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A Combination of the Orthogonal Polynomials with Least – Squares Method for Solving High-Orders Fredholm-Volterra Integro-Differential Equations

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ABSTRACT

This study introduced new technique which is based on a combination of the least-squares technique (LST) with Chebyshev and Legendre polynomials for finding the approximate solutions of higher-order linear Fredholm-Volterra integro-differential equations (FVIDEs) subject to the mixed conditions. Two examples of second and third-order linear FVIDEs are considered to illustrate the proposed method, the numerical results are comprised to demonstrate the validity and applicability of this technique, and comparisons with the exact solution are made. These results have shown that the competence and accuracy of the present technique.

1. Introduction

There are many scientific fields including biomedical and biophysics have been investigated by integro-differential equations. There are three types of these questions; Fredholm Volterra or mixed integro-differential equations have been used for solving many problems in applied mathematics, such as modelling and bioinformatics. So, the numerical methods are used for these scientific subjects. Even though, extremely difficult in the nonlinear equation, there are numerical methods to solve the exact solutions of FVIDE which are the most approximation method [8, 17]. Recently, FVIDEs have been solved by matrix methods, for instance, the method is used to solve the system of differential equations

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[2, 10]. There are many numerical methods approximation method, the Adomian decomposition method, the Chebyshev-Taylor collocation method, Haar Wavelet the Tau method, Wavelet-Galerkin method, the monotone iterative technique, the variation iteration method, the Tau method and the Walsh series method [5, 7, 9, 12, 13, 14, 15, 16, 17].

The aim of this paper is to introduce a new technique to get better and faster numerical solutions depend on the Chebyshev and Legendre polynomials for solving linear Fredholm- Volterra integro-differential equations [5].

$$u^{(k)}(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)u(t)dt + \lambda_2 \int_c^d k_2(x,t)u(t)dt\, , c \le x,t \le d, \quad (1.1)$$

under the mixed conditions

$$\sum_{i=0}^{N-1}(c_{ji}u^i(c) + d_{ji}u^i(\mathrm{d})) = \beta_j \qquad j = 0\,,1\,,\ldots,N-1 \qquad (1.2)$$

2. Numerical methods

In this paper, the standard LST has been discussed to solve equations (1.1) and (1.2) by combining with the following basis functions [1]:

* Chebyshev Polynomials * Legendre Polynomials

2.1 Chebyshev Least- Squares Technique:

We will define Chebyshev polynomials by the following equation, [1]

$$C_{i+1}(x) = 2xC_i(x) - C_{i-1}(x) \;\; , i \ge 1 \qquad (2.1)$$

where

$$C_0(x) = 1 \;\; , C_1(x) = x \; , \qquad (2.2)$$

We assume the approximate solution as

$$u(x) = u_m(x) = \sum_{i=0}^{m} a_i C_i(x) \qquad \mathrm{c} \le \mathrm{x} \le \mathrm{d} \qquad (2.3)$$

Where a_i and $C_i(x)$ are unknown constants and the Chebyshev polynomial of degrees (i) of the first kind which is for all $x \in [-1,1]$. Substituting equation (2.3) into equation (1.1), we get

$$\sum_{i=0}^{m} a_i {C_i}^{(k)}(x) = f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt \qquad (2.4)$$

The residual equation has been given by

$$R(x,a_i) = R(x,u_m(x)) = \sum_{i=0}^{m} a_i C_i^{(k)}(x) - \left\{ f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt \right\} \qquad (2.6)$$

Let

$$S(a_0,a_1,\dots,a_m) = \int_c^d [R(x,a_i)]^2 w(x) dx, \qquad (2.7)$$

Where w(x) is the positive weight function defined in the interval [c, d]. For simplicity set w (x)=1, thus,

$$S(a_0,a_1,\dots,a_m) = \int_c^d \left[\sum_{i=0}^{m} a_i C_i^{(k)}(x) - \left\{ f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt \right\} \right]^2 dx, \qquad (2.8)$$

We can get the values of a_i , $i \geq 0$ by minimizing the value of S as follows :

$$\frac{\partial S}{\partial a_i} = 0, i = 0,1,\dots,m \qquad (2.9)$$

Then from (2.8) by applying (2.9) get:

$$\frac{\partial S}{\partial a_i} = \int_c^d \left[\sum_{i=0}^{m} a_i C_i^{(k)}(x) - \left\{ f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt \right\} \right] dx \times \int_c^d \left[C_i^{(k)}(x) - \left\{ \lambda_1 \int_0^x k_1(x,t) C_i(t) dt + \lambda_2 \int_c^d k_2(x,t) C_i(t) dt \right\} \right] dx = 0 \qquad (2.10)$$

Thus, (2.10) are generated (m+1) algebraic system of equations in (m+1) unknown $a_i, i = 0,\cdots,m$, or in the matrix form as follow:

$$W = \begin{pmatrix} \int_c^d R(x,a_0)h_0 dx & \int_c^d R(x,a_1)h_0 dx & \dots & \int_c^d R(x,a_m)h_0 dx \\ \int_c^d R(x,a_0)h_1 dx & \int_c^d R(x,a_1)h_1 dx & \dots & \int_c^d R(x,a_m)h_1 dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_c^d R(x,a_0)h_m dx & \int_c^d R(x,a_1)h_m dx & \dots & \int_c^d R(x,a_m)h_m dx \end{pmatrix}, \qquad (2.11)$$

$$G = \begin{pmatrix} \int_c^d \{f(x)\} h_0 dx \\ \int_c^d \{f(x)\} h_1 dx \\ \vdots \\ \int_c^d \{f(x)\} h_m dx \end{pmatrix}, \qquad (2.12)$$

where

$$h_i = C_i^{(k)} - \left\{ \lambda_1 \int_0^x k_1(x,t) C_i dt + \lambda_2 \int_c^d k_2(x,t) C_i dt\, dx \right\} \qquad (2.13)$$

$$R(x, a_i) = \sum_{i=0}^{m} a_i {C_i}^{(k)}(x) - \left\{ \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i C_i(t)\, dt \right\}, \qquad (2.14)$$

$$WA = G \text{ or } A = [W; G]. \qquad (2.15)$$

Property: $\forall x \in \bar{\Omega}$ the matrix $w(x)$ defined in (16) is non-singular. □

