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Spectral Shifted Jacobi-Gauss-Lobatto Methodology for Solving Two-Dimensional Time-Space Fractional Bioheat Model

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Abstract. In this paper, we extend use the spectral scheme by including a new application for solving the two-dimensional time-space fractional bioheat equation (T-SFBHE) with initial/Neumann boundary conditions. To achieve this goal, we suggest the numerical algorithm that depend on shifted Jacobi-Gauss-Lobatto polynomials (SJ-GL-Ps) together with Jacobi-Gauss-Lobatto points to calculate the approximate derivatives of any order (fractional/ordinary) in the matrix form. the proposed technique of the two examples is applied in order to evidence its utility and precision. The numerical results designate that the utilized approach is very effectual and gives high accuracy and good convergence by using a few grid points.

INTRODUCTION

In lastly decades, fractional calculus of any arbitrary order, which includes essential principles of the derivatives or integrals of the fractional-order, has been utilized for describing many phenomena in engineering, physics and control its results accurately in such different fields as diffusion problems, viscoelasticity, mechanics of solids, biomedical engineering, control theory, and economics, etc.[35].

Spectral methods are powerful mathematical techniques and very useful to find numerical solutions of the differential equations of fractional/integer-orders. This efficiency comes about because of the spectral weighting coefficients, typically what approaching to zero quicker than any else algebraic power, indicating that exponential/super-exponential convergence [13].

A lot of studies exist that used spectral techniques for different types of applications that utilized the various formula for solving fractional differential equations that are that contributed to a significant rate of the ongoing research these days. For example, Pedas and Tamme [31] and Ghoreishi and Yazdani [20] in (2011), used the spline collocation methods depend on the Lagrange fundamental polynomials for solving linear differential equations in the multi-term fractional-order; studied the spectral Tau technique to provide an efficient numerical solutions construct on Chebyshev and Legendre polynomials of differential equations multi-term fractional-order. Bhrawy and Alghamdi [6] in (2012), utilized collocation methods to develop a shifted Jacobi-Gauss-Lobatto polynomials for solving the nonlinear-fractional Langevin equation. Khader [25] and Bhrawy et al. [8] in (2013), employed the spectral and Tau methods to find the numerical solutions for the differential equations of the fractional-order by utilizing the generalized Laguerre polynomials. Tian et al. [38] and Saeed and Rehman [36] in (2014), applied the Legendre/Chebyshev-Gauss-Labatto polynomials in the matrix form for the solution the advection-diffusion equation of fractional-order depending on spectral collocation method; and proposed the Hermite wavelet polynomials for solving linear/nonlinear-fractional delay differential equations. Bhrawy [7] in (2015), used spectral collocation scheme to propose new suitable techniques for solving the nonlinear-fractional sub-diffusion/reaction sub-diffusion equations based basically on the SJPs. Thamareerat et al. [39] and Alshbool and Hashim [3] in (2016), employed moving Kriging interpolation to find the numerical solutions of time fractional Navier-Stokes problems

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by using Petrov-Galerkin method: suggested a new adjustment based on the Bernstein polynomials for solving Riccati differential equation and stiff systems of the fractional-order. Rahimkhani et al. [33] and Isah and Phang [23] in (2017), suggested the operational matrix by using Bernoulli wavelet polynomials to find approximate solutions for fractional delay differential equations; introduced collocation method to proposed a new operational method based on Genocchi polynomials for solving nonlinear fractional differential equations. Agarwal and El-Sayed [1] and Bahmanpour et al. [4] in (2018), utilized both them collocation method builds on the shifted Chebyshev polynomials (SCPs) of the second kind, Müntz Legendre polynomials and Jacobi polynomials as a test function for solving models of any arbitrary fractional-order. Ali et al. [2] in (2019), applied spectral collocation formula on suggest solving the fractional-order delay-differential equations depending on Chebyshev operational form.

Pennes' suggested in (1948) the essential structure of the mathematical designing that describes temperature propagation in human tissues, the model known as the bioheat equation remains extensively used in the hyperthermal and freezing treatments [19]. The fractional bioheat model which extracted the focus of the researchers and these contributed to a significant amount of the researches based on approximate and analytic methodology, for example (Singh et al. [37] in (2011), finite difference and homotopy perturbation method, Jiang and Qi [24] in (2012), Taylor's series expansion, Damor et al. [10] in (2013), implicit finite difference method, Ezzat et al. [18] in (2014), Laplace transform mode, Ferrás et al. [19] and Kumar et al. [27] in (2015), implicit finite difference method, "backward finite difference method" and "Legendre wavelet Galerkin scheme", Qin and Wu [32] and Damor et al. [11] in (2016), quadratic spline collocation method and Fourier-Laplace transforms, Kumar and Rai [26] in (2017), finite element based on Legendre wavelet Galerkin method, Roohi et al. [34] in (2018), Galerkin scheme, Hosseininia et al. [21] in (2019), "Legendre wavelet method").

In this work, we employed the SJ-GL-Ps in the matrix form to present the numerical approach for solving two-dimensional T-SFBHE.

This article is organized in that the governing equation of distribution of the temperature in the biological tissue will be appeared in the next section. Some definitions about the essentials principles of the fractional calculus will be shown (Section 3), followed by the shifted Jacobi polynomials operational matrix for ordinary derivatives and their fractional derivatives (Section 4). The approximation of 1D, 2D and 3D temperature functions in matrix form dependent on shifted Jacobi polynomials for fractional differentiation is in (Section 5) to establish a numerical solution for T-SFBHE. After that, a method for solution is explained in (Section 6) and to determine an error bound $T(x,y,t)$ is called for in Section 7 and an efficient error estimation for the SJ-GL-Ps will be given in Section 8. final Section 9, deals with the numerical results for the T-SFBHE.

GOVERNING EQUATION

The time-space fractional version of the two-dimensional unsteady state Pennes bioheat model can be obtained by replacing the thermal derivative with a derivative of arbitrary positive real order $\gamma \in (0,1]$ and second order space derivative by Riesz-Feller derivative of fractional arbitrary positive real order $v_1, v_2 \in (1,2]$. The T-SFBHE is given according to

$$\rho c \frac{\partial^{\gamma} T(x,y,t)}{\partial t^{\gamma}} - K\left(\frac{\partial^{v_1} T(x,y,t)}{\partial x^{v_1}} + \frac{\partial^{v_2} T(x,y,t)}{\partial y^{v_2}}\right) + W_b\, c_b\left(T(x,y,t) - T_a\right) = Q_{ext}(x,y,t) + Q_{met}, \quad 0 \le t \le \mathcal{T}, 0 \le x \le \mathcal{R}_1, 0 \le y \le \mathcal{R}_2 \tag{1}$$

with initial and boundary conditions

$$T(x,y,0) = T_c, \qquad 0 < x < \mathcal{R}_1,\ 0 < y < \mathcal{R}_2 \tag{2}$$

$$-K\frac{\partial T(0,y,t)}{\partial x} = q_0, \quad 0 \le y \le \mathcal{R}_2, t > 0 \tag{3}$$

$$-K\frac{\partial T(x,0,t)}{\partial y} = q_1, \quad 0 \le x \le \mathcal{R}_1, t > 0 \tag{4}$$

$$-K\frac{\partial T(\mathcal{R}_1,y,t)}{\partial x} = 0, \quad 0 \le y \le \mathcal{R}_2, t > 0 \tag{5}$$

$$-K\frac{\partial T(x,\mathcal{R}_2,t)}{\partial y} = 0, \quad 0 \le x \le \mathcal{R}_1, t > 0 \tag{6}$$

where $\rho, c, K, T, t, x, y, T_a, q_0, W_b = \rho_b w_b, Q_{ext}$ and Q_{met} symbolizes density, specific heat, thermal conductivity, temperature, time, distances with x, y, artillery temperature, heat flux on the skin surface, blood exudation rate, metabolic heat obstetrics in lacing tissue and external heat exporter in skin tissue respectively. The units and value of the symbolizations that expressed in the above equation are tabulating in Table1 [12].

TABLE 1. The units and values utilized in this paper of the two dimentional T-SFBHE.

Symbolizations	T	T_a	t	x	ρ and ρ_b	c and c_b	K	ω_b	Q_{met}
Units	°C	°C	s	m	kg/m^3	J/kg °C	W/m °C	m^3/s/ m^3	W/m^3
values		37			1000	4000	0.5	0.0005	420

PRELIMINARIES AND NOTATIONS

This section is basically restricted to the principles essentials of the fractional calculus theory that will be used in this article.

