A weighted finite difference method involving nine-point formula for two-dimensional convection-diffusion equation

Abdul-Sattar Al-Saif
University of Basrah

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A Weighted Finite Difference Method Involving Nine-Point Formula for Two-Dimensional Convection-Diffusion Equation

A. S. J. Al-Saif and Muna O. Al-Humedi

Department of Mathematics, College of Education
University of Basrah, Iraq
ajsaif9@hotmail.com

Abstract: New issues of finite difference method are proposed in this paper. Successfully, we have generalized and extended the works in [7] and [1], respectively. By a new version" nine-point finite difference formula" more accurate results are obtained than those in the literature.

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1. Introduction

Accurate solution of the two-dimensional convection-diffusion equation is desirable. This requires the development of numerical schemes that can remain accurate even for large amount of computational time, several effective schemes have been proposed to achieve this goal [1,2,6,11,12]. In the field of finite difference methods (FDMs), there are many articles that were carried out to numerically solve the convection-diffusion equation in both one- and two-dimensional forms successfully [1-12].

During the last two decades, some new techniques involving weighted discretization and modified equivalent partial differential equation have been introduced to develop FDMs. For example, the explicit, and implicit weighted finite difference method with three-point formula[9], and five-point formula[3,5] are used to solve the one- dimensional convection-diffusion equation, for solving the two-dimensional convection-diffusion equation three-point semi-implicit and implicit formula FDMs[6,11,12,8], and five-point explicit and semi-implicit formula FDM[1] have been used successfully . The motives to present this work are: there are perhaps only few (if there are any) papers that deal with nine-point weighted FDM for solving the two-dimensional convection-diffusion equation, in addition to extending our previous work[7],and to generalizing [1].

In this paper, we extended and developed the weighted FDM. The development reported here is nine-point weighted FDM. When compared with [1,6,11,12,8],present numerical results are in agreement with the existing results, and show that our method is efficient for giving accurate solution for two-dimensional convection-diffusion equation and have reasonable stability.

2. The mathematical formulas

2.1 Governing equation

The two-dimensional convection-diffusion equation for a transport scalar function $T(x,y,t)$ is

$$\frac{\partial T(x,y,t)}{\partial t} + \underbrace{u\frac{\partial T(x,y,t)}{\partial x} + v\frac{\partial T(x,y,t)}{\partial y}}_{Convection\ \ terms} \underbrace{-\alpha_x \frac{\partial^2 T(x,y,t)}{\partial x^2} - \alpha_y \frac{\partial^2 T(x,y,t)}{\partial y^2}}_{Diffusion\ \ terms} = 0 \qquad (1)$$

where x, y are the distances, t is the time, $u,v>0$ are the constants speeds of convection in x and y direction, $\alpha_x,\alpha_y>0$ are the diffusivity in x and y direction.

Equation (1) can be solved in the interior of the region $0\le x\le a$, $0\le y\le b$, $t\ge 0$ of (x,y,t) space, where a, b are constants, subject to appropriate initial and boundary conditions defined as;

$$T(x,y,0)=h(x,y) \quad 0\le x\le a \quad 0\le y\le b\} \qquad (2)$$

$$\left.\begin{aligned} T(0,y,t) &= f_0(y,t) && 0\le y\le b && t>0 \\ T(a,y,t) &= f_a(y,t) && 0\le y\le b && t>0 \\ T(x,0,t) &= g_0(x,t) && 0\le x\le a && t>0 \\ T(x,b,t) &= g_b(x,t) && 0\le x\le a && t>0 \end{aligned}\right\} \qquad (3)$$

where h, f_0, f_a, g_0, g_b are known data.

2.2 The modified equivalent partial differential equation

Consider the finite difference equation (FDE)

$$L_\Delta[\tau^n_{j,k}] = \left.\frac{\partial \tau}{\partial t}\right|^n_{j.k} + u\left.\frac{\partial \tau}{\partial x}\right|^n_{j,k} + v\left.\frac{\partial \tau}{\partial y}\right|^n_{j,k} - \alpha_x \left.\frac{\partial^2 \tau}{\partial x^2}\right|^n_{j,k} - \alpha_y \left.\frac{\partial^2 \tau}{\partial y^2}\right|^n_{j,k} = 0 \qquad (4)$$

which is consistent with partial differential equation (1), where L_Δ is a finite difference operator and $T=\tau(i\Delta x, j\Delta y, n\Delta t)$ is the approximation values at a set of grid points $(i\Delta x, j\Delta y, n\Delta t)$, where $i=1(1)M_1-1$, $j=1(1)M_2-1$ $n=1(1)N-1$.The distance between the points on lines which parallel to x -axis , y -axis and t are $\Delta x=\frac{a}{M_1}, \Delta y=\frac{b}{M_2}$ *and* $\Delta t=\frac{\hat{T}}{N}$ respectively, where $\hat{T}$ is an optimal time and M_1, M_2, N are positive integer numbers. The equivalent partial differential equation (EPDE) to equation (1), can be obtained by converting it to finite difference equation and then expanding the resulting equation by Taylor's series around the grid point

$$\frac{\partial \tau}{\partial t}+u\frac{\partial \tau}{\partial x}+v\frac{\partial \tau}{\partial y}-\alpha_x\frac{\partial^2 \tau}{\partial x^2}-\alpha_y\frac{\partial^2 \tau}{\partial y^2}+\sum_{p=2}^{\infty}\sum_{q=0}^{p}\sum_{r=0}^{q}F_{r,q-r,p-q}\frac{\partial^P \tau}{\partial t^r\partial x^{q-r}\partial y^{p-q}}=0 \qquad (5)$$

Any (FDE) which is consistent with convection-diffusion equation(1) after expanding each term of it in Taylor series about some fixed point in the solution domain, has a modified equivalent partial differential equation(MPDE) of the form

$$\frac{\partial \tau}{\partial t}+u\frac{\partial \tau}{\partial x}+v\frac{\partial \tau}{\partial y}-\alpha_x\frac{\partial^2 \tau}{\partial x^2}-\alpha_y\frac{\partial^2 \tau}{\partial y^2}+\sum_{p=2}^{\infty}\sum_{q=0}^{p}C_{q,p-q}\frac{\partial^P \tau}{\partial x^q\partial y^{p-q}}=0 \qquad (6)$$

where, the term under the summation signs form is the truncation error.

A (FDEs) for solving (1) is said to have accuracy of r^{th} order if $C_{q,p-q}=0$ for $q=0(1)p$ is valid, for all $p=2,....,r$,where at least $C_{r+1,r+1-q}\neq 0$ $q=0,1,...,r+1$. Using the weighted difference method is useful to develop the FDMs to obtain high accurate numerical solution [1-10].

3. Accurate (FDEs) by weighted-differences

Using weighted differencing in order to construct higher-order schemes weights are used to eliminate from the MEPDE as many as possible of the terms containing the derivatives $\frac{\partial^p \tau}{\partial x^p\partial y^{p-q}}$, $q=1(1)p-1$, $p=2,3,\cdots$ to develop FDMs to higher order of accuracy than conventional methods.