The equation (1.1) corresponds to a system of $(m + 1)$ linear algebraic equations with the unknown Chebyshev coefficients $a_i, i = 0,1, \dots, m$, [23].
Another form of (2.15) by applying the conditions can be explained as

$$[U_i : \beta_i] \quad , i = 0, 1, \dots, N-1$$

where

$$U_i = [u_{i0} \ \ u_{i1} \ \ u_{i2} \ \ \dots \ \ u_{iN}], i = 0, 1, 2 \dots N-1 \qquad (2.16)$$

To get the solution of (1.1) under conditions (1.2), by changing the row matrices (2.16) by the last (m) rows of the matrix form (2.15), we get the new augmented matrix [4, 11, 18, 19, 20, 21, 22].

$$[\widetilde{W};\widetilde{G}] = \begin{pmatrix} \int_c^d R(x,a_0)h_0dx & \int_c^d R(x,a_1)h_0dx & \cdots & \int_c^d R(x,a_m)h_0dx & ; & G_0 \\ \int_c^d R(x,a_0)h_1dx & \int_c^d R(x,a_1)h_1dx & \cdots & \int_c^d R(x,a_m)h_1dx & ; & G_1 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ \int_c^d R(x,a_0)h_{m_{N_0}}dx & \int_c^d R(x,a_1)h_{m_{N_1}}dx & \cdots & \int_c^d R(x,a_m)h_{m_{N_m}}dx & ; & G_{m-N} \\ u_{00} & u_{01} & & u_{0N} & ; & \beta_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & \cdots & u_{(m-1)N} & ; & \beta_{N-1} \end{pmatrix}, (2.17)$$

$$A = \widetilde{W}^{-1}\widetilde{G},$$

therefore, the matrix A (thus the coefficients $a_0, a_1, a_2, \cdots, a_m$) is uniquely determined. Also, the equation (1.1) with conditions (1.2) has a unique solution.

2.2 Legendre Least – Squares Technique:

We will define Legendre polynomials by the following equation [1]:

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \quad, n = 0, 1 \ldots \qquad (2.17)$$

where

$$p_0(x) = 1\ , \ p_1(x) = x\ ,$$

we suppose the approximate solution as

$$u(x) = u_m(x) = \sum_{i=0}^{m} a_i p_i(x) \qquad c \le x \le d, \qquad (2.18)$$

by the same procedure in Chebyshev polynomials which have been discussed in section (**2.1**). We have $(m+1)$ algebraic linear system of equations in $(m+1)$ unknown Legendre coefficients $a_i\ ,\ i \ge 0$.

3. Convergence analysis Chebyshev (Legendre) Polynomials

Now we will review an estimate of the errors above based on the numerical methods which introduced in the second section, want to prove that as $m \to \infty$ the approximate solution $u_m(x)$ will be converge to the exact solution $u(x)$ of (1).

Note: We will suffice with proof of the convergence of the Chebyshev polynomial and likewise, the proof of the convergence of the Legendre polynomial.

Lemma (3.1): [6]. Let $u(x)$ be in the Sobolev space H^k(-1, 1) and, $\mathbb{q}_m u(x) = \sum_{i=0}^{m} a_i T_i(x)$ be the best approximation polynomial of $u(x)$ in $L_2 - norm$ .Then

$$|| u - \mathbb{q}_m u||_{L_2[-1,1]} \le cm^{-k}|| u^k||_{H(-1,1)} \ , \qquad (3.1)$$

where C is a positive constant, which depends on the selected norm and is independent of $u(x)$ and m.

Proof: By using of the transformation

$$x \in [-1,1], u(x) \to u^*(\theta) = u(\cos(\theta)), \theta \in (0,2\pi) \qquad (3.2)$$

since, $\theta = cos^{-1}(x)$, where, $\frac{d\theta}{dx} = -w(x)$, (the Chebyshev weight)

$$|| u^*||^2_{L_2[-1,1]} = \frac{1}{2}|| u^*||^2_{L_2(0,2\pi)} \qquad (3.3)$$

He is following the map $u \to u^*$ it is an analogy between $L_2[-1,1]$ and the subspace to $L_2(0,2\pi)$ even from real functions . Furthermore, it maps $H^k_w[-1,1]$ in space the periodic functions $H^k_P(0,2\pi)$. Actually, since $u \in C^{m-1}([-1,1])$, then $u^* \in C^{m-1}(\mathbb{R})$ it is a 2π - cyclic with all derivatives of the system up to -1 , whence $u^* \in H^k_P(0,2\pi)$. Lastly, since $\left|\frac{d\theta}{dx}\right| = |-sin\theta| \le 1$, we also have

$$||u^*||_{H^k(0,2\pi)} \le C||u||_{H^k_w(-1,1)} \qquad \text{for } k \ge 1 \qquad (3.4)$$

Let $\mathbb{q}_m$ denote the symmetric truncation to the grade , i.e.,

$$P^*_m(\sum_{j=-\infty}^{\infty} a_j e^{ij\theta}) = \sum_{j=-m}^{m} a_j e^{ij\theta}$$

It is easily seen that

$$(\mathbb{q}_m u)^* = \mathbb{q}^*_m u^* \qquad \forall u \in L_2[-1,1] \qquad (3.5)$$

Actually , since $u(x) = \sum_{i=0}^{\infty} a_i T_i(x)$, $u^*(\theta) = \sum_{i=0}^{\infty} a_i \ cosi\theta = \frac{\sum_{i=0}^{\infty} a_i(e^{ij\theta}+e^{-ij\theta})}{2}$,

whence (3.5). Now, from

$$|| u - \mathbb{q}_m u||_{L_2[0,2\pi]} \le Cm^{-k}|| u^k||_{L_2(0,2\pi)} \ \ \forall u \in H^k_P(0,2\pi) \qquad (3.6)$$

and (2.3) we get

$$|| u - \mathbb{q}_m u||_{L_2[-1,1]} = \frac{1}{\sqrt{2}}|| u^* - \mathbb{q}^*_m u^*||_{L_2(0,2\pi)} \le Cm^{-k}|| u^{*(k)}||_{L_2(0,2\pi)}$$

thus,

$$|| u - p_m u||_{L_2[-1,1]} \le Cm^{-k}|| u^k||_{H(-1,1)}$$

Theorem (3.2): [3]. Assume $k: x \to x$ is bounded, with x a Banach space, and assume $\lambda - k: x \to x$ is one to one and onto. Further assume

$$\|k - \mathbb{q}_m k\| \to 0 \text{ as } m \to \infty \quad (3.7)$$

Then for all sufficiently large m, say $M \geq m$ the operator $(\lambda - \mathbb{q}_m k)^{-1}$ exists as a bounded operator from $x \; to \; x$. Moreover, it is uniformly bounded:

$$\text{Sup}_{M\geq m}\|(\lambda - \mathbb{q}_m k)^{-1}\| \to \infty \quad (3.8)$$