3.1 Definition ([9], [26], [34]): *The Riemann-Liouville fractional integral of order $v > 0$ defined as:*

$$I^{v}\varphi(x) = \frac{1}{\Gamma(v)}\int_{0}^{x}(x-s)^{v-1}\varphi(s)\,ds, \qquad v > 0,$$
$$I^{0}\varphi(x) = \varphi(x) \tag{7}$$

3.2 Definition ([9], [26], [34]): *The Caputo definition of fractional differential operator defined as:*

$$D^{v}\varphi(x) = \begin{cases} \frac{1}{\Gamma(n-v)} \int_0^x \frac{\varphi^{(n)}(s)}{(x-s)^{v-n+1}}\,ds, & n-1 \le v < n, \\ \frac{d^n\varphi(x)}{dx^n} & , v = n. \end{cases} \tag{8}$$

The relation that governing the Riemann-Liouville and Caputo of fractional order given via the forms [37]:

$$D^{v}I^{v}\varphi(x) = \varphi(x),$$
$$I^{v}D^{v}\varphi(x) = \varphi(x) - \sum_{k=0}^{n-1}\varphi^{(k)}(0^{+})\frac{x^k}{k!} \tag{9}$$

For $\beta \ge 0, v \ge -1$, and constant C, Caputo fractional derivative has some fundamental properties which are needed here as follows [22]:

$$i)\ D^{v}C = 0,$$
$$ii)\ D^{v}\,x^{\beta} = \begin{cases} 0 & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil v \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-v+1)}x^{\beta-v}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \ge \lceil v \rceil \end{cases} \tag{10}$$
$$iii)\ D^{v}\left(\sum_{i=0}^{n} c_i\,\varphi_i(x)\right) = \sum_{i=0}^{n} c_i D^{v}\varphi_i(x), \quad \text{where } \{c_i\}_{i=0}^{n} \text{ are constant}$$

3.3 Definition [22]: *(generalized Taylor's formula). Suppose that $D^{iv}\varphi(t) \in \mathbb{C}(0,1)$ for $i = 0(1)(n-1)$, then one has*

$$\varphi(x) = \sum_{i=0}^{n-1}\frac{x^{iv}}{\Gamma(iv+1)}D^{iv}\varphi(0^{+}) + \frac{x^{nv}}{\Gamma(nv+1)}D^{nv}\varphi(\xi) \tag{11}$$

where $0 < \xi \le x$, $\forall x \in (0,\mathcal{R})$. Also, one has assume

$$\left|\varphi(x) - \sum_{i=0}^{n-1}\frac{x^{iv}}{\Gamma(iv+1)}D^{iv}\varphi(0^{+})\right| \le M_v\ \frac{x^{nv}}{\Gamma(nv+1)} \tag{12}$$

and $M_v \ge |D^{nv}\varphi(\xi)|$.

In case $v = 1$, the generalized Taylor's formula in Eq. (12) is the classical Taylors formula.

SHIFTED JACOBI POLYNOMIALS AND FRACTIONAL DERIVATIVES

The Jacobi polynomials which are orthogonal in the interval $[-1, 1]$ are defined as the following formula

$$P_i^{(\vartheta,\theta)}(t) = \frac{(i+\vartheta+\theta-1)\{\vartheta^2-\theta^2+t(2i+\vartheta+\theta)(2i+\vartheta+\theta-2)\}}{2i(i+\vartheta+\theta)(2i+\vartheta+\theta-2)} P_{i-1}^{(\vartheta,\theta))}(t) - \frac{(i+\vartheta-1)(i+\theta-1)(2i+\vartheta+\theta)}{i(i+\vartheta+\theta)(2i+\vartheta+\theta-2)} P_{i-2}^{(\vartheta,\theta)}(t),\ n = 2,3,\dots \quad (13)$$

where $P_0^{(\vartheta,\theta)}(t) = 1,\ P_1^{(\vartheta,\theta)}(t) = \frac{1}{2}[(\vartheta+\theta+2)t+(\vartheta-\theta)]$.

For transform Jacobi polynomials on a region $0 \le x \le \mathcal{R}$, one can work the replace of variables $t = \frac{2x}{\mathcal{R}} - 1$ in the above formula. Therefore, the shifted Jacobi polynomials (SJPs) are constructed in the relation as follows [7]

$P_i^{(\vartheta,\theta)}(\frac{2x}{\mathcal{R}} - 1) = P_{\mathcal{R},i}^{(\vartheta,\theta}(x), x \in [0,\mathcal{R}]$.

The analytic form of the SJPs $P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)$ of degree i is given by:

$$P_{\mathcal{R},i}^{(\vartheta,\theta)}(x) \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+k+\vartheta+\theta+1)\Gamma(i+\theta+1)}{\mathcal{R}^k\Gamma(i+\vartheta+\theta+1)\Gamma(k+\theta+1)(i-k)!k!} x^k, i \in N, \quad (14)$$

where $P_{\mathcal{R},i}^{(\vartheta,\theta)}(0) = (-1)^i \frac{\Gamma(i+\theta+1)}{\Gamma(\theta+1)i!}, P_{\mathcal{R},i}^{(\vartheta,\theta)}(\mathcal{R}) = \frac{\Gamma(\vartheta+i+1)}{i!\Gamma(\vartheta+1)}$.

From the SJPs, we can be obtain the formula that most utilized are the "shifted Legendre polynomials" (SLPs) $L_i(x)$; the "shifted Chebyshev polynomials" (SCPs) of the first kind $T_{\mathcal{R},i}(x)$; the SCPs of the second kind $U_{\mathcal{R},i}(x)$; the nonsymmetric SJPs, the two important special cases of SCPs of third(fourth) kinds $V_{\mathcal{R},i}(x)$ and $W_{\mathcal{R},i}(x)$; and also, the symmetric SJPs that called "Gegenbauer (ultraspherical) polynomials" $C_{\mathcal{R},i}^{\vartheta}(x)$. These orthogonal polynomials are interrelated to the SJPs by the following relations [16]

$$L_i(x) = P_{\mathcal{R},i}^{(0,0)}(x), \qquad T_{\mathcal{R},i}(x) = \frac{i!\Gamma(\frac{1}{2})}{\Gamma(i+\frac{1}{2})} P_{\mathcal{R},i}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)$$
$$U_{\mathcal{R},i}(x) = \frac{(i+1)!\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(i+\frac{3}{2}\right)} P_{\mathcal{R},i}^{\left(\frac{1}{2},\frac{1}{2}\right)}(x),\ V_{\mathcal{R},i}(x) = \frac{(2i)!!}{(2i-1)!!} P_{\mathcal{R},i}^{\left(\frac{1}{2},\frac{-1}{2}\right)}(x) \quad (15)$$
$$W_{\mathcal{R},i}(x) = \frac{(2i)!!}{(2i-1)!!} P_{\mathcal{R},i}^{(-\frac{1}{2},\frac{1}{2})}(x),\ C_{\mathcal{R},i}^{\vartheta}(x) = \frac{i!\Gamma(\vartheta+\frac{1}{2})}{\Gamma(i+\vartheta+\frac{1}{2})} P_{\mathcal{R},i}^{(\vartheta-\frac{1}{2},\theta-\frac{1}{2})}(x)$$

The orthogonal property of SJPs is given by

$$\int_0^{\mathcal{R}} P_{\mathcal{R},i}^{(\vartheta,\theta)}(x) P_{\mathcal{R},k}^{(\vartheta,\theta)}(x) \omega_{\mathcal{R},i}^{(\vartheta,\theta)} dx = \mathfrak{h}_{\mathcal{R},k} \quad (16)$$

where $\omega_{\mathcal{R},i}^{(\vartheta,\theta)} = x^\theta(\mathcal{R}-x)^\vartheta$ and $\mathfrak{h}_{\mathcal{R},k} = \begin{cases} \frac{\mathcal{R}^{\vartheta+\theta+1}}{k!(2k+\vartheta+\theta+1)} \frac{\Gamma(k+\vartheta+1)\Gamma(k+\theta+1)}{\Gamma(k+\vartheta+\theta+1)}, & i = k, \\ 0, & i \neq k. \end{cases}$

The fractional derivative of the vector $\emptyset(x) = \left[P_{\mathcal{R},0}^{(\vartheta,\theta)}(x), P_{\mathcal{R},1}^{(\vartheta,\theta)}(x), \dots, P_{\mathcal{R},N}^{(\vartheta,\theta)}(x)\right]'$ can be discusses in the lemma and theorem as following

4.1 Lemma:- *Let $P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)$ be the SJPs. Then $\mathcal{D}^v P_{\mathcal{R},i}^{(\vartheta,\theta)}(x) = 0,\ i = 0,1,2,\dots,\lceil v\rceil - 1, v > 0$.*

Proof:- Using properties (ii) and (iii) of the Eq. (14) into Eq. (16) lead us to prove the lemma.