For the stencil of Figure (1a), the first-order derivatives of $\tau(x,y,t)$ can be approximated by using three-point formulas (3pt.) at $(j,k,n)^{th}$ as follows

$$\left.\frac{\partial \tau}{\partial y}\right|_{j,k}^{n}=\frac{\tau_{j,k}^{n}-\tau_{j,k-1}^{n}}{\Delta y}+O(\Delta x) \qquad \left.\frac{\partial \tau}{\partial x}\right|_{j,k}^{n}=\frac{\tau_{j,k}^{n}-\tau_{j-1,k}^{n}}{\Delta x}+O(\Delta x)$$

$$\left.\frac{\partial \tau}{\partial t}\right|_{j,k}^{n}=\frac{\tau_{j,k}^{n+1}-\tau_{j,k}^{n}}{\Delta t}+O(\Delta t)$$

These approximations are called forward scheme (FS) and backward scheme (BS).
For the stencil of figure (1b), the first -order and the second-order derivatives of $\tau(x,y,t)$ can be approximated by using nine-point formulas (9pt.) at $(j,k,n)^{th}$ as follows

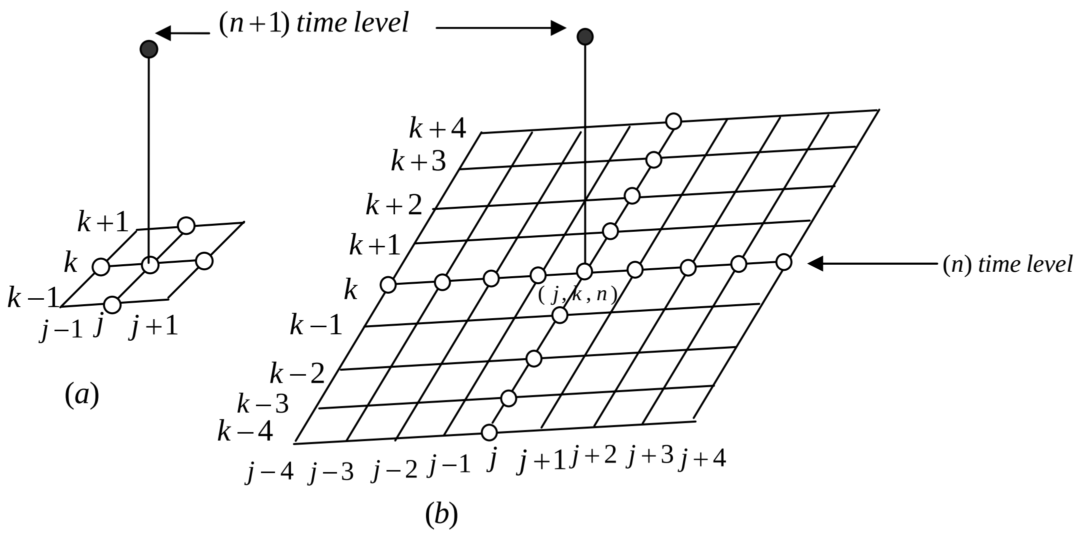


Figure (1). The stencil in (x, y) plane for : (a) three-point (b) nine-point.

$$\left.\frac{\partial \tau}{\partial x}\right|_{j,k}^{n} \cong \frac{\tau_{j-4,k}^{n} + 36\tau_{j-3,k}^{n} + 686\tau_{j-2,k}^{n} - 6524\tau_{j-1,k}^{n} + 6524\tau_{j+1,k}^{n} - 686\tau_{j+2,k}^{n} - 36\tau_{j+3,k}^{n} - \tau_{j+4,k}^{n}}{10080(\Delta x)} + O(\Delta x^{8})$$

$$\left.\frac{\partial \tau}{\partial y}\right|_{j,k}^{n} \cong \frac{\tau_{j,k-4}^{n} + 36\tau_{j,k-3}^{n} + 686\tau_{j,k-2}^{n} - 6524\tau_{j,k-1}^{n} + 6524\tau_{j,k+1}^{n} - 686\tau_{j,k+2}^{n} - 36\tau_{j,k+3}^{n} - \tau_{j,k+4}^{n}}{10080(\Delta y)} + O(\Delta y^{8})$$

$$\left.\frac{\partial^{2} \tau}{\partial x^{2}}\right|_{j,k}^{n} = \frac{-\tau_{j-4,k}^{n} - 48\,\tau_{j-3,k}^{n} - 1372\tau_{j-2,k}^{n} + 26096\tau_{j-1,k}^{n} - 49350\tau_{j,k}^{n} + 26096\tau_{j+1,k}^{n} - 1372\tau_{j+2,k}^{n} - 48\tau_{j+3,k}^{n} - \tau_{j+4,k}^{n}}{20160(\Delta x)^{2}} + O(\Delta x^{8})$$

$$\left.\frac{\partial^{2} \tau}{\partial y^{2}}\right|_{j,k}^{n} = \frac{-\tau_{j,k-4}^{n} - 48\tau_{j,k-3}^{n} - 1372\tau_{j,k-2}^{n} + 26096\tau_{j,k-1}^{n} - 49350\tau_{j,k}^{n} + 26096\tau_{j,k+1}^{n} - 1372\tau_{j,k+2}^{n} - 48\,\tau_{j,k+3}^{n} - \tau_{j,k+4}^{n}}{20160(\Delta y)^{2}} + O(\Delta y^{8})$$

These approximations are called central scheme (CS). We can approximate all the derivatives in equation (4) with weighted or non-weighted discretization as the following formulas;

3.1 The forward time- central space scheme with nine-point formulas (FTCS 9pt.)

Using a first-order forward difference approximation equation for the time derivative and eighth-order central difference approximation for the first and second spatial derivatives to approximate equation (4), leads to a new formula as,

$$
\begin{aligned}
\tau_{j,k}^{n+1} = &-\left(\frac{C_x}{10080}+\frac{S_x}{20160}\right)\tau_{j-4,k}^{n} - \left(\frac{C_x}{280}+\frac{S_x}{420}\right)\tau_{j-3,k}^{n} - \frac{343}{5040}(C_x+S_x)\tau_{j-2,k}^{n} + \frac{1631}{2520}(C_x+2S_x)\tau_{j-1,k}^{n} \\
&+\left(1-\frac{4935}{2016}(S_x+S_y)\right)\tau_{j,k}^{n} - \frac{1631}{2520}(C_x-2S_x)\tau_{j+1,k}^{n} + \frac{343}{5040}(C_x-S_x)\tau_{j+2,k}^{n} + \left(\frac{C_x}{280}-\frac{S_x}{420}\right)\tau_{j+3,k}^{n} \\
&+\left(\frac{C_x}{10080}-\frac{S_x}{20160}\right)\tau_{j+4,k}^{n} - \left(\frac{C_y}{10080}+\frac{S_y}{20160}\right)\tau_{j,k-4}^{n} - \left(\frac{C_y}{280}+\frac{S_y}{420}\right)\tau_{j,k-3}^{n} - \frac{343}{5040}(C_y+S_y)\tau_{j,k-2}^{n} \\
&+\frac{1631}{2520}(C_y+2S_y)\tau_{j,k-1}^{n} - \frac{1631}{2520}(C_y-2S_y)\tau_{j,k+1}^{n} + \frac{343}{5040}(C_y-S_y)\tau_{j,k+2}^{n} + \left(\frac{C_y}{280}-\frac{S_y}{420}\right)\tau_{j,k+3}^{n} \\
&+\left(\frac{C_y}{10080}-\frac{S_y}{20160}\right)\tau_{j,k+4}^{n}
\end{aligned} \tag{7}
$$

where $C_x = \dfrac{u\Delta t}{\Delta x}, C_y = \dfrac{v\Delta t}{\Delta y}, S_x = \dfrac{\alpha_x \Delta t}{(\Delta x)^2}, S_y = \dfrac{\alpha_y \Delta t}{(\Delta y)^2}$ are the Courant numbers and diffusion numbers respectively.

