The Numerical Solutions of 2D Time-Space Fractional Bioheat Problem by Using Fractional Quadratic Spline Method

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Abstract

In this article, a time-space fractional two-dimensional bioheat transfer model of temperature distribution in tissue has been solved Caputo fractional derivative for time-fractional derivative with α order and fractional quadratic spline for space of fractional derivative with β order. The goal of this article is to make a comparison between exact and numerical solutions. Theoretically, the stability analysis uses the Von Neumann method.

Keyword: 2D Time-Space fractional bioheat equations, Caputo fractional derivative, Fractional Quadratic Spline method, Stability, Truncation Error.

1. Introduction

The largest organ of the human body is skin, which has significant in thermoregulation. Skin works as a generator, transmitter, absorber, and conductor of heat. Heat transportation in living tissue is a complex operation and includes convection, conduction and blood perfusion, and refreshing of the human body. The heat generation results from a chain of chemical reactions occurring in the living cells, and the occurrence of blood perfusion due to energy exchange between the living tissue and blood flow through small vessels in the living tissue, and best describes the thermal distribution is Pennes bioheat equation [1] as follows:

$$\rho_t c_t T_t = \mu T_{xx} + W_b c_b (T_a - T) + Q + q_m, \qquad x \in (0, n), \ t > 0 \qquad (1)$$

The mathematical resolve of the complex thermal interaction between the vasculature and tissue has been a topic of interest for numerous physiologists, physicians, and engineers [2]. Temperature distribution in skin tissue is important for medical applications like skin cancer, skin burns, etc. [3]. The accurate solution of Pennes' equation does not exist specifically, so approximation and numerical techniques should be used to solve this equation. Karaa et al. [4] developed numerical methods for computer simulation and modeling of a 3-dimensional heat transfer problem in biological bodies. A finite difference discretization scheme is used to discretize the governing partial differential equation. Singh et al. [5] presented the solution of fractional bioheat equation using the shifted Grünwald finite difference approximation for Riemann-Liouville space fractional derivative FDM and HPM of the fractional derivative of space and the Caputo fractional for the fractional time. It has been spotted that the time possessed to achieve hyperthermia in a location is reduced as the order fractional derivative is decreased. Kunter and Seker [6] constructed a 3D bioheat transfer model of the human eye by using weighted extended B-splines as shape functions for the finite element method. The

simulation results which are verified using the values reported in the literature are pointed out to better efficiency in terms of the accuracy level. Dehghan M. Sabouri M. [7] discussed the two cases: 1D and 2D Pennes' bioheat model, in the two-dimensional case both triangular and quadrilateral elements are investigated. Through test problems, the discretization error generated from this method is reported. Damor et.al [8] discussed the approximate solution of fractional Pennes' bioheat equation with constant and Sinusoidal heat flux conditions on the skin using implicit finite difference method and the fractional time derivative is of Caputo form. Uzezi and Shariffudin [9] solved 2D Bioheat equation on a distributed system through employing Message Passing Interface (MPI) and Parallel Virtue Machine (PVM) is presented using applying domain decomposition strategy. It is noted that the efficiency is strongly dependent on the mesh size block numbers and the number of processors for both MPI and PVM. Different strategies to get better computational efficiency are proposed. Cui, et.al [3] showed a numerical solution for the time-fractional Pennes' bioheat transfer equation on skin tissue and solved using Fourier Sine transform for second-order derivative and the Caputo for the fractional time. Ezzat, et.al [10] discussed the 2D fractional bioheat equation using Laplace transforms of second-order derivative, and investigation of the numerical solutions were performed to search the temperature transfer in the skin uncovered to immediate surface heating. Some comparisons were shown to estimate the impact the fractional-order parameter α has on the temperature wave. Mishra and Rai [11] presented a numerical solution of the fractional bioheat equation when they used a finite difference of second-order derivative and the fractional derivative using Grünwald Letnikov for the fractional time, also discussed and analyzed the stability and convergence. Luis Ferràs et.al [12] studied the fractional bioheat transfer equation and the approximate solution using a finite difference of second-order derivative and the fractional time derivative using Caputo derivative and discussed the stability and convergence depending on this scheme. Pandey [13] discussed the 2D fractional bioheat equation using Galerkin FEM and he found the solution method in the cylindrical living tissue and noted that the effects of thermal conductivities have significant and more remarkable effects on temperature variation in living tissue. Damor et.al [14] studied the fractional bioheat equation when the time-space fractional derivative is in the form and solved it using Caputo fractional derivative of order $\alpha \in (0, 1]$ and Riesz–Feller fractional derivative of order $\beta \in (1, 2]$ respectively, the result in terms of Fox's H-function with some specific cases is obtained, by using Fourier–Laplace transforms. Roohi R. et.al [15] solved space-time fractional bioheat equation using fractional-order Legendre functions of fractional space order derivative and the fractional time derivative using Caputo derivative, and he noted that the quantity of the temperature at the skin surface is a strong function of the space-fractional order and conversely the impact of the time-fractional order is almost negligible.

In this paper, we will solve a time-space fractional two-dimensional bioheat transfer model of temperature distribution by Caputo fractional derivative for time-fractional derivative with α order and fractional quadratic spline for space-fractional derivative with β order.

2. 2D Pennes' bioheat transfer equation with time-space fractional derivative

The 2D time-space fractional Pennes' bioheat transfer equation for modeling skin tissue heat transfer is expressed as [3], [9-10], [15-16]

$$\rho_t c_t \frac{\partial^\alpha T(x,y,t)}{\partial t^\alpha} = \mu \left(\frac{\partial^\beta T(x,y,t)}{\partial x^\beta} + \frac{\partial^\beta T(x,y,t)}{\partial y^\beta} \right) + W_b c_b (T_a - T) + Q + q_m , \qquad (2)$$

$x \in (0,n), y \in (0,s), \ t > 0 ,$

with initial and boundary conditions,

$$T(x, y, 0) = f_1(x, y), \qquad x \in (0, n), y \in (0, s), \qquad (3)$$

$$\left. \frac{\partial}{\partial x} T(i, y, t) \right|_{i=0,n} = g_1(y, t), \quad y \in (0, s), \ t > 0 \qquad (4)$$

$$\left. \frac{\partial}{\partial y} T(x, j, t) \right|_{j=0,s} = g_2(x, t), \quad x \in (0, n), \ t > 0 \qquad (5)$$

where,

$\alpha \in (0,1)$ is fractional order of time,

$\beta \in (1,2)$ is fractional order of space,

x and y are the distances from the skin surface,

$\rho_t = 1000$ is constant representing the density tissue (kg/m^3),

$c_t = 4000$ is constant representing the specific heat of tissue (Jk/kg),

$\mu = 0.5$ is the tissue thermal conductivity (J/sm),

$W_b = 0.0005$ is the mass flow rate of blood per unit volume of tissue (kg/sm^3),

$c_b = 4000$ is the specific heat of blood (J/kg),

$Q_m = 420$ is the metabolic temperature generation per unit volume (J/m^3),

$T_a = 37$ represent the temperature of arterial blood,

T is the temperature of tissue,

Q is the metabolic heat source,

and $W_b c_b (T_a - T)$ represents the blood perfusion. It is value mentioning that Pennes' obtained constant W_b experimentally for a human forearm.

Definition 2.1: The Riemann-Liouville fractional derivative of order $\alpha \in (n-1, n), n \in \mathbb{N}, t > a$ defined by equation [3], [5], [8], [10-12], [14-17]

$$^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(n-1)} \frac{d^n}{d\psi^n} \int_a^t (t - \psi)^{n-\alpha-1} f(\psi) d\psi.$$

Definition 2.2: The Caputo fractional derivative of order $\alpha \in (n-1, n), n \in \mathbb{N}, t > a$ defined by equation [3], [5], [8], [10-12], [14-17]

$$^{C}D_a^\alpha f(t) = \frac{1}{\Gamma(n-1)} \int_a^t (t - \psi)^{n-\alpha-1} \frac{d^n}{d\psi^n} f(\psi) d\psi.$$

Definition 2.3: The Gauss hypergeometric function $_2F_1(\mu, \delta, \gamma)$ is defined as [18]

$$_2F_1(\mu, \delta, \gamma; x) = \sum_{n=0}^\infty \frac{(\mu)_n (\delta)_n}{(\gamma)_n n!} x^n,$$

where μ and δ are real or complex parameters with $\gamma \notin \mathbb{Z}_{\leq 0}$. Also

$$(\mu)_n \equiv \frac{\Gamma(\mu + n)}{\Gamma(\mu)} = \mu(\mu + 1)(\mu + 2) \dots (\mu + n - 1),$$

the generalized hypergeometric function $_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; x)$ is defined by

$$_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; x) = \, _pF_q\begin{pmatrix} a_1, a_2, \dots, a_p; \\ b_1, b_2, \dots, b_q; \end{pmatrix} = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} \frac{z^k}{k!}.$$

Definition 2.4: The Lommel function of the first kind $s_{\mu,\nu}$ can be expressed as follows [19]

$$s_{\mu,\nu} = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} \, _1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right),$$

where $\mu, \nu \in \mathbb{C}$ and $Re(\mu \mp \nu + 1) > 0$ and $z \in \mathbb{C}/(-\infty, 0]$.