□

The following theorem is generalizing the operational matrix of derivatives form an arbitrary fractional order based on SJPs that have given as

4.2 Theorem [15]:- *Let $\emptyset(x)$ be shifted Jacobi vector defined in $\emptyset(x) = \left[P_{\mathcal{R},0}^{(\vartheta,\theta)}(x), P_{\mathcal{R},1}^{(\vartheta,\theta)}(x), \dots, P_{\mathcal{R},N}^{(\vartheta,\theta)}(x)\right]'$ and assume also, $v > 0$. Then*

$$\mathcal{D}^v\emptyset(x) \cong \mathcal{D}_{\mathcal{R}}^{v}\emptyset(x) \quad (17)$$

where $\mathcal{D}_{\mathcal{R}}^{(v)4}$ is the $(N+1)\times(N+1)$ shifted Jacobi operational matrix of derivatives of order v in the Caputo formula and is defined by:

$$\mathcal{D}_{\mathcal{R}}^{(v)}=\begin{bmatrix} & 0 \quad 0 \quad 0 & & \cdots & 0 \\ & \vdots \quad \vdots \quad \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \mathfrak{D}^{v}(\lceil v\rceil,0) & \mathfrak{D}^{v}(\lceil v\rceil,1) & \mathfrak{D}^{v}(\lceil v\rceil,2) & \cdots & \mathfrak{D}^{v}(\lceil v\rceil,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathfrak{D}^{v}(i,0) & \mathfrak{D}^{v}(i,1) & \mathfrak{D}^{v}(i,2) & \cdots & \mathfrak{D}^{v}(i,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathfrak{D}^{v}(N,0) & \mathfrak{D}^{v}(N,1) & \mathfrak{D}^{v}(N,2) & \cdots & \mathfrak{D}^{v}(N,N) \end{bmatrix}$$

where

$$\mathfrak{D}^{v}(i,j)=\sum_{k=\lceil v\rceil}^{i}\delta_{ijk} \tag{18}$$

and δ_{ijk} is given by$\times \sum_{l=0}^{j}\frac{(-1)^{j-l}\Gamma(j+l+\vartheta+\theta+1)\Gamma(\vartheta+1)\Gamma(l+k+\theta-v+1)}{\Gamma(l+\theta+1)\Gamma(l+k+\vartheta+\theta-v+2)(j-l)!l!}$ (19)

Proof. Applying Eq. (20) into Eq. (42) (the SJPs $P_{R,i}^{(\vartheta,\theta)}(x)$ of degree) we obtain

$$\mathcal{D}^{v}P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)=\sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(i+\theta+1)\Gamma(i+k+\vartheta+\theta+1)}{\mathcal{R}^{k}k!\,\Gamma(k+\theta+1)\Gamma(i+\vartheta+\theta+1)(i-k)!}\,D^{v}x^{k}$$

$$=\sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+\theta+1)\Gamma(i+k+\vartheta+\theta+1)}{\mathcal{R}^{k}\Gamma(k+\theta+1)\Gamma(i+\vartheta+\theta+1)(i-k)!\,\Gamma(k-v+1)}x^{k-v},\, i=\lceil v\rceil,\lceil v\rceil+1,\ldots \tag{20}$$

Now, approximate x^{k-v} by $(N+1)$ terms of shifted Jacobi series, we get

$$x^{k-v}\approx\sum_{j=0}^{N}\mathfrak{b}_{kj}\,P_{\mathcal{R},j}^{(\vartheta,\theta)}(x) \tag{21}$$

where the coefficients $\mathfrak{b}_{kj}$ can be obtain as following

$$\mathfrak{b}_{kj}=\frac{1}{\mathfrak{h}_{\mathcal{R},j}}\int_{0}^{\mathcal{R}}x^{k-v}\,P_{\mathcal{R},j}^{(\vartheta,\theta)}(x)x^{\theta}(\mathcal{R}-x)^{\vartheta}dx$$

$$=\frac{1}{\mathfrak{h}_{\mathcal{R},j}}\int_{0}^{\mathcal{R}}x^{k-v}\,x^{\theta}(\mathcal{R}-x)^{\vartheta}\times\sum_{l=0}^{j}(-1)^{j-l}\frac{\Gamma(j+\theta+1)\Gamma(j+l+\vartheta+\theta+1)}{\mathcal{R}^{l}\Gamma(l+\theta+1)\Gamma(j+\vartheta+\theta+1)(j-l)!l!}x^{l}\,dx$$

$$=\frac{1}{h_{\mathcal{R},j}}\sum_{l=0}^{j}\frac{(-1)^{j-l}\Gamma(j+\theta+1)\Gamma(j+l+\vartheta+\theta+1)}{\mathcal{R}^{l}\Gamma(l+\theta+1)\Gamma(j+\vartheta+\theta+1)(j-l)!l!}\times\int_{0}^{\mathcal{R}}x^{\theta+l+k-v}\,(\mathcal{R}-x)^{\vartheta}dx \tag{22}$$

$$=\frac{1}{\mathfrak{h}_{\mathcal{R},j}}\sum_{l=0}^{j}\left[(-1)^{j-l}\frac{\Gamma(j+\theta+1)\Gamma(j+l+\vartheta+\theta+1)}{\mathcal{R}^{l}\Gamma(l+\theta+1)\Gamma(j+\vartheta+\theta+1)(j-l)!\,l!}\times\frac{\Gamma(\vartheta+1)\Gamma(\theta+l+k-v+1)}{\Gamma(\theta+l+k-v+2)}\mathcal{R}^{\vartheta+\theta+k+l-v+1}\right]$$

$$=\frac{\Gamma(j+\theta+1)\mathcal{R}^{\vartheta+\theta+k-v+1}}{\Gamma(j+\vartheta+\theta+1)h_{\mathcal{R},i}}\times\sum_{l=0}^{j}(-1)^{j-l}\frac{\Gamma(\vartheta+1)\Gamma(j+l+\vartheta+\theta+1)\Gamma(\theta+l+k-v+1)}{\Gamma(l+\theta+1)\Gamma(\vartheta+\theta+l+k-v+2)(j-k)!\,l!}$$

Now, substituting Eq. (25) into Eq. (27), we observe that

$$\mathcal{D}^{v}P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)=\sum_{j=0}^{N}\mathfrak{D}^{v}(i,j)P_{\mathcal{R},j}^{(\vartheta,\theta)}(x)\,,\; i=\lceil v\rceil,\lceil v\rceil+1,\ldots,N \tag{23}$$

where $\mathfrak{D}^{v}(i,j)$ is given in Eq. (26). We can write the Eq. (50) in a vector form as:

$$\mathcal{D}^{v}P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)\approx[\mathfrak{D}^{v}(i,0),\mathfrak{D}^{v}(i,1),\ldots,\mathfrak{D}^{v}(i,N)]\emptyset(x),\, i=\lceil v\rceil(1)N \tag{24}$$

Also from Lemma 4.1, we have

$$\mathcal{D}^{v}P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)\approx[0,0,\ldots,0]\emptyset(x),\, i=0(1)(\lceil v\rceil-1),\, v>0. \tag{25}$$

By a combination of the Eqs. (25-26), we obtain the desired result.