The MEPDE of finite difference equation (7) is equivalent to equation (6), and it has the coefficients of the leading (first-order) error terms given by

$$C_{2,0} = \frac{u\Delta x C_x}{2}, \quad C_{1,1} = uv\Delta t, \quad C_{0,2} = \frac{v\Delta y C_y}{2}$$

This implies that the FTCS 9pt. method introduces numerical diffusion in both the x -and y -directions. The stability of equation (7) can be established by using the Von Neumann method, where the Von Neumann amplification factor of equation (7) is given by

$$|G|^2 - 1 = f(\chi), \ \chi \in [-1,1]$$

where
$$f(\chi) = b_0 + b_1\chi + b_2\chi^2 + b_3\chi^3 + b_4\chi^4 + b_5\chi^5 + b_6\chi^6 + b_7\chi^7 + b_8\chi^8 \tag{8}$$

in which $\chi = \cos(B)$,($B = B_x = B_y$, *here* $B_x = \pi m_x \Delta x$, $B_y = \pi m_y \Delta y$ and (m_x, m_y) Fourier components)and the coefficients

$$b_0 = (4.624)^2 S^2 - 9.248\, S + (2.603)^2 C^2$$
$$b_1 = -4.811 S^2 + 10.413\, S - 2.826 C^2$$
$$b_2 = 32.126 S^2 - 1.086 S - 7.074 C^2$$
$$b_3 = -5.300 S^2 - 0.076 S + 2.872 C^2$$
$$b_4 = -0.353 S^2 - 3.175\times10^{-3} S + 9.524\times10^{-3} C^2$$
$$b_5 = 0.025 S^2 - 0.045 C^2$$
$$b_6 = 3.175\times10^{-3} S^2 - 6.702\times10^{-3} C^2$$
$$b_7 = 1.209\times10^{-4} S^2 - 3.628\times10^{-4} C^2$$
$$b_8 = 2.519\times10^{-6} S^2 - 1.008\times10^{-5} C^2$$

Here and in the next stability analysis, we setting $C_x = C_y = C$ and $S_x = S_y = S$.
To satisfy the stability criterion of equation (7), we require $f(\chi) \le 0$,for $-1 \le \chi \le 1$ .Thus, If $\chi = 0$, then for $f(0) \le 0$,we have

$$0 < S \le \frac{1260}{(2913)^2}[5826 \mp \frac{1}{21}\sqrt{14968543 - 1043606\ C^2}] \ .$$

This is applied to $0 < C \le 1.198$. and if $\chi = -1$,then for $f(-1) \le 0$,yields

$$0 < S \le \frac{630}{2347552}[13024 \mp \sqrt{169624576 + \frac{54915814}{(630)^2}C^2}] \ .$$

This condition is applied to $C > 0$.

3.2 The Upwind difference scheme with nine-point formulas (UDS 9pt.)

In this section, we use the first-order forward and backward approximation of the first time and spatial derivatives, while the second spatial derivatives are approximated by the eighth-order central difference. Thus, a new formula can be obtained as,

$$\begin{aligned}\tau_{j,k}^{n+1} = & -\frac{S_x}{20160}(\tau_{j-4,k}^n + \tau_{j+4,k}^n) - \frac{S_x}{420}(\tau_{j-3,k}^n + \tau_{j+3,k}^n) - \frac{343}{5040}S_x(\tau_{j-2,k}^n + \tau_{j+2,k}^n) \\ & + (C_x + \frac{1631}{1260}S_x)\tau_{j-1,k}^n + (1 - C_x + C_y - \frac{4935}{2016}(S_x + S_y))\tau_{j,k}^n + \frac{1631}{1260}S_x\tau_{j+1,k}^n \\ & - \frac{S_y}{20160}(\tau_{j,k-4}^n + \tau_{j,k+4}^n) - \frac{S_y}{420}(\tau_{j,k-3}^n + \tau_{j,k+3}^n) - \frac{343}{5040}S_y(\tau_{j,k-2}^n + \tau_{j,k+2}^n) \\ & + (C_y + \frac{1631}{1260}S_y)\tau_{j,k-1}^n + \frac{1631}{1260}S_y\tau_{j,k+1}^n\end{aligned} \tag{9}$$

The MEPDE of this FDE has the coefficients of the leading (first- order) error with coefficients given by

$$C_{2,0} = -\frac{u\Delta x(1 - C_x)}{2} \ , \ C_{1,1} = uv\Delta t \ , \ C_{0,2} = -\frac{v\Delta y(1 - C_y)}{2}$$

We noted that the coefficients $C_{2,0}$, $C_{1,1}$, $C_{0,2}$ are not equal to zero , therefore the upwind method introduced numerical diffusion in both the *x*- and *y*-directions.

The application of Von Neumann stability analysis shows that the equation (9) has the amplification factor G with $f(\chi)$ having the same form in equation (8) with the coefficients:

$b_0 = (4.624)^2 S^2 + (18.575C - 9.248)S + 4C^2$

$b_1 = -4.810S^2 + (10.413 - 39.321C)S + 4C - 8C^2$

$b_2 = 32.126S^2 + (11.498C - 1.086)S$

$b_3 = -5.300S^2 - (0.076 + 2.095C)S$

$b_4 = -0.353S^2 - (3.175\times10^{-3} + 0.149C)S$

$b_5 = 0.025S^2 - 6.349\times10^{-3}C^2$

$b_6 = 3.175S^2$

$b_7 = 1.209\times10^{-4}S^2$

$b_8 = 2.519\times10^{-6}S^2$

The stability criterion

$$0 < S \le \frac{1260}{(2913)^2}[(-11702C + 5826) \mp \sqrt{33942276 - 13635170C + \frac{27763319}{441}C^2}\ ]$$

for $0 < C \le 0.9$ when $f(0) \le 0$, and for $0 < C < 0.1$ when $f(-1) \le 0$ is

$$0 < S \le \frac{630}{2347552}[(-44948C + 13024) \mp \sqrt{169624576 - 1152052088C + \frac{675459848}{(630)^2}C^2}\ ]$$

Note: these schemes (FTCS 9pt.) and (UDS 9pt.) are unconditionally stable if $\chi = 1$ for all $S,C > 0$.

3.3 Weighted finite difference scheme with nine-point formulas (WFDS 9pt.)

To get this scheme, when discretizing equation (4) at the grid point $(j,k,n)^{th}$, we used weights $0 \le \phi, \gamma \le 1$ in the approximation

$$\frac{\partial\tau}{\partial x} \approx \phi\times BS + (1-\phi)\times CS\ ,\ \ \frac{\partial\tau}{\partial y} \approx \gamma\times BS + (1-\gamma)\times CS$$

with the forward time difference approximation for time derivative and eight-order central-space approximation for $\frac{\partial^2\tau}{\partial x^2}$ and $\frac{\partial^2\tau}{\partial y^2}$ the resulting (WFDE 9pt.) was

$$\begin{aligned}
\tau_{j,k}^{n+1} = &-(\frac{C_x(1-\phi)}{10080} + \frac{S_x}{20160})\tau_{j-4,k}^n - (\frac{C_x(1-\phi)}{280} + \frac{S_x}{420})\tau_{j-3,k}^n - \frac{343}{5040}(C_x(1-\phi) + S_x)\tau_{j-2,k}^n \\
&+ (C_x\phi + \frac{1631}{2520}(C_x(1-\phi) + 2S_x))\tau_{j-1,k}^n + (1 - C_x\phi - C_y\gamma - \frac{4935}{2016}(S_x + S_y))\tau_{j,k}^n \\
&- \frac{1631}{2520}(C_x(1-\phi) - 2S_x)\tau_{j+1,k}^n + \frac{343}{5040}(C_x(1-\phi) - S_x)\tau_{j+2,k}^n + (\frac{C_x(1-\phi)}{280} - \frac{S_x}{420})\tau_{j+3,k}^n \\
&+ (\frac{C_x(1-\phi)}{10080} - \frac{S_x}{20160})\tau_{j+4,k}^n - (\frac{C_y(1-\gamma)}{10080} + \frac{S_y}{20160})\tau_{j,k-4}^n - (\frac{C_y(1-\gamma)}{280} + \frac{S_y}{420})\tau_{j,k-3}^n \\
&- \frac{343}{5040}(C_y(1-\gamma) + S_y)\tau_{j,k-2}^n + (C_y\gamma + \frac{1631}{2520}(C_y(1-\gamma) + 2S_y))\tau_{j,k-1}^n - \frac{1631}{2520}(C_y(1-\gamma) - 2S_y)\tau_{j,k+1}^n \\
&+ \frac{343}{5040}(C_y(1-\gamma) - S_y)\tau_{j,k+2}^n + (\frac{C_y(1-\gamma)}{280} - \frac{S_y}{420})\tau_{j,k+3}^n + (\frac{C_y(1-\gamma)}{10080} - \frac{S_y}{20160})\tau_{j,k+4}^n
\end{aligned} \tag{10}$$