For the solutions of $(\lambda - \mathbb{q}_m k)x_m = x_m y$, $x_m \epsilon x$ and $(\lambda - k)x = y$

$$x - x_m = \lambda(\lambda - \mathbb{q}_m k)^{-1}(x - \mathbb{q}_m x) \quad (3.9)$$

$$\frac{|\lambda|}{\|\lambda - \mathbb{q}_m k\|}\|\lambda - \mathbb{q}_m k\| \leq \|x - x_m\| \leq |\lambda|\|(\lambda - \mathbb{q}_m k)^{-1}\|\|x - \mathbb{q}_m x\| \quad (3.10)$$

This leads to $\|x - x_m\| \to 0$ as $\|x - \mathbb{q}_m x\| \to 0$.

Proof:

A) We choose m search that

$$\epsilon_m \equiv \sup_{M\geq m}\|\lambda - \mathbb{q}_m k\| < \frac{1}{\|(\lambda - k)^{-1}\|}$$

Then the inverse $[I + (\lambda - k)^{-1}(k - \mathbb{q}_m k)]^{-1}$ exists and is uniformly bounded by the geometric series theorem:

$$\|[I + (\lambda - k)^{-1}(k - \mathbb{q}_m k)]^{-1}\| \leq \frac{1}{1 - \epsilon_m \|(\lambda - k)^{-1}\|}$$

Using $\lambda - \mathbb{q}_m k = (\lambda - k) + (k - \mathbb{q}_m k) = (\lambda - k)[I + (\lambda - k)^{-1}(k - \mathbb{q}_m k)]$,

$(\lambda - \mathbb{q}_m k)^{-1}$exists,

$$\|(\lambda - \mathbb{q}_m k)^{-1}\| \leq \frac{\|(\lambda - k)^{-1}\|}{1 - \epsilon_m\|(\lambda - k)^{-1}\|} \equiv m \quad (3.11)$$

This shows (3.8).

B) for the error formula (3.9) , multiply $(\lambda - k)x = y$ by $\mathbb{q}_m$, and then rearrange to obtain

$$(\lambda - \mathbb{q}_m k)x = \mathbb{q}_m y + \lambda(x - \mathbb{q}_m x) ,$$

subtract $(\lambda - \mathbb{q}_m k)x_m = \mathbb{q}_m y$ to get

$$(\lambda - \mathbb{q}_m k)(x - x_m) = \lambda(x - \mathbb{q}_m x) \quad (3.12)$$

$$x - x_m = \lambda(\lambda - \mathbb{q}_m k)^{-1}(x - \mathbb{q}_m x),$$

which is (3.9). Taking norms and using (3.10)

$$\|x - x_m\| \leq |\lambda| m \|x - \mathbb{q}_m x\| \quad (3.13)$$

Thus if $\mathbb{q}_m x \to x$,then $x_m \to x$ as $m \to \infty$

C) The upper bound in (3.10) follows directly from (3.9) as we have just seen. The minimum follows by taking the limits of (3.12), to get

$$|\lambda|\|x - \mathbb{q}_m x\| \leq \|\lambda - \mathbb{q}_m k\|\|x - x_m\|.$$

This is equivalent to the minimum in (3.10). To get a minimum that is uniform in m note that for $M \geq m$,

$$\|\lambda - \mathbb{q}_m k\| \leq \|\lambda - k\| + \|k - \mathbb{q}_m k\| \leq \|\lambda - k\| + \epsilon_m$$

The minimum in (3.11) can now be replaced by

$$\frac{|\lambda|}{\|\lambda - \mathbb{q}_m k\| + \epsilon_m}\|x - \mathbb{q}_m x\| \leq \|x - x_m\|.$$

Combining this and (3.13), we have

$$\frac{|\lambda|}{\|\lambda - \mathbb{q}_m k\| + \epsilon_m}\|x - \mathbb{q}_m x\| \leq \|x - x_m\| \leq |\lambda| m \|x - \mathbb{q}_m x\| \qquad (3.14)$$

This shows that converges to x $\leftrightarrow \mathbb{q}_m x$ converges to. Furthermore, if convergence does occur, then $\|x - \mathbb{q}_m x\|$ and $\|x - x_m\| \to 0$ exactly at the same speed. To apply the above theorem, we need to know whether $\|k - \mathbb{q}_m k\| \to 0$ as $m \to \infty$. ■

Lamma (3.3): [3]. Suppose x , y be Banach spaces, and let $W_m: x \to y, m \geq 1$ be a sequence of bounded linear operators. Assume $\{W_m x\}$ converges for all $x \in X$.Then the convergence is uniform on compact subsets of X,

Proof. By using the principle of uniform boundedness, the operators W_m are uniform bounded:

$$M \equiv \operatorname*{Sup}_{m \geq 1} \|W_m\| < \infty$$

The functions W_m are also equal:

$$\|W_m x - W_m y\| \leq M\|x - y\|$$

Suppose S is a compressed subset of X. Then $\{W_m\}$ is a set of functions with uniform and equal boundaries in the combined set S; hence the standard result of the analysis is that $\{W_m x\}$ is uniformly convergent for $x \in$ S. ■

Lamma (3.4) Suppose X is a Banach space, and let $\{\mathbb{q}_m\}$ be a set of finite projections on X with

$$\mathbb{q}_m x \to x \text{ as } m \to \infty, x \in X \qquad (3.15)$$

Let $k: x \to x$ be compact. Then

$$\|k - \mathbb{q}_m k\| \to 0 \text{ as } m \to \infty$$

Proof. by definition of operator norm,

$$\|k - \mathbb{q}_m k\| = \sup_{\|x\|\le 1} \|kx - \mathbb{q}_m kx\| = \sup_{\mathbb{z}\in k(U)} \|\mathbb{z} - \mathbb{q}_m \mathbb{z}\|$$

With $k(U) = \{kx \mid \|x\| \le 1\}$.