Notes that if $v=n\in N$, then theorem 4.2 gives the formula $\mathcal{D}^{(n)}=\left(\mathcal{D}^{(1)}\right)^{n}$, $n=1,2,\ldots$

4.3 Corollary [35]:- *If $\vartheta=\theta=0$ and $\mathcal{R}=1$, then δ_{ijk} is given as follows:*

$$\delta_{ijk}=\frac{(-1)^{i-k}\Gamma(j+1)\Gamma(i+1)\Gamma(i+k+1)}{\mathfrak{h}_{\mathcal{R},j}(i-k)!\,\Gamma(j+1)\Gamma(k+1)\Gamma(i+1)\Gamma(k-v+1)}\times\sum_{l=0}^{j}\frac{(-1)^{j-l}\Gamma(j+l+1)\Gamma(l+k-v+1)}{(j-l)!\,l!\,\Gamma(l+1)\Gamma(l+k-v+2)}$$

By the aid of properties of the SJPs with simplification, we get

$$\delta_{ijk}=\lambda_{ijk}=(2j+1)\sum_{l=0}^{j}\frac{(-1)^{i+j+k+l}(j+l)!\,(i+k)!}{(l+k-v+1)(i-k)!\,k!\,(j-l)!\,(l!)^{2}\Gamma(k-v+1)}$$

Then one can easily demonstrate that

$$\mathfrak{D}^{v}(i,j) = \sum_{k=\lceil v \rceil}^{i} \lambda_{ijk}$$

where λ_{ijk} is given as in [35]. It is clear that the SJPs for derivatives in the matrix form for fractional calculus with $\vartheta = \theta = 0$, is in complete agreement with the SLPs for derivatives in the matrix form for fractional calculus [35].

4.4 Corollary [14]:- *If* $\vartheta = \theta = -\frac{1}{2}$ *then* δ_{ijk} *given as follows:*

$$\delta_{ijk} = \frac{(-1)^{i-k}\mathcal{R}^{-v}\Gamma(j+\frac{1}{2})\Gamma(i+\frac{1}{2})\Gamma(i+k)}{\epsilon_{R,j}\Gamma(j)\Gamma(\frac{1}{2}+k)\Gamma(i)\Gamma(k-v+1)(i-k)!} \times \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+l)\Gamma(l+k-v+\frac{1}{2})}{\Gamma(l+\frac{1}{2})\Gamma(l+k-v+1)(j-l)!\,l!}$$

by the aid of properties of the SJPs with simplification, we have

$$\delta_{ijk} = \lambda_{ijk} = \frac{(-1)^{i-k}2i(i+k-1)!\,\Gamma(k-v+\frac{1}{2})}{\epsilon_{R,j}\mathcal{R}^{v}\Gamma(k+\frac{1}{2})(i-k)!\,\Gamma(k-v-j+1)\Gamma(k+j-v+1)}, j = 0(1)N.$$

Then one can easily elucidate that

$$\mathfrak{D}^{v}(i,j) = \sum_{k=\lceil v \rceil}^{i} \varphi_{ijk}$$

where λ_{ijk} and $\epsilon_{R,j}$ are given as in [14]. It is clear that the SJPs for derivatives in the matrix form for any arbitrary fractional order with $\vartheta = \theta = -\frac{1}{2}$, is complete accord with the SCPs for derivatives in the matrix form for fractional calculus obtained by [14].

SHIFTED JACOBI OPERATIONAL MATRIX OF FRACTIONAL DIFFERENTIATION

A temperature function $T(x)$ defined for $0 \le x \le \mathcal{R}$ may be expressed in terms of the SJPs as

$$T(x) = \sum_{i=0}^{\infty} c_i\, P_{\mathcal{R},i}^{(\vartheta,\theta)}(x) \tag{26}$$

where the coefficients c_i are given by

$$c_i = \frac{1}{\mathfrak{h}_{\mathcal{R},i}} \int_0^{\mathcal{R}} T(x)\, P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)\omega_{\mathcal{R},i}^{(\vartheta,\theta)}(x)dx\ , i = 0,1,2,\dots \tag{27}$$

In practice, consider the $(N+1)$-term SJPs so that

$$T(x) \approx \sum_{i=0}^{N} c_i\, P_{\mathcal{R},i}^{(\vartheta,\theta)}(x) = C'\emptyset(x) \tag{28}$$

where the shifted Jacobi coefficient vector c_i and the shifted Jacobi vector $\emptyset(x)$ are given by $C' = [c_0, c_1, \dots, c_N], \emptyset(x) = [P_{\mathcal{R},0}^{(\vartheta,\theta)}(x), P_{\mathcal{R},1}^{(\vartheta,\theta)}(x), \dots, P_{\mathcal{R},N}^{(\vartheta,\theta)}(x)]'$.

By extending the above property in two variable functions, we can approximate a two variable function $T(x,y)$ defined for $0 \le x \le \mathcal{R}$ and $0 \le t \le \mathcal{T}$ dependent on double SJPs as

$$T(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\, P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)P_{\mathcal{T},j}^{(\vartheta,\theta)}(t) \tag{29}$$

where

$$a_{ij} = \frac{1}{\mathfrak{h}_{\mathcal{R},i}\mathfrak{h}_{\mathcal{T},j}} \int_0^{\mathcal{R}}\int_0^{\mathcal{T}} T(x,t)\, P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)P_{\mathcal{T},j}^{(\vartheta,\theta)}(t)\omega^{(\vartheta,\theta)}(x,t)dtdx \tag{30}$$

such that $\omega^{(\vartheta,\theta)}(x,t) = \omega_{\mathcal{R},i}^{(\vartheta,\theta)}(x)\ \omega_{\mathcal{T},j}^{(\vartheta,\theta)}(t)$.

In practice, consider the $(N+1)$ and $(M+1)$ terms double SJPs with respect to x, t so that

$$T_{N,M}(x,t) \approx \sum_{i=0}^{N}\sum_{j=0}^{M} a_{ij}\, P_{\mathcal{R},i}^{(\vartheta,\theta)}(x)P_{\mathcal{T},j}^{(\vartheta,\theta)}(t) = \emptyset(x)'A\ \emptyset(t) \tag{31}$$

where the shifted Jacobi coefficient matrix A and the shifted Jacobi vectors $\emptyset(x)$ and $\emptyset(t)$ are given by.

$$A = \{a_{ij}\}_{i,j=0}^{N,M}, \emptyset(x) = [P_{\mathcal{R},0}^{(\vartheta,\theta)}(x), P_{\mathcal{R},1}^{(\vartheta,\theta)}(x), \dots, P_{\mathcal{R},N}^{(\vartheta,\theta)}(x)]'\ , \emptyset(t) = [P_{\mathcal{T},0}^{(\vartheta,\theta)}(t), P_{\mathcal{T},1}^{(\vartheta,\theta)}(t), \dots, P_{\mathcal{T},M}^{(\vartheta,\theta)}(t)]'.$$

Now, in order to approximate a three variable temperature function $T(x,y,t)$ defined for $0 \le x \le \mathcal{R}_1, 0 \le y \le \mathcal{R}_2$ and $0 \le t \le \mathcal{T}$ dependent on triple Jacobi series as

$$T(x,y,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \tilde{t}_{kij}\, P_{\mathcal{R}_1,i}^{(\vartheta,\theta)}(x)P_{\mathcal{R}_2,i}^{(\vartheta,\theta)}(y)P_{\mathcal{T},j}^{(\vartheta,\theta)}(t) \tag{32}$$

In practice, consider the $(N_1+1), (N_2+1)$ and $(M+1)$ terms triple SJPs with respect to x, y, t so that where

$$\tilde{t}_{kij} = \frac{1}{\mathfrak{h}_{\mathcal{R}_1,i}\mathfrak{h}_{\mathcal{R}_2,j}\mathfrak{h}_{\mathcal{T},k}} \int_0^{\mathcal{R}_1}\int_0^{\mathcal{R}_2}\int_0^{\mathcal{T}} T(x,y,t)\, P_{\mathcal{R}_1,i}^{(\vartheta,\theta)}(x) P_{\mathcal{R}_2,i}^{(\vartheta,\theta)}(y) P_{\mathcal{T},j}^{(\vartheta,\theta)}(t)\omega^{(\vartheta,\theta)}(x,y,t)dtdydx \tag{33}$$

such that $\omega^{(\vartheta,\theta)}(x,y,t) = \omega_{\mathcal{R}_1,i}^{(\vartheta,\theta)}(x)\ \omega_{\mathcal{R}_2,j}^{(\vartheta,\theta)}(y)\omega_{\mathcal{T},k}^{(\vartheta,\theta)}(t)$.