The finite difference equation has MEPDE with coefficients, which are the same as those of equation (6). Therefore, equation (10) has the first-order truncation error with coefficients

$$C_{2,0} = -\frac{u\Delta x(\phi - C_x)}{2}\ ,\ C_{1,1} = uv\Delta t\ \ ,\ C_{0,2} = -\frac{v\Delta y(\gamma - C_y)}{2}$$

The diffusion that numerical introduced by the first and last of these equations may be eliminated by setting $\phi = C_x$ *and* $\gamma = C_y$, but there remains a first-order error term involving $\frac{\partial^2 \tau}{\partial x \partial y}$. The Von Neumann amplification factor G with $f(\chi)$ has the same form in equation (8) with the coefficients

$$b_0 = (4.624)^2 S^2 + (9.287C(\phi+\gamma) - 9.248)S + (1.302)^2 C^2(2-\phi-\gamma)^2 + 2.603C^2(2-\phi-\gamma)(\phi+\gamma) + 2C^2(\phi^2+\gamma^2)$$
$$b_1 = -4.810S^2 + (10.413 - 19.660C(\phi+\gamma))S - 0.707C^2(2-\phi-\gamma)^2 - 0.542C^2(2-\phi-\gamma)^2(\phi+\gamma) + 2C(\phi+\gamma) - 2C^2(\phi+\gamma)^2$$
$$b_2 = 32.126S^2 + (5.749C(\phi+\gamma) - 1.086)S - 1.769C^2(2-\phi-\gamma)^2 - 26.603C^2(2-\phi-\gamma)(\phi+\gamma)$$
$$b_3 = -53.017S^2 - (0.076 - 1.048C(\phi+\gamma))S + 0.718C^2(2-\phi-\gamma)^2 + 0.539C^2(2-\phi-\gamma)(\phi+\gamma)$$
$$b_4 = -0.353S^2 - (3.175\times10^{-3} + 0.075\,C(\phi+\gamma))S + 2.381\times10^{-3}C^2(2-\phi-\gamma)^2 + 0.057C^2(2-\phi-\gamma)(\phi+\gamma)$$
$$b_5 = 0.025S^2 - 3.175\times10^{-3} S\,C(\phi+\gamma) - 0.011C^2(2-\phi-\gamma)^2 + 3.175\times10^{-3}C^2(2-\phi-\gamma)(\phi+\gamma)$$
$$b_6 = 3.175\times10^{-3}S^2 - 1.675\times10^{-3}C^2(2-\phi-\gamma)^2$$
$$b_7 = 1.209\times10^{-4}S^2 - 9.070\times10^{-5}C^2(2-\phi-\gamma)^2$$
$$b_8 = 2.519\times10^{-6}S^2 - 2.519\times10^{-6}C^2(2-\phi-\gamma)^2$$

To satisfy the stability criterion of equation (10), we require $f(\chi) \le 0$ for $-1 \le \chi \le 1$. Thus if $x = 0$, then for $f(0) \le 0$, we have

$$0 < S \le \frac{1260}{(2913)^2}[(5826 - 11702C^2) \mp \sqrt{33942276 - \frac{60232536}{441}C^2 + \frac{47006280}{441}C^3 + \frac{332061517}{441}C^4}\,]$$

This is applied to $0 < C \le 2$. If $x = -1$, then for $f(-1) \le 0$ we have

$$0 < S \le \frac{630}{2347552}[(13024 - 44948\ C^2) \mp$$

$$\sqrt{169624576 + \frac{13191938}{(630)^2}C^2 + \frac{181416902}{(630)^2}C^3 + \frac{422212125}{(630)^2}C^4 - \frac{1284580456}{630}C^5}\,]$$

this condition is applied to $0 < C \le 0.77$.
Note: these schemes (WFDS 9pt.) are unconditionally stable if $\chi = 1$ for all $S, C > 0$.

So, we can say that equation (10) is a generalization to the pervious schemes. We show now how this equation reduces to some well-known finite difference equation when it contains values of ϕ and γ are chosen. If substituting $\phi = \gamma = 0$,in equation (10),we get (FTCS 9pt.)equation (7).If we put $\phi = \gamma = 1$ in equation (10) UDS 9pt. is obtained, equation (9). If $\phi = \gamma = C$, we get weighted Difference scheme of nine points.

3.4 Modified weighted finite difference scheme (MWFDS. 9pt)

To obtain FDMs. of greater accuracy, more grid points must be used in order to introduce weights so that more terms in the truncation error of the MEPDE may be eliminated. Adding four extra grid points $(j-1,k-1,n)$, $(j-1,k+1,n)$, $(j+1,k-1,n)$, $(j+1,k+1,n)$ on the stencil of Figure (1b) and the error terms involving $\dfrac{\partial^2\tau}{\partial x\partial y}$ in the MEPDE of (10) may be eliminated by using the approximation

$$\left.\frac{\partial^{2\tau}}{\partial x\partial y}\right|_{j,k}^{n} \approx \frac{1}{4\Delta x\Delta y}(-\tau_{j-1,k+1}^{n} + \tau_{j+1,k+1}^{n} + \tau_{j-1,k-1}^{n} - \tau_{j+1,k-1}^{n}) \tag{11}$$

which has a truncation error of $O\{(\Delta x)^2,(\Delta y)^2\}$. The method of obtaining a more accurate FDE through the replacement of error terms in the MEPDE has been described for the one-dimensional case by [5]. Consider the finite difference equation (10). Written in the form

$$L_{\Delta}[\tau_{j,k}^{n}] = 0 \tag{12}$$

where

$$\begin{aligned} L_{\Delta}[\tau_{j,k}^{n}] \equiv \tau_{j,k}^{n+1} &- a_{-4,0}\tau_{j-4,k}^{n} - a_{-3,0}\tau_{j-3,k}^{n} - a_{-2,0}\tau_{j-2,k}^{n} - a_{-1,0}\tau_{j-1,k}^{n} - a_{1,0}\tau_{j+1,k}^{n} - \\ & a_{2,0}\tau_{j+2,k}^{n} - a_{3,0}\tau_{j+3,k}^{n} - a_{4,0}\tau_{j+4,k}^{n} - a_{0,0}\tau_{j,k}^{n} - a_{0,-4}\tau_{j,k-4}^{n} - a_{0,-3}\tau_{j,k-3}^{n} \\ & - a_{0,-2}\tau_{j,k-2}^{n} - a_{0,-1}\tau_{j,k-1}^{n} - a_{0,1}\tau_{j,k+1}^{n} - a_{0,2}\tau_{j,k+2}^{n} - a_{0,3}\tau_{j,k+3}^{n} - a_{0,4}\tau_{j,k+4}^{n} \end{aligned} \tag{13}$$

in which

$a_{-4,0}$, $a_{-3,0}$, $a_{-2,0}$, $a_{-1,0}$, $a_{1,0}$, $a_{2,0}$, $a_{3,0}$, $a_{4,0}$, $a_{0,0}$, $a_{0,-4}$, $a_{0,-3}$, $a_{0,-2}$, $a_{0,-1}$, $a_{0,1}$, $a_{0,2}$, $a_{0,3}$ *and* $a_{0,4}$ are the corresponding coefficients in (10). Using the results of MEPDE in (10) with $C_x = \phi$ and $C_y = \gamma$ the alternative differential form may be written as;

$$L_{\Delta}[\tau_{j,k}^{n}] \equiv \Delta t\left[\frac{\partial\tau}{\partial t} + u\frac{\partial\tau}{\partial x} + v\frac{\partial\tau}{\partial y} - \alpha_x\frac{\partial^2\tau}{\partial x^2} - \alpha_y\frac{\partial^2\tau}{\partial y^2} + uv\Delta t\frac{\partial^2\tau}{\partial x\partial y} + \cdots\right]\Bigg|_{j,k}^{n} \tag{14}$$