2.2 Fractional quadratic spline form for the space fractional derivative

Let us write the quadratic spline $Q_{1m}(x, t_k)$ and $Q_{2l}(y, t_k)$ in the forms:

$$Q_{1m}(x, t_k) = a_{1l}(t_k) + b_{1l}(t_k)(x - x_l) + c_{1l}(t_k)(x - x_l)^2, \quad l = 1, \dots, n, \tag{6}$$

$$Q_{2l}(y, t_k) = a_{2m}(t_k) + b_{2m}(t_k)(y - y_m) + c_{2m}(t_k)(y - y_m)^2, \quad m = 1, \dots, s, \tag{7}$$

where $a_{1l}(t_k), b_{1l}(t_k), c_{1l}(t_k), a_{2m}(t_k), b_{2m}(t_k)$ and $c_{2m}(t_k)$ are unknown coefficients, to derive expression for the coefficients of equations (6)-(7) in terms of $S_{l,m}, S_{l+1,m}, S_{l,m+1}, F_{1l,m}, F_{2l,m}, F_{1l+1,m}$ and $F_{2l,m+1}$, at first we define [20]

$$Q_{1m}(x_l, t_k) = S_{l,m}, \quad Q_{1m}(x_{l+1}, t_k) = S_{l+1,m},$$

$$Q_{1m}^{\beta}(x_{l+1}, t_k) = F_{1l+1,m}, \quad Q_{1m}^{\beta}(x_l, t_k) = F_{1l,m}, \tag{8}$$

$$Q_{2l}(y_m, t_k) = S_{l,m}, \quad Q_{2l}(y_{m+1}, t_k) = S_{l,m+1},$$

$$Q_{2l}^{\beta}(y_{m+1}, t_k) = F_{2l,m+1}, \quad Q_{2m}^{\beta}(y_l, t_k) = F_{2l,m} \tag{9}$$

From equations (6)-(9) and by Caputo fractional derivative we get

$$a_{1l}(t_k) = S_{l,m} \quad , \quad b_{1l}(t_k) = \frac{1}{h}\left(S_{l+1,m} - S_{l,m}\right) - \frac{h^{\beta-1}\Gamma(3-\beta)}{2}F_{1l+1,m}$$

$$c_{1l}(t_k) = \frac{\Gamma(3-\beta)}{2h^{2-\beta}}F_{1l+1,m}, \tag{10}$$

$$a_{2m}(t_k) = S_{l,m} \quad , \quad b_{2m}(t_k) = \frac{1}{h}\left(S_{l,m+1} - S_{l,m}\right) - \frac{h^{\beta-1}\Gamma(3-\beta)}{2}F_{2l,m+1}$$

$$c_{2m}(t_k) = \frac{\Gamma(3-\beta)}{2h^{2-\beta}}F_{2l,m+1}, \tag{11}$$

Therefore, by equations (10)-(11) and the continuity conditions $Q'_{m-1}(x_l, t_k) = Q'_m(x_l, t_k)$ and $Q'_{l-1}(y_{m-1}, t_k) = Q'_l(y_m, t_k)$, which gives the following relation respectively,

$$F_{1l+1,m} + F_{1l,m} = \delta\left(T_{l+1,m}^r - 2T_{l,m}^r + T_{l-1,m}^r\right), \qquad \text{where } \delta = \frac{2}{h^\beta \Gamma(3-\beta)} \tag{12}$$

$$F_{2l,m+1} + F_{2l,m} = \gamma\left(T_{l,m+1}^r - 2T_{l,m}^r + T_{l,m-1}^r\right), \qquad \text{where } \gamma = \frac{2}{k^\beta \Gamma(3-\beta)} \tag{13}$$

Then, we have

$$F_{1l+1,m+1} + F_{1l,m+1} = \delta\left(T_{l+1,m+1}^r - 2T_{l,m+1}^r + T_{l-1,m+1}^r\right), \tag{14}$$

$$F_{2l+1,m+1} + F_{2l+1,m} = \gamma\left(T_{l+1,m+1}^r - 2T_{l+1,m}^r + T_{l+1,m-1}^r\right), \tag{15}$$

2.3 Caputo fractional derivative for the time fractional derivative

To discretize the Caputo time-fractional derivative, we use forward Euler scheme. Let $t_r = r\Delta t, r = 0, 1, \ldots, K$, in which $\tau = \frac{G}{K}$ is the time step size, then, the Caputo time-fractional derivative at time point $t = t_{j+1}$, can be approximated, as

$$\frac{\partial^\alpha T(x,y,t_{r+1})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{r+1}} \frac{\partial T(x,y,s)}{\partial s}(t_{r+1}-s)^{-\alpha}ds + E_T^{r+1}$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^r \int_{t_j}^{t_{j+1}} \frac{\partial T(x,y,s)}{\partial s}(t_{r+1}-s)^{-\alpha}ds + E_T^{r+1}$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^r \frac{T(x,y,t_{j+1}) - T(x,y,t_j)}{\tau} \int_{t_j}^{t_{j+1}} (t_{r+1}-s)^{-\alpha}ds + E_T^{r+1}$$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{1}{\tau^\alpha \Gamma(2-\alpha)}\left[T^{r+1} + T^r(u_1 - 1) + \sum_{j=1}^{r-1} T^{r-j}\left(u_{j+1} - u_j\right) - u_r T^0\right] + E_T^{r+1}, \tag{16}$$

where $u_j = (j+1)^{1-\alpha} - (j)^{1-\alpha}$, for all $j = 0, \ldots, r$

Now, we define the semi discrete fractional differential operator $Y_t^\alpha(x, y, t_{r+1})$ as,

$$Y_t^\alpha(x, y, t_{r+1}) = \frac{1}{\tau^\alpha \Gamma(2-\alpha)}\left[T^{r+1} + T^r(u_1 - 1) + \sum_{j=1}^{r-1} T^{r-j}\left(u_{j+1} - u_j\right) - u_r T^0\right], \tag{17}$$

can be written equation (16) as,

$$\frac{\partial^\alpha T(x,y,t_{r+1})}{\partial t^\alpha} = Y_t^\alpha(x, y, t_{r+1}) + E_T^{r+1} \tag{18}$$

where E_T^{r+1} is truncation error between $\frac{\partial^\alpha T(x,y,t_{r+1})}{\partial t^\alpha}$ and $Y_t^\alpha(x, y, t_{r+1})$, also is bounded

$$|E_T^{r+1}| \le c\tau^2, \tag{19}$$

where the constant c depending on T.