$$T_{M,N_1,N_2}(x,y,t) \approx \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{M} \tilde{t}_{kij} P_{\mathcal{R}_1,i}^{(\vartheta,\theta)}(x) P_{\mathcal{R}_2,j}^{(\vartheta,\theta)}(y) P_{\mathcal{T},k}^{(\vartheta,\theta)}(t) = \emptyset(t)' \ddot{T}\, \emptyset(x) \otimes \emptyset(y) \tag{34}$$

where the symbol $\otimes$ is the Kronecker tensor product, the shifted Jacobi vectors $\emptyset(x)$, $\emptyset(y)$ and $\emptyset(t)$ are given by

$$\left.\begin{aligned} \emptyset(x) &= \left[P_{\mathcal{R}_1,0}^{(\vartheta,\theta)}(x), P_{\mathcal{R}_1,1}^{(\vartheta,\theta)}(x), \dots, P_{\mathcal{R}_1,N_1}^{(\vartheta,\theta)}(x)\right]' \\ \emptyset(y) &= \left[P_{\mathcal{R}_2,0}^{(\vartheta,\theta)}(y), P_{\mathcal{R}_2,1}^{(\vartheta,\theta)}(y), \dots, P_{\mathcal{R}_2,N_2}^{(\vartheta,\theta)}(y)\right]' \\ \emptyset(t) &= \left[P_{\mathcal{T},0}^{(\vartheta,\theta)}(t), P_{\mathcal{T},1}^{(\vartheta,\theta)}(t), \dots, P_{\mathcal{T},M}^{(\vartheta,\theta)}(t)\right]' \end{aligned}\right\} \tag{35}$$

Also shifted Jacobi coefficient matrix $\ddot{T}$ is given in a block form as follows

$$\ddot{T} = \begin{bmatrix} \tilde{t}_{000} & \tilde{t}_{001} & \cdots & \tilde{t}_{00N_2} & \tilde{t}_{010} & \tilde{t}_{011} & \cdots & \tilde{t}_{0N_1N_2} \\ \tilde{t}_{100} & \tilde{t}_{101} & \ddots & \tilde{t}_{10N_2} & \tilde{t}_{110} & \tilde{t}_{111} & \ddots & \tilde{t}_{1N_1N_2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \tilde{t}_{M00} & \tilde{t}_{M01} & \cdots & \tilde{t}_{M0N_2} & \tilde{t}_{M10} & \tilde{t}_{M11} & \cdots & \tilde{t}_{MN_1N_2} \end{bmatrix} \tag{36}$$

DESCRIPTION OF THE METHOD

The selection of collocation points is playing a significant role in the efficiency and convergence of the "collocation method". For boundary value problems, the "Gauss-Lobatto" points represent one of the most important keys utilized of approximation. It should be renowned that for a differential equation with the singularity at $x = 0$ in the region $[0,\ \mathcal{R}]$ one is unable to apply the "collocation method" with "Jacobi-Gauss-Lobatto" points because the two assigned abscissas 0 and $\mathcal{R}$ are necessary to be used as two points from the collocation nodes. We use the "collocation method" with "Jacobi-Gauss-Lobatto" nodes to treat the two dimensional T-SFBHE; i.e., we collocate this equation only at the $M \times (N_1 - 1) \times (N_2 - 1)$ "Jacobi-Gauss-Lobatto" points $(0,\ \mathcal{T}), (0,\ \mathcal{R}_1)$ and $(0,\ \mathcal{R}_2)$ respectively. These equations and with initial, boundary conditions generate $(M+1) \times (N_1+1) \times (N_2+1)$ nonlinear algebraic equations by using one of the iteration methods can be solved.

Now, we set $\boldsymbol{P}_{N_1}(0,\mathcal{R}_1) = span\left\{P_{\mathcal{R}_1,0}^{(\vartheta,\theta)}(x), P_{\mathcal{R}_1,1}^{(\vartheta,\theta)}(x), \dots, P_{\mathcal{R}_1,N_1}^{(\vartheta,\theta)}(x)\right\}$.

We recall the "Jacobi-Gauss-Lobatto" generators. Such that N_1 is any positive integer, $\boldsymbol{P}_{N_1}(0,\mathcal{R}_1)$ stands for the group of all algebraic polynomials from degree at most N_1. If we denoting $x_{N_1,i}$ by $x_{\mathcal{R}_1,N_1,i}$, and $\omega_{N_1,i}^{(\vartheta,\theta)}$ by $\omega_{\mathcal{R}_1,N_1,i}^{(\vartheta,\theta)}$, $0 \le i \le N_1$, to the grid points and "Ghristoffel numbers" [13] of the standard or "shifted Jacobi-Gauss-Lobatto" quadrature on the $(-1,1)$ or $(0,\mathcal{R}_1)$ respectively.

$$x_{\mathcal{R}_1,N_1,i} = \frac{\mathcal{R}_1}{2}(x_{N_1,i}+1),\ \ 0 \le i \le N_1 \tag{37}$$

$$\omega_{\mathcal{R}_1,N_1,i}^{(\vartheta,\theta)} = (\frac{\mathcal{R}_1}{2})^{\vartheta+\theta+1}\omega_{\mathcal{R}_1,i}^{(\vartheta,\theta)}$$

For any $\emptyset(x) \in \boldsymbol{P}_{N_1}(0,\mathcal{R}_1)$, we have

$$\int_0^{\mathcal{R}_1} \omega_{\mathcal{R}_1,i}^{(\vartheta,\theta)}\emptyset(x)dx = \left(\frac{\mathcal{R}_1}{2}\right)^{\vartheta+\theta+1}\int_{-1}^{1}(1-x)^{\vartheta}(1+x)^{\theta}\, P\left(\frac{\mathcal{R}_1}{2}(x+1)\right)$$

$$= \left(\frac{\mathcal{R}_1}{2}\right)^{\vartheta+\theta+1}\sum_{i=0}^{N_1}\omega_{\mathcal{R}_1,i}^{(\vartheta,\theta)}(x_{N_1,i})\,\emptyset\left(\frac{\mathcal{R}_1}{2}(x_{N_1,i}+1)\right) \tag{38}$$

$$= \sum_{i=0}^{N_1}\ \omega_{\mathcal{R}_1,N_1,i}^{(\vartheta,\theta)}\,\emptyset(x_{\mathcal{R}_1,N_1,i})$$

where $x_{\mathcal{R}_1,N_1,i}$ and $\omega_{\mathcal{R}_1,N_1,i}^{(\vartheta,\theta)}$ are the grid points and equivalent weights of the "shifted Jacobi-Gauss-quadrature" technique on the region $[0,\ \mathcal{R}_1]$ respectively. In the same procedure on the intervals $[0,\ \mathcal{R}_2]$ and $[0,\ \mathcal{T}]$ then one can readily show that

$$y_{\mathcal{R}_2,N_2,j} = \frac{\mathcal{R}_2}{2}\left(y_{N_2,j}+1\right),\ 0 \le j \le N_2, \tag{39}$$
$$t_{\mathcal{T},M,k} = \frac{\mathcal{T}}{2}\left(t_{M,k}+1\right),\ 0 \le k \le M, \tag{40}$$
$$\omega^{(\vartheta,\theta)}_{\mathcal{R}_2,N_2,j} = \left(\frac{\mathcal{R}_2}{2}\right)^{\vartheta+\theta+1}\omega^{(\vartheta,\theta)}_{\mathcal{R}_2,j} \text{ and } \omega^{(\vartheta,\theta)}_{\mathcal{T},M,k} = \left(\frac{\mathcal{T}}{2}\right)^{\vartheta+\theta+1}\omega^{(\vartheta,\theta)}_{\mathcal{T},k}.$$