Subtracting the error term $uv(\Delta t)^2\dfrac{\partial^2\tau}{\partial x\partial y}$ from (14) is equivalent to subtracting from (15) the term

$$C_x C_y(-\tau_{j-1,k+1}^{n} + \tau_{j+1,k+1}^{n} + \tau_{j-1,k-1}^{n} - \tau_{j+1,k-1}^{n})/4$$

The largest error terms in the brackets on the right-hand side of (14) are now $O\{(\Delta x)^2,(\Delta y)^2\}$. Rearrangement gives the second-order FDE

$$\tau_{j,k}^{n+1} = \sum_{p=-4}^{4}\sum_{q=-4}^{4} a_{p,q}\tau_{j+p,k+q}^{n} \tag{15}$$

In which $a_{p,q} = 0 \; for \;\; p = q = -4(1)4,$ except $a_{p,0}, a_{0,q}$ and

$$a_{-1,-1}(C_x, C_y) = -a_{-1,1}(C_x, C_y) = a_{1,-1}(C_x, C_y) = a_{1,1}(C_x, C_y) = C_x C_y / 4$$

The MEPDE of equation (15) contains no first-order error terms and has the following coefficients of the second-order error terms:

$$C_{0,3} = \frac{(\Delta y)^2}{6} v(C_y - (C_y)^2) - \Delta y C_y \alpha_y \quad C_{1,3} = \Delta x \Delta y C_x C_y \alpha_y$$

$$C_{1,2} = -\Delta x C_x \alpha_y - \frac{(\Delta y)^2}{2} u(C_y)^2$$

$$C_{4,0} = \frac{\Delta x}{2}\alpha_x C_x\left(\frac{\alpha_x}{u} + \Delta x C_x\right) \qquad C_{2,2} = \frac{\Delta x}{u} C_x C_y \alpha_x \alpha_y + \frac{(\Delta x)^2}{2}(C_x)^2 \alpha_y + \frac{(\Delta y)^2}{2}(C_y)^2 \alpha_x$$

$$C_{3,1} = \Delta x \Delta y C_x C_y \alpha_x \qquad C_{2,1} = -\Delta y C_y \alpha_x - \frac{(\Delta x)^2}{2} v(C_x)^2$$

$$C_{0,4} = \frac{\Delta y}{2}\alpha_y C_y\left(\frac{\alpha_y}{v} + \Delta y C_y\right) \qquad C_{3,0} = \frac{(\Delta x)^2}{6} u(C_x - (C_x)^2) - \Delta x C_x \alpha_x$$

Setting $S_x = S_y = S$, $C_x = C_y = C$ and carrying out a numerical stability analysis yields the stability of second order weighted difference scheme which is identical to the stability of WFDS 9pt. equation (13).

4. Adjacent boundary domain

When applying the difference schemes which depend on formulas of the nine-point to solve the two-dimensional transport equation (1), we find that there are difficulties in computing the values of the numerical approximation for the transport variable $T(x, y, t)$ at the adjacent points of boundaries Figure (2). We cannot use the formulas of these schemes directly to compute these values, so the method of computing the approximate values of the transport variable in equation (1) has been done on two stages, the first stage is done by using the formulas of schemes explained previously (for the bounded area by S, W, N, E), whereas the second stage is by computing the approximate values for the transport variable at the adjacent points of the boundaries in $(N, E, S, W, NS, SE, SW, NW)$. It is suggested that there may be an implicit scheme resulting as an idea that the value of the function at any point is equal to the average of the function value at the adjacent points to it when $\Delta x = \Delta y$ .This scheme will be illustrated in the following algorithms.

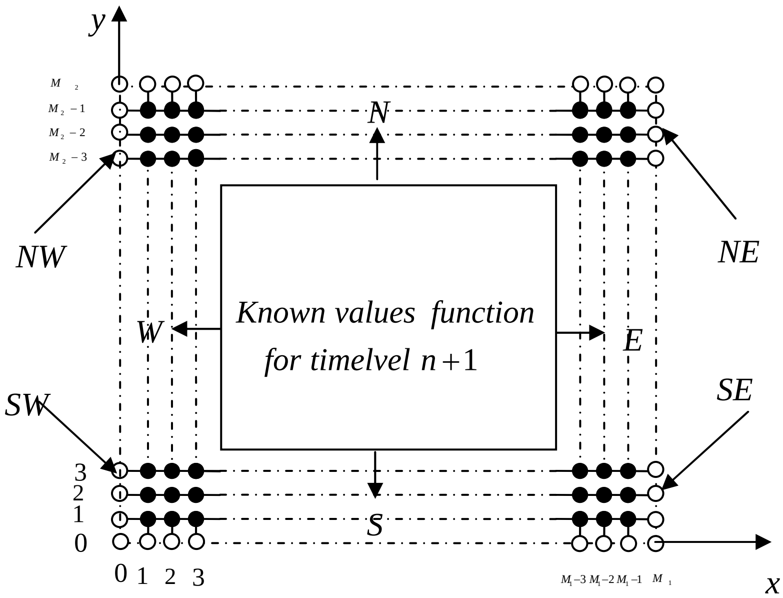


Figure (2) Stencil of Adjacent points of boundary

Algorithms for computation of adjacent boundary points

For region of boundary layers E *and* W :

For the time level $n+1$; $n = 0,1,2......$
do $k = 4,5,.........,M_2 - 4$
To compute values function at grid of layers E & W
do $j = 3,2,-1$
$\tau^{n+1}_{j,k} = 2\tau^{n+1}_{j+1,k} - \tau^{n+1}_{j+2,k} \quad : \quad \tau^{n+1}_{M_1-j,k} = 2\tau^{n+1}_{M_1-(j+1),k} - \tau^{n+1}_{M_1-(j+2),k}$
enddo
for $j = 1$ *or* $j = M_1 - 1$: $\tau^{n+1}_{j,k} = (\tau^{n+1}_{j+1,k} + \tau^{n+1}_{j-1,k})/2$
enddo

For region of boundary layers at the corners SW *and* NW :

For the time level $n+1$; $n = 0,1,2......$
do $k = 3,\ 1,-1$
To compute values function at grid of layers SW & NW
do $j = 3,2,-1$
$\tau^{n+1}_{j,k} = 2\tau^{n+1}_{j+1,k} - \tau^{n+1}_{j+2,k} \quad : \quad \tau^{n+1}_{M_1-j,k} = 2\tau^{n+1}_{M_1-(j+1),k} - \tau^{n+1}_{M_1-(j+2),k}$
enddo
for $j = 1$ *or* $j = M_1 - 1$: $\tau^{n+1}_{j,k} = (\tau^{n+1}_{j+1,k} + \tau^{n+1}_{j-1,k})/2$
enddo

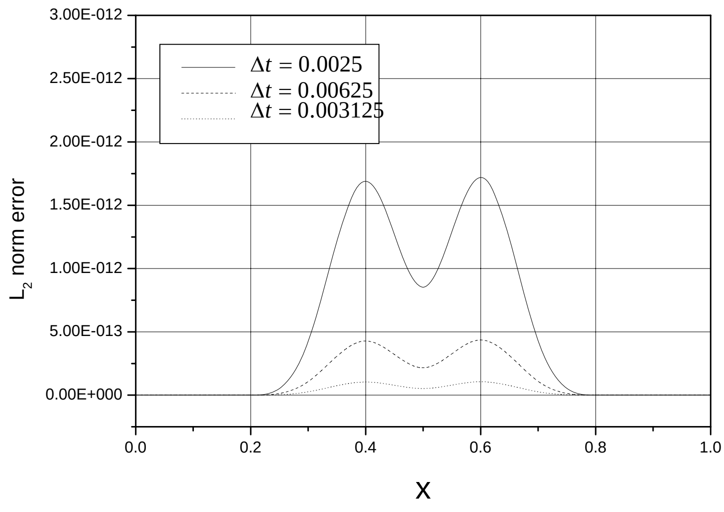
A similar procedure can be used for other regions $N, S, NE, and\ SE$. These algorithms were applied to compute the adjacent points of the boundaries in the nine-point (weighted and non-weighted) finite difference schemes .These schemes can be regarded as general ones that can be applied to treat the same problems that occur when the five-point finite difference schemes are used . This procedure may be a useful tool to treat other schemes.