Theorem [17]: Let T be the exact solution of equation (2) and $\{T^k\}_{k=0}^K$ be the time discrete solution of equation (17) with initial condition $T^0 = f_1(x,y), x \in (0,n), y \in (0,s)$, then, it holds

$$\|T(t_k) - T^k\|_2 \leq cT^\alpha \tau^{2-\alpha}, \quad k = 1, 2, \ldots, K \tag{20}$$

3. Analysis of the method

3.1 Derivation of 2D Pennes' bioheat transfer equation

The equation (2) of the 2D time-space fractional Penne's bioheat transfer, can be write as

$$\frac{\partial^\beta T(x,y,t)}{\partial x^2} + \frac{\partial^\beta T(x,y,t)}{\partial y^2} = \frac{\rho_t c_t}{\mu} \frac{\partial^\alpha T(x,y,t)}{\partial t^\alpha} - \frac{W_b c_b}{\mu}(T_a - T) - \frac{Q}{\mu} - \frac{q_m}{\mu}$$
(21)

Using equations (8)-(9) and (17) in the equation (21), gives

$$F_{1l,m} + F_{2l,m} = \frac{\rho_t c_t}{\mu} \frac{1}{\Delta\tau^\alpha \Gamma(2-\alpha)} \sum_{j=0}^r u_j \left(T_{l,m}^{r-j+1} - T_{l,m}^{r-j}\right) - \frac{W_b c_b}{\mu}\left(T_a - T_{l,m}^r\right) - \frac{Q_{l,m}^r}{\mu} - \frac{q_m}{\mu}$$
(22)

Now, equation (22) can be written, as

$$F_{1l,m} + F_{2l,m} = AT_{l,m}^{r+1} + BT_{l,m}^r + A\sum_{j=1}^{r-1} T_{l,m}^{r-j} w_j - Au_r T_{l,m}^0 + CQ_{l,m}^r + D \tag{23}$$

where $A = \frac{\rho_t c_t}{\mu\tau^\alpha \Gamma(2-\alpha)}$, $B = A(u_1 - 1) + \frac{W_b c_b}{\mu}$, $C = -\frac{1}{\mu}$,

$$D = C(W_b c_b T_a + q_m), \qquad w_j = u_{j+1} - u_j$$

from (23) we conclude that

$$F_{1l+1,m} + F_{2l+1,m} = AT_{l+1,m}^{r+1} + BT_{l+1,m}^r + A\sum_{j=1}^{r-1} T_{l+1,m}^{r-j} w_j - Au_r T_{l+1,m}^0 + CQ_{l+1,m}^r + D,$$

(24)

$$F_{1l,m+1} + F_{2l,m+1} = AT_{l,m+1}^{r+1} + BT_{l,m+1}^r + A\sum_{j=1}^{r-1} T_{l,m+1}^{r-j} w_j - Au_r T_{l,m+1}^0 + CQ_{l,m+1}^r + D,$$

(25)

and

$$F_{1l+1,m+1} + F_{2l+1,m+1} = AT_{l+1,m+1}^{r+1} + BT_{l+1,m+1}^r + A\sum_{j=1}^{r-1} T_{l+1,m+1}^{r-j} w_j - Au_r T_{l+1,m+1}^0 + CQ_{l,m+1}^r + D,$$

(26)

now, by adding the equations (23)-(26), we obtain

$$F_{1l,m} + F_{1l+1,m} + F_{2l,m} + F_{2l,m+1} + F_{1l,m+1} + F_{1l+1,m+1} + F_{2l+1,m} + F_{2l+1,m+1} =$$
$$A\left(T_{l,m}^{r+1} + T_{l+1,m}^{r+1} + T_{l,m+1}^{r+1} + T_{l+1,m+1}^{r+1}\right) + B\left(T_{l,m}^r + T_{l+1,m}^r + T_{l,m+1}^r + T_{l+1,m+1}^r\right) +$$
$$A\left(\sum_{j=1}^r \left(T_{l,m}^{r-j+1} - T_{l,m}^{r-j} + T_{l+1,m}^{r-j+1} - T_{l+1,m}^{r-j} + T_{l,m+1}^{r-j+1} - T_{l,m+1}^{r-j} + T_{l+1,m+1}^{r-j+1} - T_{l+1,m+1}^{r-j}\right)\right)w_j -$$

$$A\big(T_{l,m}^{0} + T_{l+1,m}^{0} + T_{l,m+1}^{0} + T_{l+1,m+1}^{0}\big) + C\big(Q_{l+1,m+1}^{r} + Q_{l,m+1}^{r} + Q_{l+1,m}^{r} + Q_{l,m}^{r}\big) + 4D$$
(27)

by substituting equations (12)-(15) in equation (27), we obtain,

$$\delta\big(T_{l+1,m}^{r} - 2T_{l,m}^{r} + T_{l-1,m}^{r} + T_{l+1,m+1}^{r} - 2T_{l,m+1}^{r} + T_{l-1,m+1}^{r}\big) + \gamma\big(T_{l,m+1}^{r} - 2T_{l,m}^{r} + T_{l,m-1}^{r} +$$
$$T_{l+1,m+1}^{r} - 2T_{l+1,m}^{r} + T_{l+1,m-1}^{r}\big) = A\big(T_{l,m}^{r+1} + T_{l+1,m}^{r+1} + T_{l,m+1}^{r+1} + T_{l+1,m+1}^{r+1}\big) + B\big(T_{l,m}^{r} +$$
$$T_{l+1,m}^{r} + T_{l,m+1}^{r} + T_{l+1,m+1}^{r}\big) + A\big(\sum_{j=1}^{r}\big(T_{l,m}^{r-j+1} - T_{l,m}^{r-j} + T_{l+1,m}^{r-j+1} - T_{l+1,m}^{r-j} + T_{l,m+1}^{r-j+1} -$$
$$T_{l,m+1}^{r-j} + T_{l+1,m+1}^{r-j+1} - T_{l+1,m+1}^{r-j}\big)\big)w_j - A\big(T_{l,m}^{0} + T_{l+1,m}^{0} + T_{l,m+1}^{0} + T_{l+1,m+1}^{0}\big) + C\big(Q_{l+1,m+1}^{r} +$$
$$Q_{l,m+1}^{r} + Q_{l+1,m}^{r} + Q_{l,m}^{r}\big) + 4D,$$
(28)

equation (28) contain of $(n-1) \times (s-1)$ linear algebraic equations in the $(n+1) \times (s+1)$ unknowns $T_{l,m}^{r}, l = 1, \dots, n-1, \ m = 1, \dots, s-1$, so we need $(2n + 2s)$ boundary equation when $l, m = 0$ and $l = n, m = s$ can be use Taylor series, these equations are

$$\delta\big(T_{1,m}^{r} - T_{0,m}^{r} + hT_{0(1),m}^{r} + T_{1,m+1}^{r} - T_{0,m+1}^{r} + hT_{0(1),m+1}^{r}\big) + \gamma\big(T_{0,m+1}^{r} - 2T_{0,m}^{r} + T_{0,m-1}^{r} +$$
$$T_{1,m+1}^{r} - 2T_{1,m}^{r} + T_{1,m-1}^{r}\big) = A\big(T_{0,m}^{r+1} + T_{1,m}^{r+1} + T_{0,m+1}^{r+1} + T_{1,m+1}^{r+1}\big) + B\big(T_{0,m}^{r} + T_{1,m}^{r} +$$
$$T_{0,m+1}^{r} + T_{1,m+1}^{r}\big) + A\big(\sum_{j=0}^{r}\big(T_{1,m}^{r-j+1} - T_{1,m}^{r-j} + T_{1,m}^{r-j+1} - T_{1,m}^{r-j} + T_{0,m+1}^{r-j+1} - T_{0,m+1}^{r-j} +$$
$$T_{1,m+1}^{r-j+1} - T_{1,m+1}^{r-j}\big)\big)w_j - A\big(T_{0,m}^{0} + T_{1,m}^{0} + T_{0,m+1}^{0} + T_{1,m+1}^{0}\big) + C\big(Q_{1,m+1}^{r} + Q_{0,m+1}^{r} + Q_{1,m}^{r} +$$
$$Q_{0,m}^{r}\big) + 4D \qquad (29)$$

$$\delta\big(hT_{n(1),m}^{r} - T_{n,m}^{r} + T_{n-1,m}^{r} + hT_{n(1),m+1}^{r} - T_{n,m+1}^{r} + T_{n-1,m+1}^{r}\big) + \gamma\big(2T_{n,m+1}^{r} - 4T_{n,m}^{r} +$$
$$2T_{n,m-1}^{r} + hT_{n(1),m+1}^{r} - 2hT_{n(1),m}^{r} + hT_{n(1),m-1}^{r}\big) = A\big(2T_{n,m}^{r+1} + hT_{n(1),m}^{r+1} + 2T_{n,m+1}^{r+1} +$$
$$hT_{n(1),m+1}^{r+1}\big) + B\big(2T_{n,m}^{r} + hT_{n(1),m}^{r} + 2T_{n,m+1}^{r} + hT_{n(1),m+1}^{r}\big) + A\left(\sum_{j=0}^{r}\big(2T_{n,m}^{r-j+1} - 2T_{n,m}^{r-j} +\right.$$
$$hT_{n(1),m}^{r-j+1} - hT_{n(1),m}^{r-j} + 2T_{n,m+1}^{r-j+1} - 2T_{n,m+1}^{r-j} + hT_{n(1),m+1}^{r-j+1} - hT_{n(1),m+1}^{r-j}\big)\bigg)w_j - A\big(2T_{n,m}^{0} +$$
$$hT_{n(1),m}^{0} + 2T_{n,m+1}^{0} + hT_{n(1),m+1}^{0}\big) + C\big(hQ_{n(1),m+1}^{r} + 2Q_{n,m+1}^{r} + hQ_{n(1),m}^{r} + 2Q_{n,m}^{r}\big) + 4D$$