We will structure the numerical solution algorithm of Eq. (1) based on SJ-GL-Ps, under the given conditions, in the series or matrix form by utilizing Eqs. (41-44) into the shifted Jacobi vectors $\emptyset(x),\ \emptyset(y)\ and\ \emptyset(t)$ define by Eq. (50). In addition, the "shifted Jacobi-Gauss-Labatto" coefficient matrix $\ddot{T}$ is given by Eq. (51).
We can approximate the 1st spatial derivatives and their temporal/spatial fractional derivatives as

$$\left.\begin{aligned}
\frac{\partial T(x,y,t)}{\partial x} &= \emptyset'(t)\ddot{T}\left[\mathcal{D}^{(1)}_{\mathcal{R}_1}\emptyset(x)\right]\otimes\emptyset(y)\\
\frac{\partial T(x,y,t)}{\partial y} &= \emptyset'(t)\ddot{T}\emptyset(x)\otimes\left[\mathcal{D}^{(1)}_{\mathcal{R}_2}\emptyset(y)\right]\\
\frac{\partial^{\gamma} T(x,y,t)}{\partial t^{\gamma}} &= \left[\mathcal{D}^{(\gamma)}_{\mathcal{T}}\emptyset(t)\right]'\ddot{T}\emptyset(x)\otimes\emptyset(y),\\
\frac{\partial^{v_1} T(x,y,t)}{\partial x^{v_1}} &= \emptyset'(t)\ddot{T}\left[\mathcal{D}^{(v_1)}_{\mathcal{R}_1}\emptyset(x)\right]\otimes\emptyset(y),\\
\frac{\partial^{v_2} T(x,y,t)}{\partial y^{v_2}} &= \emptyset'(t)\ddot{T}\emptyset(x)\otimes\left[\mathcal{D}^{(v_2)}_{\mathcal{R}_2}\emptyset(y)\right]
\end{aligned}\right\} \tag{41}$$

By appling solution method for the two dimensional T-SFBHE based on Jacobi-Gauss-Labatto in the matrix form that given in Eq. (1), we get

$$\rho c\emptyset'(t)\left[\mathcal{D}^{(\gamma)}_{\mathcal{T}}\right]'\ddot{T}\emptyset(x)\bigotimes(\emptyset(y) - K\left(\emptyset'(t)\ddot{T}\left[\mathcal{D}^{(v_1)}_{\mathcal{R}_1}\emptyset(x)\right]\bigotimes\quad\emptyset(y) + \emptyset'(t)\ddot{T}\emptyset(x)\otimes\left[\mathcal{D}^{(v_2)}_{\mathcal{R}_2}\emptyset(y)\right]\right)$$
$$+ W_b\, c_b\emptyset'(t)\,\ddot{T}\emptyset(x)\otimes\emptyset(y) = G(x,y,t), \tag{42}$$

where, $G(x,y,t) = Q_{ext}(x,y,t) + Q_{met} + W_b c_b T_a I$. We collocate Eq. (66) at $M\times(N_1-1)\times(N_2-1)$ point, as

$$\begin{aligned}
\rho c\emptyset'\left(t_{\mathcal{T},M,k}\right)&\left[\mathcal{D}^{(\gamma)}_{\mathcal{T}}\right]'\ddot{T}\emptyset\left(x_{\mathcal{R}_1,N_1,i}\right)\bigotimes\quad\emptyset\left(y_{\mathcal{R}_2,N_2,j}\right)\\
&- K\left(\emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\left[\mathcal{D}^{(v_1)}_{R_1}\emptyset\left(x_{\mathcal{R}_1,N_1,i}\right)\right]\bigotimes\quad\emptyset\left(y_{\mathcal{R}_2,N_2,j}\right)\right.\\
&\left.+ \emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\emptyset\left(x_{\mathcal{R}_1,N_1,i}\right)\otimes\left[\mathcal{D}^{(v_2)}_{\mathcal{R}_2}\emptyset\left(y_{\mathcal{R}_2,N_2,j}\right)\right]\right)\\
&+ W_b\, c_b\emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\emptyset\left(x_{\mathcal{R}_1,N_1,i}\right)\bigotimes\quad\emptyset\left(y_{\mathcal{R}_2,N_2,j}\right)\\
&= G\left(x_{\mathcal{R}_1,N_1,i}, y_{\mathcal{R}_2,N_2,j}, t_{\mathcal{T},M,k}\right).
\end{aligned} \tag{43}$$

For $i = 1(1)(N_1-1)$, $j = 1(1)(N_2-1)$ and $k = 1(1)M$.

where $x_{\mathcal{R}_1,N_1,i}$ $(0 \le i \le N_1)$ and $y_{\mathcal{R}_2,N_2,j}$ $(0 \le j \le N_2)$ are the shifted Jacobi-Gauss-Lobatto quadrature of $P^{(\vartheta,\theta)}_{\mathcal{R}_1,i}(x)$ and $P^{(\vartheta,\theta)}_{\mathcal{R}_2,j}(y)$ respectively, while $t_{\mathcal{T},M,k}$ $(0 \le k \le M)$ are the roots of $P^{(\vartheta,\theta)}_{\mathcal{T},k}(t)$, that generates a system of $M\times(N_1-1)\times(N_2-1)$ nonlinear algebraic equations in the unknown extension coefficients, $\tilde{t}_{kij}$, $i = 1(1)(N_1-1)$, $j = 1(1)(N_2-1)$ and $k = 1(1)M$, and the rest of this system is obtained from the initial, boundary conditions by utilize Eqs. (2-6), as

$$\left.\begin{aligned}
\emptyset'(0)\,\ddot{T}\emptyset\left(x_{\mathcal{R}_1,N_1,i}\right)\otimes\emptyset\left(y_{\mathcal{R}_2,N_2,j}\right) &= f_0\left(x_{\mathcal{R}_1,N_1,i}, y_{\mathcal{R}_2,N_2,j}\right), 0 \le i \le N_1, 0 \le j \le N_2\\
\emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\left[\mathcal{D}^{(1)}_{\mathcal{R}_1}\emptyset(0)\right]\otimes\emptyset\left(y_{\mathcal{R}_2,N_2,j}\right) &= g_{10}\left(y_{\mathcal{R}_2,N_2,j}, t_{\mathcal{T},M,k}\right), 0 \le j \le N_2, 0 \le k \le M\\
\emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\left[\mathcal{D}^{(1)}_{\mathcal{R}_1}\emptyset(\mathcal{R}_1)\right]\otimes\emptyset\left(y_{\mathcal{R}_2,N_2,j}\right) &= g_{1\mathcal{R}_1}\left(y_{\mathcal{R}_2,N_2,j}, t_{\mathcal{T},M,k}\right), 0 \le j \le N_2, 0 \le k \le M\\
\emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\emptyset\left(x_{\mathcal{R}_1,N_1,i}\right)\otimes\left[\mathcal{D}^{(1)}_{\mathcal{R}_2}\emptyset(0)\right] &= g_{20}\left(x_{\mathcal{R}_1,N_1,i}, t_{\mathcal{T},M,k}\right), 0 \le i \le N_1, 0 \le k \le M\\
\emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\emptyset\left(x_{\mathcal{R}_1,N_1,i}\right)\otimes\left[\mathcal{D}^{(1)}_{\mathcal{R}_2}\emptyset(\mathcal{R}_2)\right] &= g_{2\mathcal{R}_2}\left(x_{\mathcal{R}_1,N_1,i}, t_{\mathcal{T},M,k}\right), 0 \le i \le N_1, 0 \le k \le M
\end{aligned}\right\} \tag{44}$$

This generates $(M+1)\times(N_1+1)\times(N_2+1)$ nonlinear algebraic equations, which can be solve by using Levenberg-Marquardt algorithm, taking $\ddot{T}$ as its variable, with an initial guess of all zeros, to reduce Eqs. (40-44), consequently, the approximate solution $T_{M,N_1,N_2}(x,y,t)$ at the point $\left(x_{\mathcal{R}_1,N_1,i}, y_{\mathcal{R}_2,N_2,j}, t_{\mathcal{T},M,k}\right)$ given in Eq. (39) can be calculate.