5. Discussion of numerical results

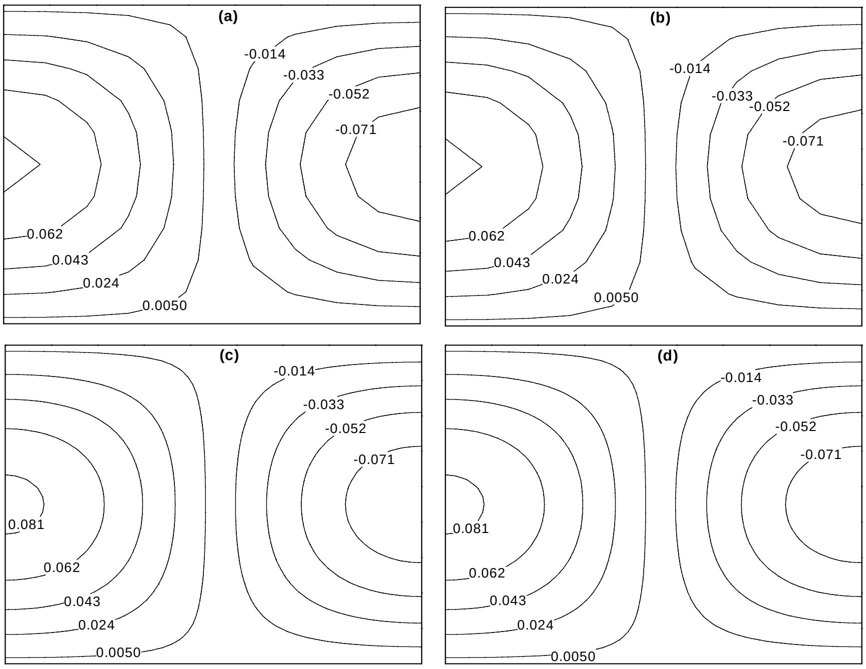
In this section, we test our difference method, which depend on the formulas of the nine-point (with weighted and without weighted) to solve unsteady two-dimensional convection – diffusion problem, in order to demonstrate the validity and effectiveness of this method. Results are presented for two problems.

Problem 1.

The first test for our propose method is the diffusion problem in the unit square $[0,1]\times[0,1]$, which is obtained by setting $\alpha_x = \alpha_y = 1$ and $u = v = 0$ in the equation (1). The exact solution of this problem is given by; $u(x,y,t) = e^{-2\pi^2 t}\sin(\pi x)\cos(\pi y)$. This problem is given in[12].The initial and boundary conditions(2-3) are directly taken from this solution. Computations were carried out at different times over the problem domain [0,1]. The results are documented in Table 1 and 2. L_2 norm error of our method is documented at $t = 0.125$ in table 1 for various mesh size, and at $t = 0.25$ in table 2 and figure 3 for various time step size. We conclude that the accuracy of present method is increased with increasing number of gird points in space and levels in time respectively. The present simulation exhibited more accurate results comparison to the rival Tain and Ge method [12]. Figure 4 show that the FD9pt. method has a good solution agreement with the exact solution. All comparison shows that the current method offers better results than the other methods.



Figure(3)L_2norm errors at diagonal of a square region [0,1]x[0,1] for t=1.25



Figure(4) Comparison between exact solution and approximation solution at t=0.125, contours plots of;
(a) Exact solution with 11×11 grids (b) Numerical solution with 11×11 grids
(c) Exact solution with 41×41 grids (d) Numerical solution with 41×41 grids

Table-1:- L_2 norm error at $t=0.125$ with $\Delta t=(\Delta x)^2$

Grid	Tain&Ge [12] L_2 **norm error**	FD9pt. method L_2 **norm error**
11x11	8.55134E-05	3.066732E-08
21x21	5.19160E-06	1.142410E-09
41x41	3.17475E-07	2.777872E-10

Table-2:- L_2 norm error at $\Delta x=\Delta y=0.1$, $t=0.25$

Δt	Tain&Ge [12] L_2 **norm error**	FD9pt. method L_2 **norm error**
0.0025	2.64692E-05	4.017735E-12
0.00625	5.40813E-06	1.032878E-12
0.003125	7.18040E-07	2.515195E-13

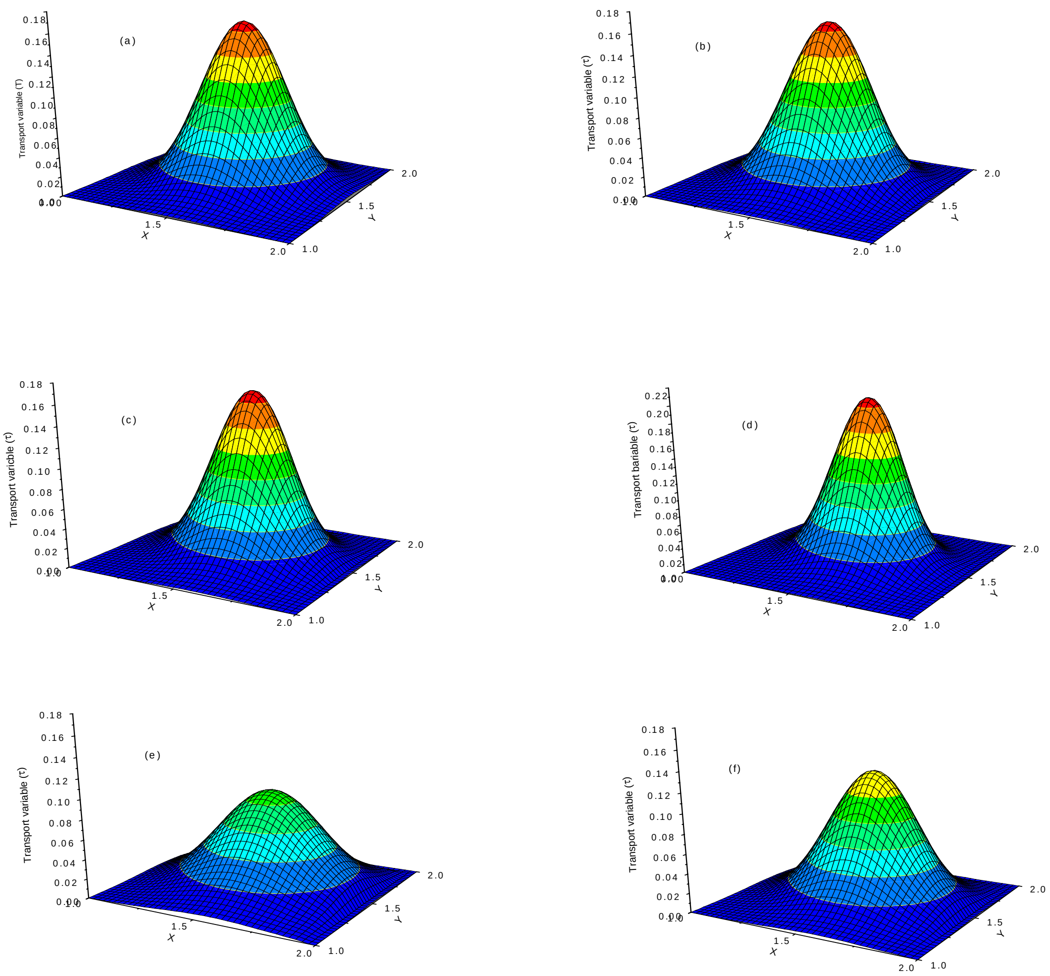
Problem 2.