(30)

$$\delta\big(T_{l+1,0}^{r} - T_{l,0}^{r} + kT_{l,0(1)}^{r} + T_{l+1,1}^{r} - T_{l,1}^{r} + kT_{l,1(1)}^{r}\big) + \gamma\big(T_{l,1}^{r} - T_{l,0}^{r} - kT_{l,0(1)}^{r} + T_{l+1,1}^{r} -$$
$$T_{l+1,0}^{r} + kT_{l+1,0(1)}^{r}\big) = A\big(T_{l,0}^{r+1} + T_{l+1,0}^{r+1} + T_{l,1}^{r+1} + T_{l+1,1}^{r+1}\big) + B\big(T_{l,0}^{r} + T_{l+1,0}^{r} + T_{l,1}^{r} + T_{l+1,1}^{r}\big) +$$
$$A\big(\sum_{j=0}^{r}\big(T_{l,0}^{r-j+1} - T_{l,0}^{r-j} + T_{l+1,0}^{r-j+1} - T_{l+1,0}^{r-j} + T_{l,1}^{r-j+1} - T_{l,1}^{r-j} + T_{l+1,1}^{r-j+1} - T_{l+1,1}^{r-j}\big)\big)w_j -$$
$$\big(T_{l,0}^{0} + T_{l+1,0}^{0} + T_{l,1}^{0} + T_{l+1,1}^{0}\big) + C\big(Q_{l+1,1}^{r} + Q_{l,1}^{r} + Q_{l+1,0}^{r} + Q_{l,0}^{r}\big) + 4D \qquad (31)$$

$$\delta\big(2T_{l+1,s}^{r} - 4T_{l,s}^{r} + 2T_{l-1,s}^{r} + kT_{l+1,s(1)}^{r} - 2kT_{l,s(1)}^{r} + kT_{l-1,s(1)}^{r}\big) + \gamma\big(kT_{l,s(1)}^{r} - T_{l,s}^{r} +$$
$$T_{l,s-1}^{r} + kT_{l+1,s(1)}^{r} - T_{l+1,s}^{r} + T_{l+1,s-1}^{r}\big) = A\big(2T_{l,s}^{r+1} + 2T_{l+1,s}^{r+1} + kT_{l,s(1)}^{r+1} + kT_{l+1,s(1)}^{r+1}\big) +$$
$$B\big(2T_{l,s}^{r} + 2T_{l+1,s}^{r} + kT_{l,s(1)}^{r} + kT_{l+1,s(1)}^{r}\big) + A\left(\sum_{j=0}^{r}\big(2T_{l,s}^{r-j+1} - 2T_{l,s}^{r-j} + 2T_{l+1,s}^{r-j+1} -\right.$$
$$2T_{l+1,s}^{r-j} + kT_{l,s(1)}^{r-j+1} - kT_{l,s(1)}^{r-j} + kT_{l+1,s(1)}^{r-j+1} - kT_{l+1,s(1)}^{r-j}\big)\bigg)w_j - A\big(2T_{l,s}^{0} + 2T_{l+1,s}^{0} + kT_{l,s(1)}^{0} +$$
$$kT_{l+1,s(1)}^{0}\big) + C\big(kQ_{l+1,s(1)}^{r} + kQ_{l,s(1)}^{r} + 2Q_{l+1,s}^{r} + 2Q_{l,s}^{r}\big) + 4D \qquad (32)$$

$$\delta\big(2T_{1,s}^{r} - 2T_{0,s}^{r} - hT_{0(1),s}^{r} + kT_{1,s(1)}^{r} - kT_{0,s(1)}^{r} - hkT_{0(1),s(1)}^{r}\big) + \gamma\big(kT_{0,s(1)}^{r} - T_{0,s}^{r} + T_{0,s-1}^{r} +$$
$$kT_{1,s(1)}^{r} - T_{1,s}^{r} + T_{1,s-1}^{r}\big) = A\big(2T_{0,s}^{r+1} + 2T_{1,s}^{r+1} + kT_{0,s(1)}^{r+1} + kT_{1,s(1)}^{r+1}\big) + B\big(2T_{0,s}^{r} + 2T_{1,s}^{r} +$$
$$kT_{0,s(1)}^{r} + kT_{1,s(1)}^{r}\big) + A\left(\sum_{j=0}^{r}\big(2T_{0,s}^{r-j+1} - 2T_{0,s}^{r-j} + 2T_{1,s}^{r-j+1} - 2T_{1,s}^{r-j} + kT_{0,s(1)}^{r-j+1} -\right.$$

$kT_{0,s(1)}^{r-j} + kT_{1,s(1)}^{r-j+1} - kT_{1,s(1)}^{r-j})) w_j - A(2T_{0,s}^0 + 2T_{1,s}^0 + kT_{0,s(1)}^0 + kT_{1,s(1)}^0) + C(2Q_{0,s}^r + 2Q_{1,s}^r + kQ_{0,s(1)}^r + kQ_{1,s(1)}^r) + 4D,$

(33)

$\delta(hT_{n(1),0}^r - T_{n,0}^r + T_{n-1,0}^r + hT_{n(1),1}^r - T_{n,1}^r + T_{n-1,1}^r) + \gamma(2T_{n,1}^r + 2T_{n,0}^r - 2kT_{n,0(1)}^r + hT_{n(1),1}^r - 4T_{n,0}^r - 2hT_{n(1),0}^r + hT_{n(1),0}^r - khT_{n(1),0(1)}^r) = A(2T_{n,0}^{r+1} + hT_{n(1),0}^{r+1} + 2T_{n,1}^{r+1} + hT_{n(1),1}^{r+1}) + B(2T_{n,0}^r + hT_{n(1),0}^r + 2T_{n,1}^r + hT_{n(1),1}^r) + A\left(\sum_{j=0}^r (2T_{n,0}^{r-j+1} - 2T_{n,0}^{r-j} + hT_{n(1),0}^{r-j+1} - hT_{n(1),0}^{r-j} + 2T_{n,1}^{r-j+1} - 2T_{n,1}^{r-j} + hT_{n(1),1}^{r-j+1} - hT_{n(1),1}^{r-j})\right) w_j - A(2T_{n,0}^0 + hT_{n(1),0}^0 + 2T_{n,1}^0 + hT_{n(1),1}^0) + C(2Q_{n,0}^r + hQ_{n(1),0}^r + 2Q_{n,1}^r + hQ_{n(1),1}^r) + 4D,$

(34)

$\delta(T_{1,0}^r - T_{0,0}^r + hT_{0(1),0}^r + T_{1,1}^r - T_{0,1}^r + hT_{0(1),1}^r) + \gamma(T_{0,1}^r - T_{0,0}^r + kT_{0,0(1)}^r + T_{1,1}^r - T_{1,0}^r - kT_{1,0}^r) =$
$A(T_{0,0}^{r+1} + T_{1,0}^{r+1} + T_{0,1}^{r+1} + T_{1,1}^{r+1}) + B(T_{0,0}^r + T_{1,0}^r + T_{0,1}^r + T_{1,1}^r) + A(\sum_{j=0}^r (T_{1,0}^{r-j+1} - T_{1,0}^{r-j} + T_{1,0}^{r-j+1} - T_{1,0}^{r-j} + T_{0,1}^{r-j+1} - T_{0,1}^{r-j} + T_{1,1}^{r-j+1} - T_{1,1}^{r-j}))w_j - A(T_{0,0}^0 + T_{1,0}^0 + T_{0,1}^0 + T_{1,1}^0) + C(Q_{1,1}^r + Q_{0,1}^r + Q_{1,0}^r + Q_{0,0}^r) + 4D,$　　　　　(35)