The group $k(U)$ is compact. So, by (3.15) and the Lemma (3.3),

$$\sup_{\mathbb{z}\in k(U)} \|\mathbb{z} - \mathbb{q}_m \mathbb{z}\| \to 0 \text{ as } m \to \infty. \qquad \blacksquare$$

4. Numerical Experiment

In this paragraph, we have investigated the combination of LST for solving high-orders linear FVIDEs with Chebyshev and Legendre's polynomials as the basis functions. The examples are solved to explain them precisely, and time of accomplishment of the method. The absolute error has been defined

$$\text{Error} = |u(x) - u_m(x)| \qquad c \le x \le d \ , m = 1,2, \dots$$

where $u(x)$ is the exact solution and $u_m(x)$ is the approximate solution .

Example 1. we considered second order FVIDE given as [1]

$$u''(x) = f(x) + \int_0^x u(t)dt + \int_{-1}^{1} (1 - 2xt)u(t)dt \ , -1 \le x \le 1$$

$$u(0) = 2, \quad u'(0) = 6$$

The exact solution is given as $u(x) = 2 + 6x - 3x^2$.

Where, $f(x) = -8 + 6x - 3x^2 + x^3 , k_1(x,t) = 1 , k_2(x,t) = 1 - 2xt , \lambda_1 = \lambda_2 = 1$.

Firstly, an approximate solution $u(x)$ will be found using the combination of least-squares with the Chebyshev polynomial, which defined in the form

$$u(x) = \sum_{i=0}^{m} a_i C_i(x) \quad , -1 \le x \le 1$$

if $m = 6$.

$$G = [\frac{158}{5} \ \ \frac{137}{15} \ \ -\frac{8546}{105} \ \ \frac{16859}{175} \ \ -\frac{21542}{63} \ \ \frac{15053}{45} \ \ -\frac{180350}{231}]'$$

$$W = \begin{bmatrix} \frac{63}{3} & \frac{-2}{9} & \frac{-286}{15} & \frac{-50}{3} & \frac{-1354}{21} & \frac{-2962}{63} & \frac{-1298}{9} \\ \frac{-2}{9} & \frac{347}{270} & \frac{-46}{45} & \frac{12893}{630} & \frac{-4406}{315} & \frac{8053}{126} & \frac{-4586}{135} \\ \frac{-286}{15} & \frac{-46}{45} & \frac{13802}{315} & \frac{302}{25} & \frac{142238}{945} & \frac{594}{35} & \frac{129566}{385} \\ \frac{-50}{3} & \frac{12893}{630} & \frac{302}{25} & \frac{1131139}{3150} & \frac{3382}{175} & \frac{1099391}{990} & \frac{60722}{1575} \\ \frac{-1354}{21} & \frac{-4406}{315} & \frac{142238}{945} & \frac{3382}{175} & \frac{22442986}{10395} & \frac{-34}{735} & \frac{26909158}{5005} \\ \frac{-2962}{63} & \frac{8053}{126} & \frac{594}{35} & \frac{1099391}{990} & \frac{-34}{735} & \frac{3072051937}{378378} & \frac{-8122}{945} \\ \frac{-1298}{9} & \frac{-4586}{135} & \frac{129566}{385} & \frac{60722}{1575} & \frac{26909158}{5005} & \frac{-8122}{945} & \frac{7609658362}{315315} \end{bmatrix}$$

From the given conditions, the augmented matrices are obtained respectively, as follows:

$U_0 = [\,1 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0 \quad -1\,]$ and $U_1 = [\,0 \quad 1 \quad 0 \quad -3 \quad 0 \quad 5 \quad 0\,]$.

If we replace the last two rows of the matrices W and G by the values of U_0 and U_1 in, then

$$\tilde{G} = [\tfrac{158}{5} \quad \tfrac{137}{15} \quad -\tfrac{8546}{105} \quad \tfrac{16859}{175} \quad -\tfrac{21542}{63} \quad 2 \quad 6\,]' ,$$

$$\widetilde{W} = \begin{bmatrix} \frac{63}{3} & \frac{-2}{9} & \frac{-286}{15} & \frac{-50}{3} & \frac{-1354}{21} & \frac{-2962}{63} & \frac{-1298}{9} \\ \frac{-2}{9} & \frac{347}{270} & \frac{-46}{45} & \frac{12893}{630} & \frac{-4406}{315} & \frac{8053}{126} & \frac{-4586}{135} \\ \frac{-286}{15} & \frac{-46}{45} & \frac{13802}{315} & \frac{302}{25} & \frac{142238}{945} & \frac{594}{35} & \frac{129566}{385} \\ \frac{-50}{3} & \frac{12893}{630} & \frac{302}{25} & \frac{1131139}{3150} & \frac{3382}{175} & \frac{1099391}{990} & \frac{60722}{1575} \\ \frac{-1354}{21} & \frac{-4406}{315} & \frac{142238}{945} & \frac{3382}{175} & \frac{22442986}{10395} & \frac{-34}{735} & \frac{26909158}{5005} \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 \end{bmatrix}.$$

Thus, Chebyshev coefficients are calculated as:

$$A = \widetilde{W}^{-1}\tilde{G} = [\tfrac{1}{2} \quad 6 \quad -\tfrac{3}{2} \; 0\;0\;0\;0]'.$$

Therefore, the approximate solution of the problem taking $m = 6$ is the exact solution under the given conditions as follows:

$$u_6(x) = (a_0 - a_2 + a_4 - a_6) + (a_1 - 3a_3 + 5a_5)x + (2a_2 - 8a_4 + 18a_6)x^2 + (4a_3 - 20a_5)x^3 + (8a_4 - 48a_6)x^4 + 16a_5x^5 + 32a_6x^6$$

which represent the exact solution, $u(x) = 2 + 6x - 3x^2$.