An analytic solution in [1,12,8] of the problem is applied to this numerical test given by

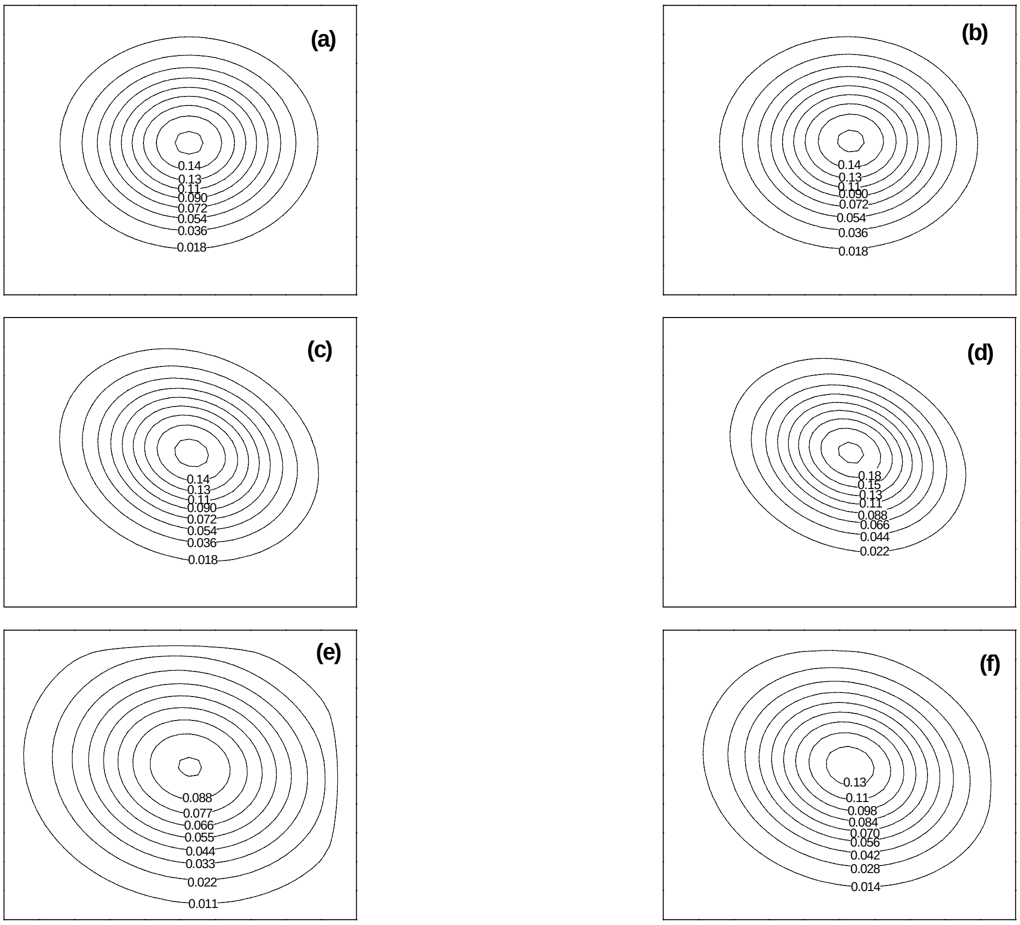
$$T(x,y,t)=\frac{1}{4t+1}\exp\left(-\frac{(x-0.5-ut)^2}{\alpha_x(4t+1)}-\frac{(y-0.5-vt)^2}{\alpha_y(4t+1)}\right) \quad , \quad t\geq 0$$

The initial and boundary conditions (2-3) are directly taken from this solution.

It is noticed that when comparing the altitude of Gauss pulse, it has the largest numerical approximation to $T(x, y, t)$ with Gauss pulse height Figures (5 a-f), explained also by contour drawings of the Figures (6 a-f). The WFD 9pt. and MWFDS 9pt. produces an approximate value identical to the analytic value and better than the schemes FTCS 9pt. and Upwind 9pt. The main reason for this can be attributed to the fact that the effect of the numerical diffusion in WFDS 9pt. scheme is small and in the MWFDS 9pt.scheme there is no numerical diffusion. Table (3) explains the average error and the maximum error for the nine-point finite difference schemes at time $t = 1.25$, $u = v = 0.8$, $C_x = C_y = 0.2$, $S_x = S_y = 0.1$ and $\Delta t = 0.00625$. Figures (5,6), and Table (3) show that the weighted modified difference schemes for the nine-point (MWFDS 9pt.) are better than other schemes because of their second-order accuracy, and their having absolute error less than the other schemes since the value of the numerical solution is approximate to that of the analytic one Figures (5 a,b)



Figure(5) The surface plots of (a) exact solution ,(b) (MWFD 9pt.), (c)(WFD 9pt. $\phi = \gamma = C$), (d) (FTCS 9pt.), (e) (Upwind 9pt.) and (f) (WFD 9pt. $\phi = \gamma = 0.5$) at $t = 1.25$ in plane [1,2]



Figure(6) The contours plots of (a) exact solution ,(b) (MWFD 9pt.), (c)(WFD 9pt. $\phi = \gamma = C$),(d) (FTCS 9pt.), (e) (Upwind 9pt.) and (f) (WFD 9pt. $\phi = \gamma = 0.5$) at $t = 1.25$ in plane [1,2]

and this is clear from contour drawings as Figures (6 a and b), Furthermore , when compared to the other schemes found in this table with MWFDS 9pt., it is noticed that all these schemes have first-order error accuracy . Generally, it is clear that one kind of nine-point schemes, is identical in identifying that the absolute error value and the WFDS are the most accurate when $\phi = \gamma = C$ and they have less error than the other schemes in terms of average absolute error and maximum absolute error.Table (4) explains the efficiency of these schemes for solving the two-dimensional transport equation (1) when taking different values for Reynolds number ($R = \frac{C}{S}$) following its difference in dimension steps $\Delta x, \Delta y$ and time step Δt , noticing that the accuracy of these schemes on the behavior of the numerical solution becomes better gradually through values changeability of these parameters.

Figures (7 a, b) show the drawing of average absolute errors curves against the time for every new FDS suggested in this study. From Figure (7 a), we notice that the error average of WFDS 9pt. for all the time-level with the weight $\phi = \gamma = C$ is better than the other schemes which are based on the nine-point formula. Figure (7 b) explains the drawing of average absolute error for the MWFDS 9pt. From this Figure, we can notice that the MWFDS 9pt. with the weight has $\phi = \gamma = C$ less error average for all the time-levels than the other schemes. Generally, it is observed that FDS based on the weights $\phi = \gamma = C$ has less errors average than the other schemes.

