.8914
0.7	2.9025	2.3861	2.4589	2.8296
0.8	2.8164	2.4722	2.5207	2.7678
0.9	2.7303	2.5582	2.5825	2.7061
1.0	2.6443	2.6443	2.6443	2.6443



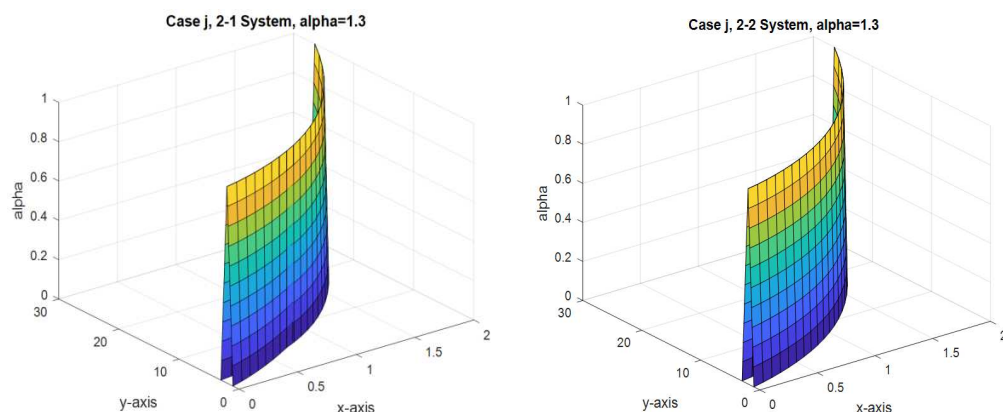


Figure 4: The numerical solutions at Case **jj**; (1-1), (1-2), (2-1) and (2-2) systems of FFODEs (13)-(14) for $\alpha = 1.3$.

7. Conclusions

In the present paper, we obtained the solutions of FFODEs involving fuzzy Caputo's H-difference according to the method proposed by [12]. A new property of fuzzy natural transform for fuzzy Caputo's H-difference has been introduced. The new property has been used to establish a procure for solving linear FFODEs. These equations have been solved numerically to show that FNT is functional. For future research, we contemplate to apply FNT on nonlinear fuzzy differential equations of fractional order. Finally, we demonstrated that the proposed approach has good effectiveness and accuracy to compute the fuzzy solution set and for applications to FFODEs.

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