Secondly, an approximate solution $u(x)$ will be found using the combination of least-squares with the Legendre polynomial, which defined in the form

$$u_6(x) = \sum_{i=0}^{6} a_i p_i(x)$$

$$G = [\frac{158}{5} \quad \frac{137}{15} \quad -\frac{372}{7} \quad \frac{8909}{140} \quad -\frac{65516}{315} \quad \frac{1545}{8} \quad -450\,]'$$

$$W = \begin{bmatrix} \frac{26}{3} & \frac{-2}{9} & \frac{-182}{15} & \frac{-21}{2} & -40 & \frac{-111}{4} & -84 \\ \frac{-2}{9} & \frac{347}{270} & -\frac{37}{45} & \frac{11149}{840} & -8 & \frac{1793}{48} & -19 \\ -\frac{182}{15} & -\frac{37}{45} & \frac{1894}{105} & \frac{11}{4} & \frac{18898}{315} & \frac{13}{8} & 126 \\ -\frac{21}{2} & \frac{11149}{840} & \frac{11}{4} & \frac{1512443}{10080} & \frac{9}{2} & \frac{18627019}{44352} & \frac{29}{4} \\ -40 & -8 & \frac{18898}{315} & \frac{9}{2} & \frac{478174}{693} & \frac{3}{4} & \frac{2162158}{1287} \\ -\frac{111}{4} & \frac{1793}{48} & \frac{13}{8} & \frac{18627019}{44352} & \frac{3}{4} & \frac{380541959}{164736} & -\frac{5}{8} \\ -84 & -19 & 126 & \frac{29}{4} & \frac{2162158}{1287} & -\frac{5}{8} & \frac{13513504}{2145} \end{bmatrix}$$

For the given conditions then augmented matrices are obtained respectively, as follows

$$U_0 = [\,1 \quad 0 \quad -\tfrac{1}{2} \quad 0 \quad \tfrac{3}{8} \quad 0 \quad -\tfrac{5}{16}\,] \;,\; U_1 = \left[\,0 \quad 1 \quad 0 \quad -\tfrac{3}{2} \quad 0 \quad \tfrac{15}{8} \quad 0\,\right]$$

If we replace the last two rows of the matrices W and G by the values of U_0 and U_1 in, then

$$\tilde{G} = [\frac{158}{5} \quad \frac{137}{15} \quad -\frac{372}{7} \quad \frac{8909}{140} \quad -\frac{65516}{315} \quad 2 \quad 6\,]'$$

$$\widetilde{W}=\begin{bmatrix} \frac{26}{3} & \frac{-2}{9} & \frac{-182}{15} & \frac{-21}{2} & -40 & \frac{-111}{4} & -84 \\ \frac{-2}{9} & \frac{347}{270} & -\frac{37}{45} & \frac{11149}{840} & -8 & \frac{1793}{48} & -19 \\ -\frac{182}{15} & -\frac{37}{45} & \frac{1894}{105} & \frac{11}{4} & \frac{18898}{315} & \frac{13}{8} & 126 \\ -\frac{21}{2} & \frac{11149}{840} & \frac{11}{4} & \frac{1512443}{10080} & \frac{9}{2} & \frac{18627019}{44352} & \frac{29}{4} \\ -40 & -8 & \frac{18898}{315} & \frac{9}{2} & \frac{478174}{693} & \frac{3}{4} & \frac{2162158}{1287} \\ 1 & 0 & -\frac{1}{2} & 0 & -\frac{3}{2} & 0 & -\frac{5}{16} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & -\frac{15}{8} & 0 \end{bmatrix}$$

Thus, Legendre coefficients are calculated as

$$A=\widetilde{W}^{-1}\tilde{G}=[\,1 \quad 6 \quad -2 \quad 0 \quad 0 \quad 0 \quad 0\,]'$$

$u_6 = 2+6x-3x^2$

which is the exact solution (1).

Example 2. Consider a second order FVIDE given as [1]

$$u'''(\mathrm{x})=f(x)+\int_0^x u(t)dt+\int_{-\pi}^{\pi} xu(t)dt\ ,$$

With conditions, $u(0)=1, u'(0)=1, u''(0)=-1$

The exact solution is given as $u(x)=x+cosx$.

Where $f(x)=\frac{1}{2}x^2\ , k_1(x,t)=1\ , k_2(x,t)=x\ , \lambda_1=\lambda_2=1$

Firstly, an approximate solution $u(x)$ will be found using the Chebyshev polynomial-least-squares technique, which defined in the form

$$u(x)=\sum_{i=0}^{m} a_i C_i(x)$$

$R_0=-x-2\pi x\ ,\ R_1=\frac{-1}{2}x^2\ ,\ R_2=x-\frac{2}{3}x^3-\frac{2\pi x(2\pi^2-3)}{3},\ \ R_3=24-x^2\left(x^2-\frac{3}{2}\right)$

$R_4 = 191x + \frac{8}{3}x^3 - \frac{8}{5}x^5 - \frac{2\pi x(24\pi^4-40\pi^2+15)}{15}$, $R_5 = 960x^2 - \frac{x^2(16x^4-30x^2+15)}{6} - 120$

$R_6 = 3834x^3 - 1151x + \frac{48x^5}{5} - \frac{32x^7}{7} - \frac{2\pi x(210\pi^2+160\pi^6-35)}{35}$,

Thus,

$G = [0 \;\; -30.6020 \;\; 0 \;\; -91.6143 \;\; 0 \;\; 5.0688e+4 \;\; 0]$,

$$W = \begin{bmatrix} 1.0965e+3 & 0 & 5.7218e+3 & 0 & 1.0240e+5 & 0 & 2.4321e+5 \\ 0 & 30.6020 & 0 & 91.6143 & 0 & -5.0688e+4 & 0 \\ 5.7218e+3 & 0 & 2.9920e+4 & 0 & 5.3574e+5 & 0 & 9.5004e+5 \\ 0 & 91.6143 & 0 & 3.5427e+3 & 0 & -1.3361e+5 & 0 \\ 1.0240e+5 & 0 & 5.3574e+5 & 0 & 9.5961e+6 & 0 & 1.5679e+7 \\ 0 & -5.0688e+4 & 0 & -1.3361e+5 & 0 & 8.4321e+7 & 0 \\ 2.4321e+5 & 0 & 9.5004e+5 & 0 & 1.5679e+7 & 0 & 1.7171e+9 \end{bmatrix}$$

For the given conditions $u(0) = 1$, $u'(0) = 1$ and $u''(0) = -1$, the augmented matrices are obtained respectively, as

$U_0 = [\,1 \;\; 0 \;\; -1 \;\; 0 \;\; 1 \;\; 0 \;\; -1\,]$, $U_1 = [\,0 \;\; 1 \;\; 0 \;\; -3 \;\; 0 \;\; 5 \;\; 0\,]$,

$U_2 = [\,0 \;\; 0 \;\; 4 \;\; 0 \;\; -16 \;\; 0 \;\; 36]$.