ERROR BOUND

We will present an analytic expression for the error norm of the preferable approximation for a smooth temperature function $T(x,y,t)\in\Omega$, where $\Omega\equiv[0,\mathcal{R}_1]\times[0,\mathcal{R}_2]\times[0,\mathcal{T}]$ by its expansion employing triple Jacobi polynomials. This shows an upper bound on the error expected in the numerical solutions. Let at first examine the space

$$\mathcal{H}^{\vartheta,\theta}_{M,N_1,N_2}=span\left\{P^{(\vartheta,\theta)}_{\mathcal{R}_1,i}(x)P^{(\vartheta,\theta)}_{\mathcal{R}_2,j}(y)P^{(\vartheta,\theta)}_{\mathcal{T},k}(t)\right\}, i=0(1)N_1,\qquad j=0,(1)N_2,\qquad k=0(1)M.$$

Assume that $T_{M,N_1,N_2}(x,y,t)$ belong to $\mathcal{H}^{\vartheta,\theta}_{M,N_1,N_2}$, be the preferable approximation of temperature function $T(x,y,t)$. Then depending on the qualifier of the best approximation, have $\forall\Theta_{M,N_1,N_2}(x,y,t)\in\mathcal{H}^{\vartheta,\theta}_{M,N_1,N_2}$

$$\left\|T(x,y,t)-T_{M,N_1,N_2}(x,y,t)\right\|_\infty\le\left\|T(x,y,t)-\Theta_{M,N_1,N_2}(x,y,t)\right\|_\infty. \tag{45}$$

It is appeared that the previous inequality also be correct if $\Theta_{M,N_1,N_2}(x,y,t)$ denotes the interpolating polynomial for $T(x,y,t)$ at points $(x_{\mathcal{R}_1,N_1,i},y_{\mathcal{R}_2,N_2,j},t_{\mathcal{T},M,k})$, where $x_{R_1,N_1,i},(0\le i\le N_1)$ are the roots of $P^{(\vartheta,\theta)}_{\mathcal{R}_1,N_1+1}(x)$, $y_{\mathcal{R}_2,N_2,j},(0\le j\le N_2)$ are the roots of $P^{(\vartheta,\theta)}_{\mathcal{R}_2,N_2+1}(y)$ and $t_{\mathcal{T},M,k},(0\ \le\ k\ \le M)$ are the roots of $P^{(\vartheta,\theta)}_{\mathcal{T},M+1}(t)$. Thus, by the applying similar procedures as in [7]

$$T(x,y,t)-\Theta_{M,N_1,N_2}(x,y,t)=$$
$$\frac{\partial^{N_1+1}T(\eta,y,t)}{\partial x^{N_1+1}(N_1+1)!}\prod_{i=0}^{N_1}\left(x-x_{\mathcal{R}_1,N_1,i}\right)+\frac{\partial^{N_2+1}T(x,\xi,t)}{\partial y^{N_2+1}(N_2+1)!}\prod_{j=0}^{N_2}\left(y-y_{\mathcal{R}_2,N_2,j}\right)+\frac{\partial^{M+1}T(x,y,\mu)}{\partial t^{M+1}(M+1)!}\prod_{k=0}^{M}\left(t-t_{\mathcal{T},M,k}\right)-\frac{\partial^{N_1+N_2+M+3}T(\tilde\eta,\tilde\xi,\tilde\mu)}{\partial x^{N_1+1}\partial y^{N_2+1}\partial t^{M+1}}\times$$
$$\frac{\prod_{i=0}^{N_1}(x-x_{\mathcal{R}_1,N_1,i})\prod_{j=0}^{N_2}(y-y_{\mathcal{R}_2,N_2,j})\prod_{k=0}^{M}(t-t_{\mathcal{T},M,k})}{(N_1+1)!(N_2+1)!(M+1)!} \tag{46}$$

where $\eta,\tilde\eta\in[0,\mathcal{R}_1],\xi,\tilde\xi\in[0,\mathcal{R}_2]$ and $\mu,\tilde\mu\in[0,\mathcal{T}]$, and we can obtain:

$$\left\|T(x,y,t)-\Theta_{M,N_1,N_2}(x,y,t)\right\|_\infty\le$$
$$\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_1+1}T(\eta,y,t)}{\partial x^{N_1+1}}\right|\frac{\left\|\prod_{i=0}^{N_1}(x-x_{\mathcal{R}_1,N_1,i})\right\|_\infty}{(N_1+1)!}+\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_2+1}T(x,\xi,t)}{\partial y^{N_2+1}}\right|\frac{\left\|\prod_{j=0}^{N_2}(y-y_{\mathcal{R}_2,N_2,j})\right\|_\infty}{(N_2+1)!}+$$
$$\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{M+1}T(x,y,\mu)}{\partial t^{M+1}}\right|\frac{\left\|\prod_{k=0}^{M}(t-t_{\mathcal{T},M,k})\right\|_\infty}{(M+1)!}+$$
$$\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_1+N_2+M+3}T(\tilde\eta,\tilde\xi,\tilde\mu)}{\partial x^{N_1+1}\partial y^{N_2+1}\partial t^{M+1}}\right|\times\frac{\left\|\prod_{i=0}^{N_1}(x-x_{\mathcal{R}_1,N_1,i})\right\|_\infty\left\|\prod_{j=0}^{N_2}(y-y_{\mathcal{R}_2,N_2,j})\right\|_\infty\left\|\prod_{k=0}^{M}(t-t_{\mathcal{T},M,k})\right\|_\infty}{(N_1+1)!(N_2+1)!(M+1)!}. \tag{47}$$

Since $T(x,y,t)$ is a smooth temperature function on Ω, then there exist a constants C_1,C_2,C_3 and C_4, such that:

$$\left.\begin{aligned}\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_1+1}T(\eta,y,t)}{\partial x^{N_1+1}}\right|&\le C_1\\ \max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_2+1}T(x,\xi,t)}{\partial y^{N_2+1}}\right|&\le C_2\\ \max_{(x,y,t)\in\Omega}\left|\frac{\partial^{M+1}T(x,y,\mu)}{\partial t^{M+1}}\right|&\le C_3\\ \max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_1+N_2+M+3}T(\tilde\eta,\tilde\xi,\tilde\mu)}{\partial x^{N_1+1}\partial y^{N_2+1}\partial t^{M+1}}\right|&\le C_4\end{aligned}\right\} \tag{48}$$

The factor $\left\|\prod_{i=0}^{N_1}\left(x-x_{\mathcal{R}_1,N_1,i}\right)\right\|_\infty$ minimized as follows: Let utilize the one-to-one mapping $x=\frac{R_1}{2}(z+1)$ between the intervals $[-1,1]$ and $[0,\mathcal{R}_1]$ to deduce that

$$\min_{x_{\mathcal{R}_1,N_1,i}\in[0,\mathcal{R}_1]}\max_{x\in[0,\mathcal{R}_1]}\prod_{i=0}^{N_1}\left(x-x_{\mathcal{R}_1,N_1,i}\right)=\min_{z_{\mathcal{R}_1,N_1,i}\in[-1,1]}\max_{z\in[-1,1]}\left|\prod_{i=0}^{N_1}\frac{\mathcal{R}_1}{2}\left(z-z_{\mathcal{R}_1,N_1,i}\right)\right|$$
$$=\left(\frac{\mathcal{R}_1}{2}\right)^{N_1+1}\min_{z_{\mathcal{R}_1,N_1,i}\in[-1,1]}\max_{z\in[-1,1]}\left|\prod_{i=0}^{N_1}\left(z-z_{\mathcal{R}_1,N_1,i}\right)\right| \tag{49}$$
$$=\left(\frac{\mathcal{R}_1}{2}\right)^{N_1+1}\min_{z_{\mathcal{R}_1,N_1,i}\in[-1,1]}\max_{z\in[-1,1]}\left|\frac{P^{(\vartheta,\theta)}_{N_1+1}(z)}{\kappa^{(\vartheta,\theta)}_{N_1}}\right|,$$

where $\kappa^{(\vartheta,\theta)}_{N_1}=\frac{\Gamma(2N_1+\vartheta+\theta+1)}{2^{N_1}(N_1)!\,\Gamma(N_1+\vartheta+\theta+1)}$ is the leading coefficient of $P^{(\vartheta,\theta)}_{N_1+1}(z)$ and $z_{\mathcal{R}_1,N_1,i}$ are the roots of $P^{(\vartheta,\theta)}_{N_1+1}(z)$. It is a well-famed reality [29], that the Jacobi polynomials satisfy

$$\max_{z\in[-1,1]} \left|P_{N_1+1}^{(\vartheta,\theta)}(z)\right| \leq C_5(N_1+1)^q, \qquad \vartheta,\theta > -1,$$

where $q = max(\vartheta, \theta, -\frac{1}{2})$ and C_5 is a favorable constant, and reach the maximum of their absolute value on the interval $[-1, 1]$, at $z = -1$ provided that $\vartheta \geq \theta$ and $\vartheta \geq -\frac{1}{2}$ from [5]

$$\max_{z\in[-1,1]} \left|P_{N_1+1}^{(\vartheta,\theta)}(z)\right| = P_{N_1+1}^{(\vartheta,\theta)}(1) = \frac{\Gamma(N_1+\vartheta+2)}{(N_1+1)!\ \Gamma(\vartheta+1)} = \mathcal{O}\big((N_1+1)^{\vartheta}\big),$$