$\delta(2hT_{n(1),s}^r - 2T_{n,s}^r + 2T_{n-1,s}^r + hkT_{n(1),s(1)}^r - kT_{n,s(1)}^r + kT_{n-1,s(1)}^r) + \gamma(2kT_{n,s(1)}^r - 6T_{n,s}^r + 2T_{n,s-1}^r + hkT_{n(1),s(1)}^r - hT_{n(1),s}^r + hT_{n(1),s-1}^r) = A(4T_{n,s}^{r+1} + 2hT_{n(1),s}^{r+1} + 2kT_{n,s(1)}^{r+1} + hkT_{n(1),s(1)}^{r+1}) + B(4T_{n,s}^r + 2hT_{n(1),s}^r + 2kT_{n,s(1)}^r + hkT_{n(1),s(1)}^r) + A\left(\sum_{j=0}^r (4T_{n,s}^{r-j+1} - 4T_{n,s}^{r-j} + 2hT_{n(1),s}^{r-j+1} - 2hT_{n(1),s}^{r-j} + 2kT_{n,s(1)}^{r-j+1} - 2kT_{n,s(1)}^{r-j} + hkT_{n(1),s(1)}^{r-j+1} - hkT_{n(1),s(1)}^{r-j})\right) w_j - A(4T_{n,s}^0 + 2hT_{n(1),s}^0 + 2kT_{n,s(1)}^0 + hkT_{n(1),s(1)}^0) + C(hkQ_{n(1),s(1)}^r + 2kQ_{n,s(1)}^r + 2hQ_{n(1),s}^r + 4Q_{n,s}^r) + 4D,$

(36)

where $T_{i(1),j} = \dfrac{\partial T}{\partial x}\Big|_{x=i,y=j}, i = 0,1, n-1, n$ and $j = 0,1, s-1, s$

$T_{i,j(1)}^r = \dfrac{\partial T}{\partial y}\Big|_{x=i,y=j}, i = 0,1, n-1, n$ and $j = 0,1, s-1, s$

and $T_{i(1),j(1)}^r = \dfrac{\partial^2 T}{\partial x \partial y}\Big|_{x=i,y=j}, i = 0,1, n-1, n$ and $j = 0,1, s-1, s.$

4. Stability analysis and Truncation Error

The stability of numerical schemes can be using the Von Neumann method. We consider

$$T_{l,m}^r = \xi_r e^{i\sigma l\theta} e^{i\alpha m\theta} \qquad (37)$$

where $i = \sqrt{-1}$, and ξ represents the time dependence of the solution and the exponential represents the spatial dependence. In the exponential $l\theta$ and $m\theta$ represents the position along the grid, σ and α are the spatial wave numbers, we can rewrite equation (28) as

$\delta(T_{l+1,m}^r - 2T_{l,m}^r + T_{l-1,m}^r + T_{l+1,m+1}^r - 2T_{l,m+1}^r + T_{l-1,m+1}^r) + \gamma(T_{l,m+1}^r - 2T_{l,m}^r + T_{l,m-1}^r + T_{l+1,m+1}^r - 2T_{l+1,m}^r + T_{l+1,m-1}^r) =$

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$B\left(T_{l,m}^r + T_{l+1,m}^r + T_{l,m+1}^r + T_{l+1,m+1}^r\right) + A\left(\sum_{j=0}^r \left(T_{l,m}^{r-j+1} - T_{l,m}^{r-j} + T_{l+1,m}^{r-j+1} - T_{l+1,m}^{r-j} + T_{l,m+1}^{r-j+1} - T_{l,m+1}^{r-j} + T_{l+1,m+1}^{r-j+1} - T_{l+1,m+1}^{r-j}\right)\right)w_j + R$

(38)

where $R = C\left(Q_{l+1,m+1}^r + Q_{l,m+1}^r + Q_{l+1,m}^r + Q_{l,m}^r\right) + 4D$, substituting (37) into (38), we obtain

$\delta\left(\xi_r e^{(l+1)i\sigma\theta}e^{mi\alpha\theta} - 2\xi_r e^{li\sigma\theta}e^{mi\alpha\theta} + \xi_r e^{(l-1)i\sigma\theta}e^{mi\alpha\theta} + \xi_r e^{(l+1)i\sigma\theta}e^{(m+1)i\alpha\theta} - 2\xi_r e^{li\sigma\theta}e^{(m+1)i\alpha\theta} + \xi_r e^{(l-1)i\sigma\theta}e^{(m+1)i\alpha\theta}\right) + \gamma\left(\xi_r e^{li\sigma\theta}e^{(m+1)i\alpha\theta} - 2\xi_r e^{li\sigma\theta}e^{mi\alpha\theta} + \xi_r e^{li\sigma\theta}e^{(m-1)i\alpha\theta} + \xi_r e^{(l+1)i\sigma\theta}e^{(m+1)i\alpha\theta} - 2\xi_r e^{(l+1)i\sigma\theta}e^{mi\alpha\theta} + \xi_r e^{(l+1)i\sigma\theta}e^{(m-1)i\alpha\theta}\right) = B\left(\xi_r e^{li\sigma\theta}e^{mi\alpha\theta} + \xi_r e^{(l+1)i\sigma\theta}e^{mi\alpha\theta} + \xi_r e^{li\sigma\theta}e^{(m+1)i\alpha\theta} + \xi_r e^{(l+1)i\sigma\theta}e^{(m+1)i\alpha\theta}\right) + A\left(\sum_{j=0}^r \left(\xi_{r-j+1}e^{li\sigma\theta}e^{mi\alpha\theta} - \xi_{r-j}e^{li\sigma\theta}e^{mi\alpha\theta} + \xi_{r-j+1}e^{(l+1)i\sigma\theta}e^{mi\alpha\theta} - \xi_{r-j}e^{(l+1)i\sigma\theta}e^{mi\alpha\theta} + \xi_{r-j+1}e^{li\sigma\theta}e^{(m+1)i\alpha\theta} - \xi_{r-j}e^{li\sigma\theta}e^{(m+1)i\alpha\theta} + \xi_{r-j+1}e^{(l+1)i\sigma\theta}e^{(m+1)i\alpha\theta} - \xi_{r-j}e^{(l+1)i\sigma\theta}e^{(m+1)i\alpha\theta}\right)\right)w_j,$ (39)

one can see that the component R is omitted because the constant value does not affect the stability of this scheme [21-22].

by divided equation (39) by $\xi_r e^{i\sigma l\theta}e^{i\alpha m\theta}$, we have

$\xi_r\delta\left(-4\sin^2\frac{\sigma\theta}{2}\left(1 + e^{i\alpha\theta}\right)\right) + \xi_r\gamma\left(-4\sin^2\frac{\alpha\theta}{2}\left(1 + e^{i\sigma\theta}\right)\right) = \xi_r B\left(1 + e^{i\sigma\theta}\right)\left(1 + e^{i\alpha\theta}\right) + A\left(\sum_{j=0}^r(\xi_{r-j+1} - \xi_{r-j})\left(1 + e^{i\sigma\theta}\right)\left(1 + e^{i\alpha\theta}\right)\right)w_j$

Now, when $r = 0$

$\xi_0\left(\delta\left(-4\sin^2\frac{\sigma\theta}{2}\left(1 + e^{i\alpha\theta}\right)\right) - \gamma\left(4\sin^2\frac{\alpha\theta}{2}\left(1 + e^{i\sigma\theta}\right)\right) + (A - B)\left(1 + e^{i\sigma\theta}\right)\left(1 + e^{i\alpha\theta}\right)\right) = A\xi_1\left(1 + e^{i\sigma\theta}\right)\left(1 + e^{i\alpha\theta}\right)$

It is easily seen that $\quad |\xi_1| < \left|\frac{2\delta + 2\gamma + B - A}{A}\right||\xi_0|$

The absolute value of $\frac{2\delta + 2\gamma + B - A}{A}$ less than one. Therefor, $|\xi_1| < |\xi_0|$.
(40)

Now, when $r = 1$

$\xi_1\delta\left(-4\sin^2\frac{\sigma\theta}{2}\left(1 + e^{i\alpha\theta}\right)\right) + \xi_1\gamma\left(-4\sin^2\frac{\alpha\theta}{2}\left(1 + e^{i\sigma\theta}\right)\right) - \xi_1 B\left(1 + e^{i\sigma\theta}\right)\left(1 + e^{i\alpha\theta}\right) + A\left(\xi_1\left(1 + e^{i\sigma\theta}\right)\left(1 + e^{i\alpha\theta}\right)\right)(1 - u_1) + A\left(\xi_0\left(1 + e^{i\sigma\theta}\right)\left(1 + e^{i\alpha\theta}\right)\right)u_1 = A\left(\xi_2\left(1 + e^{i\sigma\theta}\right)\left(1 + e^{i\alpha\theta}\right)\right),$

$$|\xi_2| < \left|\frac{2\delta + 2\gamma + B - A + Au_1}{A}\right||\xi_1| - u_1|\xi_0|$$

but $|\xi_1| < |\xi_0|$.