Table-3:- Error measurements at $t = 1.25$ with $\Delta t = 0.00625$ $u = v = 0.8$, and $C_x = C_y = 0.2$

The Method values	**Average |error|**	**Maximum |error|**	**Type of formula**	**weight**
Upwind 9pt.	9.787177E-05	4.420168E-03	*first – order*	-----
FTCS 9pt.	5.826575E-05	1.418220E-03	*first – order*	-----
WFDS 9pt	4.925594E-05	1.067123E-03	*first – order*	$\phi = \gamma$ =0.5
	2.794456E-05	1.607237E-04	*first – order*	$\phi = \gamma$ =C
MWFDS 9pt.	9.054745E-06	2.557348E-05	*Second – order*	$\phi = \gamma$ =C

Table-4:- Error measurements at $t = 1.25$ with different $R = \frac{c}{s}$, Δt *and* $\Delta = \Delta x = \Delta y$

The Method	**Average |error|**	**Maximum |error|**	Δt	Δ	**R**
Upwind 9pt.	9.787177E-05	4.420168E-03	0.00625	0.025	2
	2.828835E-04	8.997762E-03	0.0125	0.05	4
	7.431901E-04	1.225278E-02	0.025	0.1	8
	1.661878E-03	2.684112E-02	0.0375	0.15	12
FTCS 9pt.	5.826575E-05	1.418220E-03	0.00625	0.025	2
	3.283737E-04	1.442082E-02	0.0125	0.05	4
	3.586615E-03	1.669250E-01	0.025	0.1	8
	3.586615E-03	1.669250E-01	0.0375	0.15	12
WFDS 9pt.	2.794456E-05	1.607237E-04	0.00625	0.025	2
	1.128239E-04	5.783730E-04	0.0125	0.05	4
	5.630792E-04	3.575699E-03	0.025	0.1	8
	1.378246E-03	1.946828E-02	0.0375	0.15	12
MWFDS 9pt.	9.054745E-06	2.557348E-05	0.00625	0.025	2
	3.419029E-05	1.195652E-04	0.0125	0.05	4
	2.819292E-04	1.335344E-03	0.025	0.1	8
	1.392667E-03	1.966503E-02	0.0375	0.15	12

6. Comparisons with the results of other researchers

Confidence in the present results is gained by the comparison of the results obtained using the present numerical methods with those previously published in the literature. These comparisons are given in the study of error measurements (average absolute error and maximum absolute error), type of formula and number of weights given in tables (5) and (6). Table (5) shows a comparison of the explicit finite difference schemes of three , five and nine points formula kind at time $t = 1.25$, when $\Delta t = 0.0125$, $u = v = 0.8$, $C_x = C_y = 0.4$ and $\alpha_x = \alpha_y = 0.01$. This table reveals that the results of MWFD 9pt. and WFDS 9pt. are better in accuracy than the other researcher's schemes .

Table (6) illustrates a comparison of explicit FDS and implicit for a formula including three, five and nine points when $\Delta t = 0.00625$, $C_x = C_y = 0.2$, noticing that the MWFD 9pt. is better than the others in accuracy. The new suggested schemes in this study are better in the measurement errors than that of the traditional and suggested schemes by other researchers. Based on these results, our conclusions are made in the next section.

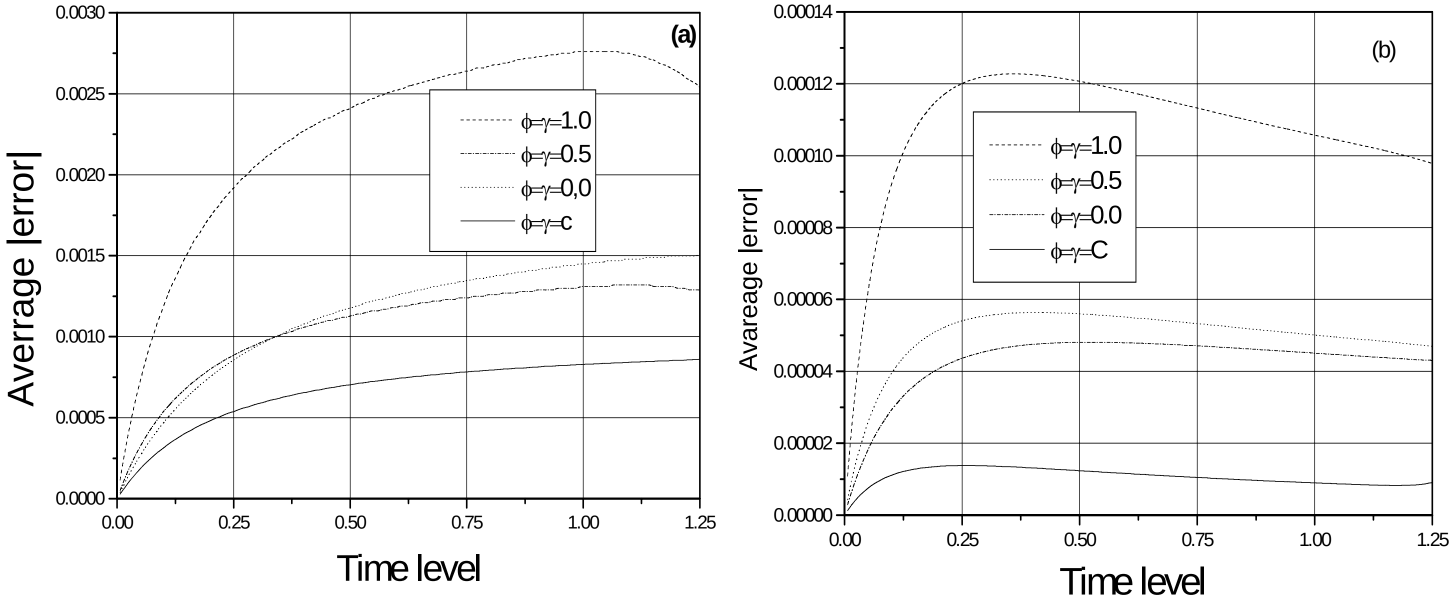


Figure (7)Comparison between average absolute error for (a) WFD 9pt. (b) MWFD 9pt.

Table-5:- Errors at $t = 1.25$ with $\Delta t = 0.0125$, $u = v = 0.8$, and $C_x = C_y = 0.4$

The Method	Average |error|	Maximum |error|	No. of weights
FTCS 3pt.	3.94E-03	1.12E-01	-----
FTCS 5pt. **[1]**	2.22E-03	8.00E-02	-----
WDF 5pt. **[1]**	3.88E-04	3.64E-03	4
Noye **[6]**	3.33E-04	6.03E-03	2
WFDS 9pt.	3.17E-04	5.47E-03	2
MWFDS 9pt.	7.31E-05	3.53E-04	2

Table-6:- Errors at $t = 1.25$ with $\Delta t = 0.00625$ $u = v = 0.8$, and $C_x = C_y = 0.2$

The Method *	Average |error|	Maximum |error|	No. of weights
Upwind 3pt.	2.65E-03	6.63E-02	-----
Upwind 5pt.**[1]**	2.62E-03	6.69E-02	-----
ADI **[11]**	9.22E-06	5.93E-05	-----
EC-ADI **[12]**	9.66E-06	6.19E-05	-----
Dehghan&Mohebbi**[8]**	9.48E-06	2.47E-04	-----
Noye **[6]**	1.43E-05	4.84E-04	8
WFDS 9pt.	2.79E-05	1.61E-04	2
MWFDS 9pt.	9.05E-06	2.56E-05	2

* The methods in [6,11,12,8] are implicit type**.**

7. Conclusions

From the numerical results, we can conclude the following:

The WFDS have been used successfully with ϕ *and* γ to introduce explicit new schemes with high accuracy and the results were better at error measurements. It is preferable to choose ideal values for the weights to give us numerical results with high accuracy to solve the two-dimensional transition equation compared with the other schemes explained in Figures(7) and Tables (5)and (6).The numerical accuracy depends on the number of the selected points. The tables and Figures show that increasing the number of the grid points gives more accurate results. We notice that's the MPDE method has been successfully used with weighted to develop several new explicit finite difference method for solving transport equation, and the use of the modified equivalent equation permits a proper determination of the accuracy order of the finite difference method. We have developed an improved finite difference schemes with weighted by the MEPDE, since it is allowed to apply two equations of finite equation which have the same order to give high order and more accurate schemes. Numerical treatment for the adjacent boundary is useful and successful to handle difficulties which appear through using new methods for solving two-dimensional transport equation. Moreover, it is applicable to obtain excellent results in the accuracy and the stability. The new explicit FDS based on nine-point formula with weights (ϕ,γ) have produced better results compared to other results of other researchers. Further study is apply WFDS to solve Navier-stokes equations.

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