If we replace the last three rows of the matrices W and G by the values of U_0, U_1 and U_2 in, then

$$\tilde{G} = [0 \;\; -30.6020 \;\; 0 \;\; -91.6143 \;\; 1 \;\; 1 \;\; -1]',$$

$$\widetilde{W} = \begin{bmatrix} 1.0965e+3 & 0 & 5.7218e+3 & 0 & 1.0240e+5 & 0 & 2.4321e+5 \\ 0 & 30.6020 & 0 & 91.6143 & 0 & -5.0688e+4 & 0 \\ 5.7218e+3 & 0 & 2.9920e+4 & 0 & 5.3574e+5 & 0 & 9.5004e+5 \\ 0 & 91.6143 & 0 & 3.5427e+3 & 0 & -1.3361e+5 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 & -16 & 0 & 36 \end{bmatrix}$$

Thus, the Chebyshev coefficients are calculated as

$$A = \widetilde{W}^{-1}\tilde{G} = [\,0.7645 \;\; 0.9745 \;\; -0.2307 \;\; -0.0065 \;\; 0.0048 \;\; 0.0012 \;\; 0\,]'$$

Therefore, the approximate solution of the problem taking $m = 6$ is the exact solution under the given conditions as follows:

$a_0 = 0.7645, a_1 = 0.9745, a_2 = -0.2307, a_3 = -0.0065, a_4 = 0.0048$,

$a_5 = 0.0012, a_6 = 0$.

$$u_6(x) = (a_0 - a_2 + a_4 - a_6) + (a_1 - 3a_3 + 5a_5)x + (2a_2 - 8a_4 + 18a_6)x^2 + (4a_3 - 20a_5)x^3 + (8a_4 - 48a_6)x^4 + 16a_5x^5 + 32a_6x^6$$

$$u_6 = 1 + x - 0.4998x^2 - 0.05x^3 + 0.0384x^4 + 0.0192x^5$$

Secondly, an approximate solution $u(x)$ will be found using the combination of least-squares with the Legendre polynomial, which defined in the form

$$u_6(x) = \sum_{i=0}^{6} a_i p_i(x)$$

$R_0 = -x - 2\pi x$, $R_1 = \frac{-1}{2}x^2$, $R_2 = -\frac{x(x^2-1)}{2} - \pi x(\pi^2 - 1)$, $R_3 = 15 - \frac{x^2(5x^2-6)}{8}$,

$R_4 = 105x - \frac{x(7x^2-3)(x^2-1)}{8} - \frac{\pi x(7\pi^2-3)(\pi^2-1)}{4}$, $R_5 = \frac{945x^2}{2} - \frac{x^2(21x^4-35x^2+15)}{16} - \frac{105}{2}$

$$R_6 = \frac{3465x^3}{2} - \frac{945x}{2} - \frac{x(x^2-1)(33x^4-30x^2+5)}{16} - \frac{\pi x(\pi^2-1)(33\pi^4-30\pi^2+5)}{8}$$

Then,

$$G = [\,0 \quad -30.6020 \quad 0 \quad -68.7347 \quad 0 \quad 2.4916e+4 \quad 0\,]',$$

and,

$$W = \begin{bmatrix} 1.0965e+3 & 0 & 4.5655e+3 & 0 & 5.7943e+4 & 0 & 1.3621e+5 \\ 0 & 30.6020 & 0 & 68.7347 & 0 & -2.4916e+4 & 0 \\ 4.5655e+3 & 0 & 1.9044e+4 & 0 & 2.4184e+5 & 0 & 4.5943e+5 \\ 0 & 68.7374 & 0 & 1.4311e+3 & 0 & -4.9825e4 & 0 \\ 5.7943e+4 & 0 & 2.4184e+5 & 0 & 3.0722e+6 & 0 & 5.4218e+6 \\ 0 & -2.4916e+4 & 0 & -4.9825e+4 & 0 & 2.0379e+7 & 0 \\ 1.3621e+5 & 0 & 4.5943e+5 & 0 & 5.4218e+6 & 0 & 3.5385e+8 \end{bmatrix}$$

For the given conditions, the augmented matrices are obtained respectively, as

$U_0 = [\,1 \quad 0 \quad -1/2 \quad 0 \quad 3/8 \quad 0 \quad -5/16\,]$, $U_1 = [\,0 \quad 1 \quad 0 \quad -3/2 \quad 0 \quad 15/8 \quad 0\,]$
$U_2 = [\,0 \quad 0 \quad 3 \quad 0 \quad -15/2 \quad 0 \quad 105/8]$.