This   led   to $|\xi_2| < |\xi_1|$

(41)

with same way, we get,

$$|\xi_r| < \left|\frac{2\delta+2\gamma+B-A}{A}\right||\xi_{r-1}| \implies |\xi_r| < |\xi_{r-1}|.$$ (42)

Therefore, from equations (39)-(41) get $|\xi_r| < |\xi_0|$, hence, this scheme is stable.

Now, when the expanding equation (28) with Taylor series in terms of $T(x_l, y_m, t_r)$ is obtained:

$$E^{r+1} = \delta\left(\left(2h^2D_x^2 + \frac{h^4}{6}D_x^4 + h^2kD_yD_x^2 + \frac{h^4k}{12}D_yD_x^4 + \frac{h^2k^2}{2}D_y^2D_x^2 + \frac{h^4k^2}{24}D_y^2D_x^4 + \right.\right.$$

$$\left.\frac{h^2k^3}{6}D_y^3D_x^2 + \frac{h^4k^3}{72}D_y^3D_x^4 + \frac{h^2k^4}{24}D_y^4D_x^2 + \frac{h^4k^4}{288}D_y^4D_x^4 + \cdots\right)T_{l,m}^r\right) + \gamma\left(\left(k^2D_y^2 + \frac{k^4}{12}D_y^4 + \right.\right.$$

$$\left.\cdots\right)T_{l,m}^r + \left(k^2D_y^2 + hk^2D_y^2D_x + \frac{h^2k^2}{2}D_y^2D_x^2 + \frac{h^3k^2}{6}D_y^2D_x^3 + \frac{h^4k^2}{12}D_y^2D_x^4 + \frac{k^4}{12}D_y^4 + \right.$$

$$\left.\left.\frac{hk^4}{12}D_y^4D_x + \frac{h^2k^4}{24}D_y^4D_x^2 + \frac{h^3k^4}{72}D_y^4D_x^3 + \frac{h^4k^4}{288}D_y^4D_x^4 + \cdots\right)T_{l,m}^r\right),$$

$$-A\left(4 + 2hD_x + h^2D_x^2 + \frac{h^3}{3}D_x^3 + \frac{h^4}{12}D_x^4 + 2kD_y + hkD_yD_x + \frac{h^2k}{2}D_yD_x^2 + \frac{h^3k}{6}D_yD_x^3 + \right.$$

$$\frac{h^4k}{24}D_yD_x^4 + k^2D_y^2 + \frac{hk^2}{2}D_y^2D_x + \frac{h^2k^2}{4}D_y^2D_x^2 + \frac{h^3k^2}{12}D_y^2D_x^3 + \frac{h^4k^2}{48}D_y^2D_x^4 + \frac{k^3}{3}D_y^3 + $$

$$\frac{hk^3}{6}hD_y^3D_x + \frac{h^2k^3}{12}D_y^3D_x^2 + \frac{h^3k^3}{36}D_y^3D_x^3 + \frac{h^4k^3}{144}D_y^3D_x^4 + \frac{k^4}{12}D_y^4 + \frac{hk^4}{24}D_y^4D_x + \frac{h^2k^4}{48}D_y^4D_x^2 + $$

$$\left.\frac{h^3k^4}{144}D_y^4D_x^3 + \frac{h^4k^4}{576}D_y^4D_x^4 + \cdots\right)T_{l,m}^{r+1} - B\left(4 + 2hD_x + h^2D_x^2 + \frac{h^3}{3}D_x^3 + \frac{h^4}{12}D_x^4 + 2kD_y + \right.$$

$$hkD_yD_x + \frac{h^2k}{2}D_yD_x^2 + \frac{h^3k}{6}D_yD_x^3 + \frac{h^4k}{24}D_yD_x^4 + k^2D_y^2 + \frac{hk^2}{2}D_y^2D_x + \frac{h^2k^2}{4}D_y^2D_x^2 + $$

$$\frac{h^3k^2}{12}D_y^2D_x^3 + \frac{h^4k^2}{48}D_y^2D_x^4 + \frac{k^3}{3}D_y^3 + \frac{hk^3}{6}hD_y^3D_x + \frac{h^2k^3}{12}D_y^3D_x^2 + \frac{h^3k^3}{36}D_y^3D_x^3 + \frac{h^4k^3}{144}D_y^3D_x^4 + $$

$$\left.\frac{k^4}{12}D_y^4 + \frac{hk^4}{24}D_y^4D_x + \frac{h^2k^4}{48}D_y^4D_x^2 + \frac{h^3k^4}{144}D_y^4D_x^3 + \frac{h^4k^4}{576}D_y^4D_x^4 + \cdots\right)T_{l,m}^r,$$

$$-A\left(\sum_{j=1}^r\left(4 + 2hD_x + h^2D_x^2 + \frac{h^3}{3}D_x^3 + \frac{h^4}{12}D_x^4 + 2kD_y + hkD_yD_x + \frac{h^2k}{2}D_yD_x^2 + \right.\right.$$

$$\frac{h^3k}{6}D_yD_x^3 + \frac{h^4k}{24}D_yD_x^4 + k^2D_y^2 + \frac{hk^2}{2}D_y^2D_x + \frac{h^2k^2}{4}D_y^2D_x^2 + \frac{h^3k^2}{12}D_y^2D_x^3 + \frac{h^4k^2}{48}D_y^2D_x^4 + \frac{k^3}{3}D_y^3 + $$

$$\frac{hk^3}{6}hD_y^3D_x + \frac{h^2k^3}{12}D_y^3D_x^2 + \frac{h^3k^3}{36}D_y^3D_x^3 + \frac{h^4k^3}{144}D_y^3D_x^4 + \frac{k^4}{12}D_y^4 + \frac{hk^4}{24}D_y^4D_x + \frac{h^2k^4}{48}D_y^4D_x^2 + $$

$$\left.\frac{h^3k^4}{144}D_y^4D_x^3 + \frac{h^4k^4}{576}D_y^4D_x^4 + \cdots\right)\left(T_{l,m}^{r-j+1} - T_{l,m}^{r-j}\right)\right)u_j + A\left(4 + 2hD_x + h^2D_x^2 + \frac{h^3}{3}D_x^3 + \right.$$

$$\frac{h^4}{12}D_x^4 + 2kD_y + hkD_yD_x + \frac{h^2k}{2}D_yD_x^2 + \frac{h^3k}{6}D_yD_x^3 + \frac{h^4k}{24}D_yD_x^4 + k^2D_y^2 + \frac{hk^2}{2}D_y^2D_x + $$

$$\frac{h^2k^2}{4}D_y^2D_x^2 + \frac{h^3k^2}{12}D_y^2D_x^3 + \frac{h^4k^2}{48}D_y^2D_x^4 + \frac{k^3}{3}D_y^3 + \frac{hk^3}{6}hD_y^3D_x + \frac{h^2k^3}{12}D_y^3D_x^2 + \frac{h^3k^3}{36}D_y^3D_x^3 + $$

$$\left.\frac{h^4k^3}{144}D_y^3D_x^4 + \frac{k^4}{12}D_y^4 + \frac{hk^4}{24}D_y^4D_x + \frac{h^2k^4}{48}D_y^4D_x^2 + \frac{h^3k^4}{144}D_y^4D_x^3 + \frac{h^4k^4}{576}D_y^4D_x^4 + \cdots\right)T_{l,m}^0,$$

$$-C\left(4 + 2hD_x + h^2D_x^2 + \frac{h^3}{3}D_x^3 + \frac{h^4}{12}D_x^4 + 2kD_y + hkD_yD_x + \frac{h^2k}{2}D_yD_x^2 + \frac{h^3k}{6}D_yD_x^3 + \right.$$

$$\frac{h^4k}{24}D_yD_x^4 + k^2D_y^2 + \frac{hk^2}{2}D_y^2D_x + \frac{h^2k^2}{4}D_y^2D_x^2 + \frac{h^3k^2}{12}D_y^2D_x^3 + \frac{h^4k^2}{48}D_y^2D_x^4 + \frac{k^3}{3}D_y^3 + $$

$$\frac{hk^3}{6}hD_y^3D_x + \frac{h^2k^3}{12}D_y^3D_x^2 + \frac{h^3k^3}{36}D_y^3D_x^3 + \frac{h^4k^3}{144}D_y^3D_x^4 + \frac{k^4}{12}D_y^4 + \frac{hk^4}{24}D_y^4D_x + \frac{h^2k^4}{48}D_y^4D_x^2 +$$

$$\frac{h^3k^4}{144}D_y^4D_x^3 + \frac{h^4k^4}{576}D_y^4D_x^4 + \cdots\Big)\big(\rho_t c_t D_t^{2-\alpha} - \mu D_x^{4-\beta} - \mu D_y^{4-\beta} + W_b c_b\big) - 4D, \qquad (43)$$

From equations (20) and (43), the scheme obtained is of $\mathcal{O}\Big(D_t^{2-\alpha} + D_x^{4-\beta} + D_y^{4-\beta}\Big)$.

5. Numerical Implementations

In this section, we will apply the Caputo fractional derivative and fractional quadratic spline method for the following two examples of Pennes' bioheat problem to check the efficiency of this method. All calculations are implemented with MAPLE software.

Example 1: Consider Pennes' bioheat equation (2) with the conditions:

$T(x, y, 0) = (5x^3 + y^3)\cos(1) + 37,$

$\left.\dfrac{\partial}{\partial x}T(i, y, t)\right|_{i=0,1} = 15i^2\cos(1 + t^2), \quad y \in (0,1)$

$\left.\dfrac{\partial}{\partial y}T(x, j, t)\right|_{j=0,1} = 3j^2\cos(1 + t^2), \quad x \in (0,1)$

where, by choosing the source function

$Q = -28849315.1\, t^{2-\alpha}x^3\; {}_3F_4([0.75, 1., 1.25], [0.575, .825, 1.075, 1.325], -.25\, t^4) - 14643424.66\, t^{2+\alpha}x^3\; {}_3F_4([1., 1.25, 1.75], [1.075, 1.325, 1.575, 1.825], -.25\, t^4) - 5769863.02\, t^{2-\alpha}y^3\; {}_3F_4([0.75, 1., 1.25], [0.575, .825, 1.075, 1.325], -.25\, t^4) - 2928684.933\, t^{2+\alpha}y^3\; {}_3F_4([1., 1.25, 1.75], [1.075, 1.325, 1.575, 1.825], -.25\, t^4) - 7.744523840\, x^{3-\beta}\,\cos t^2 + 12.06138125\, x^{3-\beta}\,\sin t^2 - 1.548904768\, y^{3-\beta}\,\cos t^2 + 2.412276250\, y^{3-\beta}\,\sin t^2 + \big(2000\,(5x^3 + y^3)\big)\cos(t^2 + 1) - 420.$

so, the exact solution given as $T(x, y, t) = (5x^3 + y^3)\cos(t^2 + 1) + 37,$

τ	α	β	$L_2 - error$	$L_\infty - error$
		1.4	3.6682028e-3	9.0253717e-4
	0.1	1.6	3.6674357e-3	9.0241298e-4
		1.8	3.6666368e-3	9.0229499e-4
		1.4	2.3549982e-4	5.7942161e-5
	0.5	1.6	2.3545522e-4	5.7934941e-5
0.001		1.8	2.3540877e-4	5.7928081e-5
		1.4	3.7954773e-5	9.3374428e-6
	0.9	1.6	3.7951752e-5	9.3369537e-6
		1.8	3.7948604e-5	9.3364891e-6
	1	2	3.1158610e-5	7.6657776e-6
		1.4	1.8266272e-3	4.4943014e-4
0.000001	0.1	1.6	1.8262427e-3	4.4936790e-4
		1.8	1.8258423e-3	4.4930877e-4
	0.5	1.4	6.7008060e-6	1.6486912e-6
		1.6	6.6993956e-6	1.6484628e-6

	1.8	6.6979267e-6	1.6482459e-6
	1.4	2.8660189e-8	7.0516583e-9
0.9	1.6	2.8654161e-8	7.0506824e-9
	1.8	2.8647884e-8	7.0497554e-9
1	2	7.5797332e-9	1.8654321e-9

Table 1: The L_2 and maximum norm errors of Numerical Solutions for Various Values of for Example 1 at $\alpha = 0.1, 0.5, 0.9, 1$ and $\beta = 1.1, 1.5, 1.9, 2$
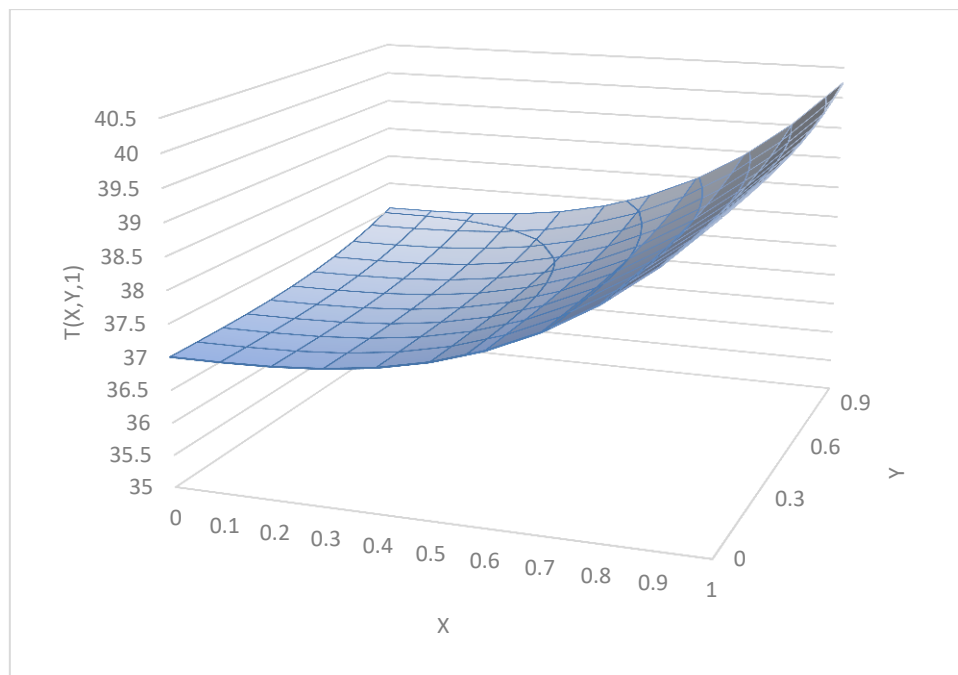

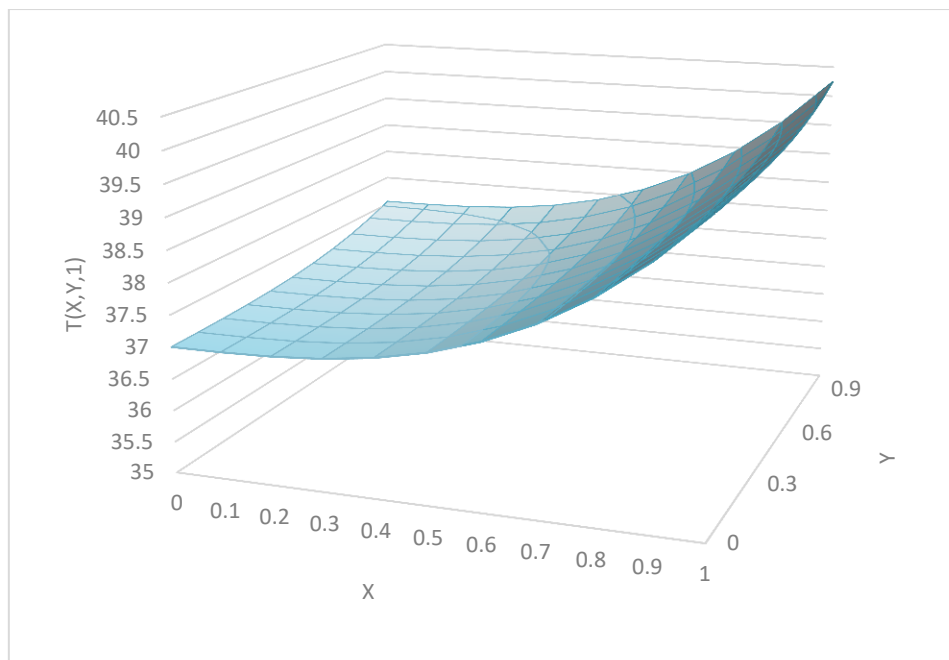
Fig. 1: Exact Solutions for Example1

Fig. 2: Numerical Solutions for Example1 at $\alpha = 0.1,\ \beta = 1.4$, and $\tau = 0.001$