If we replace the last three rows of the matrices W and G by the values of U_0, U_1 and U_2 in, then

$$\tilde{G} = [\,0 \quad -30.6020 \quad 0 \quad 1 \quad 1 \quad -1\,]'$$

$$\widetilde{W}=\begin{bmatrix} 1.0965e+3 & 0 & 4.5655e+3 & 0 & 5.7943e+4 & 0 & 1.3621e+5 \\ 0 & 30.6020 & 0 & 68.7347 & 0 & -2.4916e+4 & 0 \\ 4.5655e+3 & 0 & 1.9044e+4 & 0 & 2.4184e+5 & 0 & 4.5943e+5 \\ 0 & 68.7374 & 0 & 1.4311e+3 & 0 & -4.9825e4 & 0 \\ 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} & 0 & -\frac{5}{16} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{15}{8} & 0 \\ 0 & 0 & 3 & 0 & -\frac{15}{2} & 0 & \frac{105}{8} \end{bmatrix}$$

$$A=\widetilde{W}^{-1}\tilde{G}=[\,0.8411\;\;0.9782\;\;-0.3113\;\;-0.0115\;\;0.0087\;\;0.0024\;\;-0.0001]'$$

Therefore, the approximate solution of the problem taking $m=6$ is the exact solution under the given conditions as follows:

$a_0 = 0.8411, a_1 = 0.9782\,, a_2 = -0.3113\,, a_3 = -0.0115\,, a_4 = 0.0087$,

$a_5 = 0.0024, a_6 = -0\,.0001$

$$u_6(x)=\left(a_0-\frac{a_2}{2}+\frac{3a_4}{8}-\frac{5a_6}{16}\right)+\left(a_1-\frac{3a_3}{2}+\frac{15a_5}{8}\right)x+\left(\frac{3a_2}{2}-\frac{15a_4}{4}+\frac{105a_6}{16}\right)x^2$$
$$+\left(\frac{5a_3}{2}-\frac{35a_5}{4}\right)x^3+\left(\frac{35a_4}{8}-\frac{315a_6}{16}\right)x^4+\left(\frac{63a_5}{8}\right)x^5+\left(\frac{231a_6}{16}\right)x^6$$

Finally, we get the approximate solution:-

$$u_6(x)=1+0.999x-0.5002x^2-0.0497x^3+0.04x^4+0.0189x^5-0.0014x^6$$

X	**Exact solution**	standard LST				Method[1]	
		Legendre Poly		Chebyshev Poly		Legendre Poly	Chebyshev Poly
		N=6	Error	N=6	Error	Error	Error
0	1	1	0	1	0	0	0
0.1	1.095004165278	1.0949424876	6.1678e-5	1.094956032	4.8133e-5	0.00486	0.00497
0.2	1.1800665778412	4435841.1796	4.2222e-4	1.179675584	3.9099e-4	0.01969	0.01988
0.3	1.2553364891256	1.2539790064	0.0014	1.254025696	0.0013	0.04452	0.04460
0.4	1.3210609940029	1.3179590016	0.0031	1.318011648	0.0030	0.07865	0.07889
0.5	1.3775825618904	1.37175625	0.0058	1.3718	0.0058	0.1222	0.1224
0.6	1.4253356149097	1.4157211456	0.0096	1.415741632	0.0096	0.1730	0.1746
0.7	1.4648421872845	1.4504007144	0.0144	1.450394784	0.0144	0.2346	0.2351
0.8	1.4967067093472	1.4765557504	0.0202	1.476548096	0.0202	0.3027	0.3033
0.9	1.5216099682707	1.4951769436	0.0264	1.495243648	0.0264	0.3774	0.3783
1	1.5403	1.5075	0.0328	1.5078	0.0325	0.4550	0.4597