Example 2: Consider Pennes' bioheat equation (2) with the conditions:

$T(x, y, 0) = x^{3+\alpha} + y^{3+\alpha} + 37,$

$$\left.\frac{\partial}{\partial x} T(i, y, t)\right|_{i=0,1} = (3 + \alpha)\, i^{2+\alpha}, \quad y \in (0,1)$$

$$\left.\frac{\partial}{\partial y} T(x, j, t)\right|_{j=0,1} = (3 + \alpha)\, j^{2+\alpha}, \quad x \in (0,1)$$

where, by choosing the source function
$Q = 8970483.264\, t^{1+\beta-\alpha} - 3.109419701 \left(x^{3+\alpha-\beta} + y^{3+\alpha-\beta}\right) + 2000\left(x^{3+\alpha} + y^{3+\alpha} + t^{1+\beta}\right) - 420.$

so, the exact solution given as $T(x, t) = x^{3+\alpha} + y^{3+\alpha} + t^{1+\beta} + 37,$

α	β	τ	$L_\infty - error$	$L_\infty - error$
		0.000001	1.5475640e-8	4.5166604e-9
	1.1	0.001	2.2616972e-6	7.7413749e-6
		0.01	1.6396255e-5	5.1455543e-5
		0.1	1.2822234e-3	1.3522203e-2
		0.000001	1.5464677e-8	4.5140604e-9
0.9	1.5	0.001	2.2603941e-6	7.7358842e-6
		0.01	1.6385905e-5	5.1416479e-5
		0.1	1.2822176e-3	1.3522507e-2
		0.000001	1.5449100e-8	4.5106490e-9
	1.9	0.001	2.2586844e-6	7.7280838e-6
		0.01	1.6372324e-5	5.1362675e-5
		0.1	1.2822138e-3	1.3522980e-2

Table 2: The L_2 and maximum norm errors of Numerical Solutions for Example 2 and Various Values of $\alpha = 0.9$, $\beta = 1.1, 1.5, 1.9$ and $\tau = 0.001, 0.01, 0.1$
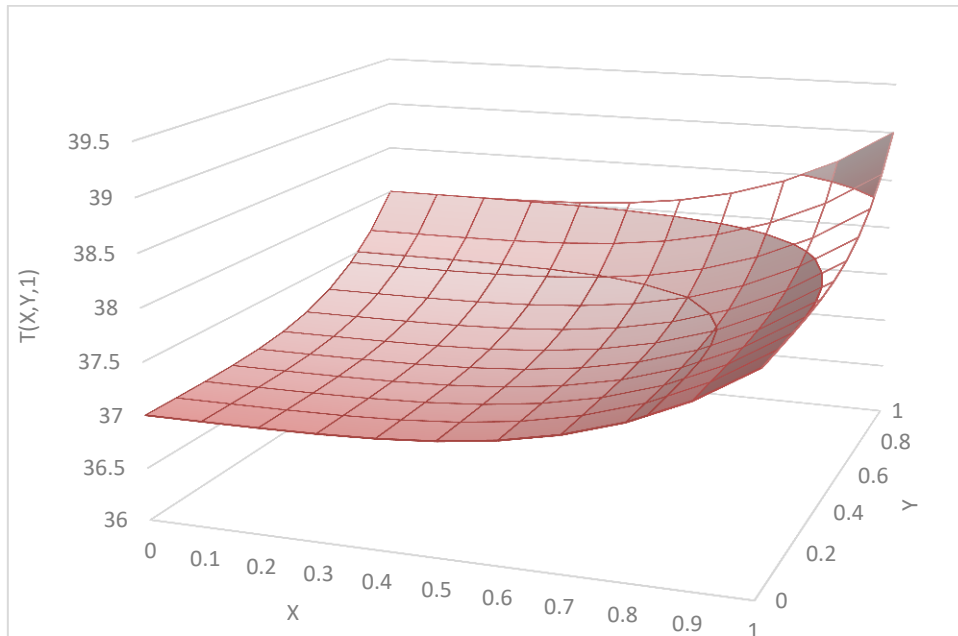

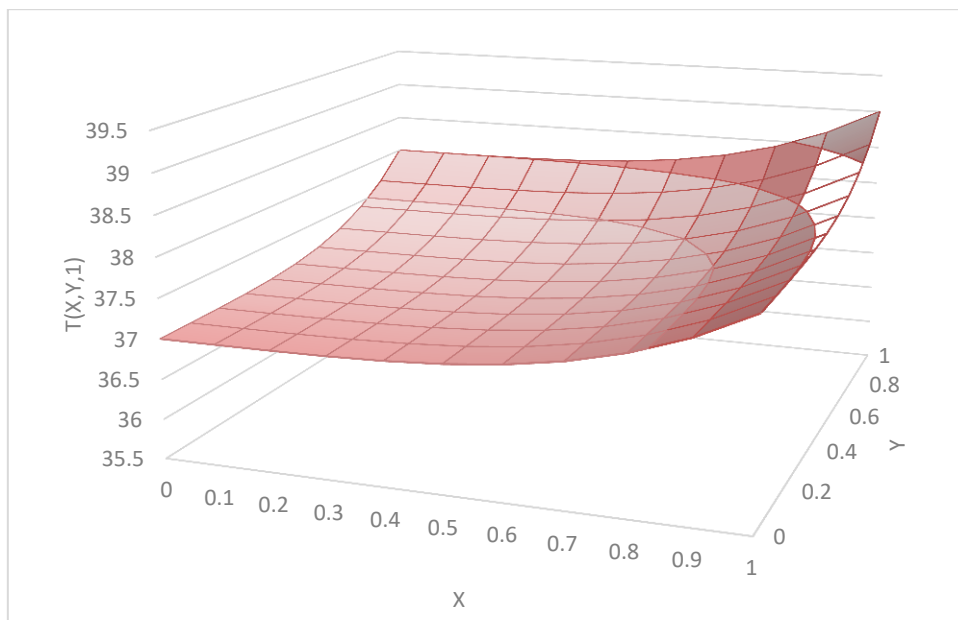Fig. 3 : Exact Solutions for Example 2

Fig. 4: Numerical Solutions for Example 2 at $\alpha = 0.9$, $\beta = 1.9$, and $t = 0.001 \, s$

Discussion and Conclusions

   The objective of this article is to compare the achievement of the model approach based on fractional quadratic spline method, which has been considered for finding the numerical solutions of 2D time-space Pennes' bioheat problem through using Caputo fractional

derivative for the fractional time derivative and fractional quadratic spline for the fractional derivative of space in this equation. Generally speaking, it can be concluded from L_2 and maximum norm errors of the numerical approximations, which tabulated in Tables 1, 2 and shown in figures 1, 2, 3, 4 compared with the accurate solution, that the proposed method is powerful, effective, highly accurate and needs little recurrence. Furthermore, the present algorithm is simply applicable as well as the results clarify the activity of the suggested method. The analysis of stability using the Von Neumann of this method has been discussed to clarify that this scheme unconditionally stable and, it can be seen that the numerical solution converges to the exact solution with order $\mathcal{O}\left(\tau^{2-\alpha} + h^{4-\beta} + k^{4-\beta}\right)$.

Acknowledgements

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الحلول العددية لمسألة الأنتشار الحراري الحيوي ثنائية البعد الكسورية للزمان ــالمكان بأستخدام طريقة الشريحة التربيعية الكسورية

عمار مسلم عبد الحسين　　و　أ. م. د. حميدة عودة الحميدي

المستخلص

في هذا البحث، تم حل مسألة الأنتشار الحراري الحيوي ثنائية البعد الكسورية للزمان – المكان لتوزيع درجة الحرارة في الأنسجة عددياً بأستخدام طريقة الشريحة التربيعية الكسورية من الرتبة α للمشتقة الزمانية بأستخدام مشتقة كابوتو ومن الرتبة β للمشتقة المكانية بأستخدام الشريحة التربيعية الكسورية بعدها تمت المقارنة بين الحلول الدقيقة والعددية, واخيرا ناقشنا تحليل الاستقرارية بطريقة فون نيومان.