Table 1. The Numerical results for Example 2

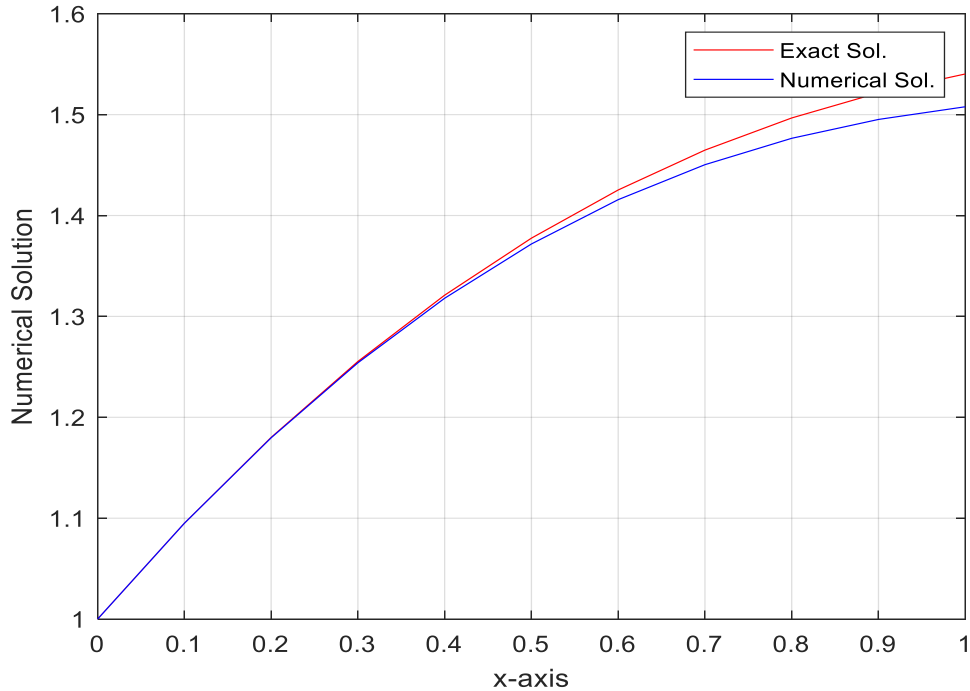

Figure 2.1: Chebyshev -Least-Squares Technique

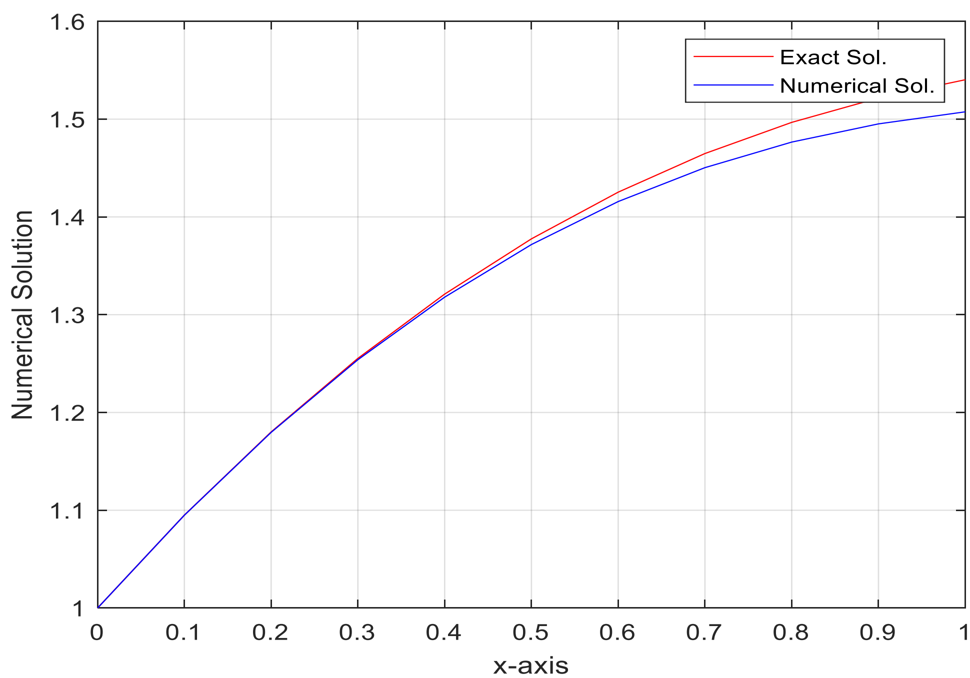

Figure 2.2: Legendre-Least-Squares Technique

6. Conclusions

In this paper, we have studied Chebyshev and Legendre polynomials. Then, the solution of higher orders linear FVDEs of the second and third type using LST method it is considered polynomial as basic functions. We found that the combination of Chebyshev and Legendre polynomials with LST method is better than [1] and through the obtained Absolute error shown that the accuracy and efficient method. Furthermore, Chebyshev method is better than Legendre Polynomials in absolute error.

References

[1] A. Abubakar, and O. Taiwo, Integral collocation Approximation method for the Numerical solution of High-order Linear Fredholm-Volterra Integro Differential Equations, American J. of comput. and Appl. math.,4(4) (2014),111- 117

[2] A. Akyuz, and M. Sezer, Chebyshev polynomial solutions of system of High order Linear Differential Equations with variable coefficients, Appl. Math. Comput. 144(2003),237-247.

[3] K. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press, 1997

[4] N. Baykuş, and M. Sezer, Solution of high-order linear Fredholm integro-differential equations with piecewise intervals, Numer. Methods Partial Differential Equations, 27(5) (2009), 327-1339.

[5] N. Bildik, A. Konuralp, and S. Yalçınbaş, Comparison of Legendre polynomial approximation and variational iteration method for the solutions of general linear Fredholm integro- differential equations, Comput. and Math. with Appl. 5(2010), 1909-1917.

[6] C. Canuto, M. Y. Hussaini, A. Quarteroni and T.A. Zang, Spectral Methods on Fluid Dynamics, Springer-Verlag, 1988.

[7] L.M. Deves, J.L. Mohmed, Computational methods for Integral Equations. Cambridge University press. Cambridge, 1985.

[8] R. Ezzati, and S. Najafalizadeh, Numerical solution of Volterra-Fredholm Integral Equation by Using Chebysher polynomials. Math. Sciences, 5(1) (2011), 1-12.

[9] S.M. Hosseini, and S. Shahmorad, Numerical solution of a class of integro- differential equations by the tau method with an error estimation, Appl. Math. Comput. 136(2003), 559–570.

[10] A. Karamete, and M. Sezer, A Taylor collocation method for the solution of Linear Integro – Differentioal Equations, int. J. Comput. Math. 79(9) (2002), 987-1000.

[11] N. Kurt, and M. Sezer, Polynomial solution of high-order linear Fredholm integro - differential equations with constant coefficients, J. Frankin Inst. 345(2008), 839– 850.

[12] K. Maleknejad, and N. Aghazadeh, Numerical solutions of Volterra integral equations of the second kind with convolution kernel by using Taylor –series expansion method, Appl. Math. Comput. 161(3) (2005), 915-922

[13] E.L. Ortiz, and H. Samara, An operational approach to the tau method for the numerical solution of nonlinear differential equations. Comput. Springer -Verlag 27(1) (1981), 15-25.

[14] J. Pour-Mahmoud, M.Y. Rahimi-Ardabili, and S. Shahmorad, Numerical solution of the system of Fredholm integro-differential equations by the tau method, Appl. Math. Comput. 168(2005), 465–478.

[15] M. Razzaghi, and S. Yousefi, Legendre wavelets method for the nonlinear Volterra Fredholm integral equations, Math. Comput. Simul. 70(2005), 1–8.

[16] Y.Ren, B. Zhang, and H. Qiao, A Simple Taylor – series expansion method for a class of second kind integral, J. Comput. Math. 110(1999), 15-24.

[17] Y. Salih, and S. Mehmet, The Approximate solution of Higher order Linear Volterra – Fredholm Integro- Differential Equations in Term of Taylor polynomials.Appl. Math. Comput. 112(2000), 291-308.

[18] M. Sezer, and M. Gulsu, Polynomial solution of the most general linear Fredholm integro-differential-difference equation by means of Taylor matrix method, Int. J. Complex Var. 50(5)(2005), 367–382.

[19] S. Yalçınbaş, M. Sezer, and H.H. Sorkun, Legendre polynomial solutions of high-order linear Fredholm integro-differential equations, Appl. Math. Comput. 210(2009), 334-349.

[20] Ş. Yüzbaşı, N. Şahin, and M. Sezer, A Bessel collocation method for numerical solution of generalized pantograph equations, Numer. Methods Partial Differential Equations, 28(4) (2011), 1105-1123.

[21] Ş. Yüzbaşı, N. Şahin, and M. Sezer, A Bessel polynomial approach for solving linear neutral delay differential equations with variable coefficients, J. Adv. Res. Differential Equations 3(2011), 81–101.

[22] Ş. Yüzbaşı, and M. Sezer, A collocation approach to solve a class ofLane– Emden type equations, J. Adv. Res. Appl. Math. 3 (2) (2011), 58–73.

[23] C. Zuppa, Error estimates for moving least square approximations, Bull.
Braz. Math. Soc. New Series 34 (2) (2003), 231-249.