from Eqs. (48-49), we get

$$\left\|T(x,y,t) - T_{M,N_1,N_2}(x,y,t)\right\|_\infty \leq$$

$$\widetilde{C_1}\frac{\left(\frac{\mathcal{R}_1}{2}\right)^{N_1+1}(N_1+1)^q}{\kappa_{N_1}^{(\vartheta,\theta)}(N_1+1)!} + \widetilde{C_2}\frac{\left(\frac{\mathcal{R}_2}{2}\right)^{N_2+1}(N_2+1)^q}{\kappa_{N_2}^{(\vartheta,\theta)}(N_2+1)!} + \widetilde{C_3}\frac{\left(\frac{\mathcal{T}}{2}\right)^{M+1}(M+1)^q}{\kappa_{M}^{(\vartheta,\theta)}(M+1)!} +$$

$$\widetilde{C_3}\,\frac{\left(\frac{\mathcal{R}_1}{2}\right)^{N_1+1}\left(\frac{\mathcal{R}_2}{2}\right)^{N_2+1}\left(\frac{\mathcal{T}}{2}\right)^{M+1}(N_1+1)^q}{\kappa_{N_1}^{(\vartheta,\theta)}\kappa_{N_2}^{(\vartheta,\theta)}\kappa_{M}^{(\vartheta,\theta)}(N_1+1)!(N_2+1)!(M+1)!} \tag{50}$$

Hence, an upper bound of the maximum absolute errors achieved for the approximate solution. The convergence of the recommended method depends fundamentally on the above error bound. Moreover, the speed of convergence of "Jacobi collocation methods" was proved be fast for any choice of shifted Jacobi parameters [17, 28].

ESTIMATION OF THE ERROR FUNCTION

In this section, we will give an efficient error estimation for the SJ-GL-Ps and also a technique to obtain the corrected solution of the T-FBHE as in Eq. (1) under the Eqs. (2-6) by using the residual correction method and thus the approximate solution Eq. (49) is corrected by the proposed method [40].

For our aim, we define $e_{M,N_1,N_2}(x,y,t) = T(x,y,t) - T_{M,N_1,N_2}(x,y,t)$ as the error function of the Collocation approximation $T_{M,N_1,N_2}(x,y,t)$ to $T(x,y,t)$, where $T(x,y,t)$ is the exact solution for the Eq. (1) under Eqs. (1-6). Hence, $T_{M,N_1,N_2}(x,y,t)$ satisfies the following system:

$$L\left[T_{M,N_1,N_2}(x,y,t)\right] = \rho c\frac{\partial T_{M,N_1,N_2}(x,y,t)}{\partial t} - K\left(\frac{\partial^{v_1}T_{M,N_1,N_2}(x,y,t)}{\partial x^{v_1}} + \frac{\partial^{v_2}T_{M,N_1,N_2}(x,y,t)}{\partial y^{v_2}}\right) + W_b\, c_b T_{M,N_1,N_2}(x,y,t)$$
$$= Q_{ext}(x,y,t) + Q_{met} + W_b\, c_b T_a + \Re_{M,N_1,N_2} \tag{51}$$

with the initial and boundary conditions

$$T_{M,N_1,N_2}(x,y,0) = T_c, \qquad 0 < x < \mathcal{R}_1,\ 0 < y < \mathcal{R}_2 \tag{52}$$
$$-K\frac{\partial T_{M,N_1,N_2}(0,y,t)}{\partial x} = q_0, \qquad 0 \leq y \leq \mathcal{R}_2, t > 0 \tag{53}$$
$$-K\frac{\partial T_{M,N_1,N_2}(x,0,t)}{\partial y} = q_1, \qquad 0 \leq x \leq \mathcal{R}_1, t > 0 \tag{54}$$
$$-K\frac{\partial T_{M,N_1,N_2}(\mathcal{R}_1,y,t)}{\partial x} = 0, \qquad 0 \leq y \leq \mathcal{R}_2, t > 0 \tag{55}$$
$$-K\frac{\partial T_{M,N_1,N_2}(x,\mathcal{R}_2,t)}{\partial y} = 0, \qquad 0 \leq x \leq \mathcal{R}_1, t > 0 \tag{56}$$

Here, $\Re_{M,N_1,N_2}(x,y,t)$ is the residual function of the T-SFBHE (2.1) which is obtained by substituting the approximate solution $T_{M,N_1,N_2}(x,y,t)$ into Eqs. (2--5).

Now, let us subtract Eqs. (81-86) from Eqs. (2-6) respectively. The we obtain the error problem:

$$\rho c\frac{\partial e_{M,N_1,N_2}(x,y,t)}{\partial t} - K\left(\frac{\partial^{v_1}e_{M,N_1,N_2}(x,y,t)}{\partial x^{v_1}} + \frac{\partial^{v_2}e_{M,N_1,N_2}(x,y,t)}{\partial y^{v_2}}\right) + W_b\, c_b\, e_{M,N_1,N_2}(x,y,t)$$
$$= -\Re_{M,N_1,N_2} \tag{57}$$

with the homogeneous conditions:

$$e_{M,N_1,N_2}(x,y,0) = 0, \qquad 0 < x < \mathcal{R}_1,\ 0 < y < \mathcal{R}_2 \tag{58}$$
$$-K\frac{\partial e_{M,N_1,N_2}(0,y,t)}{\partial x} = 0, \qquad 0 \leq y \leq \mathcal{R}_2, t > 0 \tag{59}$$
$$-K\frac{\partial e_{M,N_1,N_2}(x,0,t)}{\partial y} = 0, \qquad 0 \leq x \leq \mathcal{R}_1, t > 0 \tag{60}$$
$$-K\frac{\partial e_{M,N_1,N_2}(R_1,y,t)}{\partial x} = 0, \qquad 0 \leq y \leq \mathcal{R}_2, t > 0 \tag{61}$$
$$-K\frac{\partial e_{M,N_1,N_2}(x,R_2,t)}{\partial y} = 0, \qquad 0 \leq x \leq \mathcal{R}_1, t > 0 \tag{62}$$

Finally, we solve the error Eqs. (87-92) in the same method which is suggestion in Section 6 and thus we get the approximation to e_{M,N_1,N_2} as following:

$$\mathrm{e}_{M,N_1,N_2}(x,y,t) \approx \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{M} \tilde{\tilde{t}}_{kij}\, P_{\mathcal{R}_1,i}^{(\theta_1,\theta_2)}(x) P_{\mathcal{R}_2,j}^{(\theta_1,\theta_2)}(y) P_{\mathcal{T},k}^{(\theta_1,\theta_2)}(t)$$

$$= \emptyset(t)'\, \tilde{\tilde{T}}\, \emptyset(x) \otimes \emptyset(y). \qquad (63)$$

By examining the Eq. (1), while the theoretical solution is not known, the maximum absolute error can be estimated approximately by using

$$\mathrm{E}_{M,N_1,N_2}(x,y,t) = max\{\mathrm{e}_{M,N_1,N_2}(x,y,t), 0 \le t \le \mathcal{T}, 0 \le x \le \mathcal{R}_1, 0 \le y \le \mathcal{R}_2\}. \qquad (64)$$

The above error estimation has established on the convergence rates of expansion in Jacobi polynomial [29]. Therefore, it provided reasonable convergence rates in temporal or spatial discretization.

ILLUSTRATIVE TEST PROBLEM

In this section, we apply the approach which has been presented in section 6 for solving the two dimensional T-SFBHE in the two examples based on SJ-GL-Ps. The two dimensional T-SFBHE transformed into non-linear algebraic Eqs. (67-68) respectively. A Levenberg-Marquardt technique, taking $\ddot{T}$ as its variable, which used to minimize these equations as a set of least squares problems. This $\ddot{T}$ is then used in Eq. (65) to acquire our approximate surface of $T(x,y,t)$. In these examples, we takeng $\mathcal{R}_1 = \mathcal{R}_2 = 1$, $\vartheta = \theta = 0$ and using Gauss-Labatto points. Tables 2 and 3 shows that the maximum errors satisfy from solving the problem under SJ-G-LPs study on $x \in [0,\mathcal{R}_1], y \in [0,\mathcal{R}_2]$ and $t \in [0,\mathcal{T}]$ when $N_1 = N_2 = M = 2,3,4,5,6,7,8,9$ and 10.

Example1:

Consider the two dimensional T-SFBHE (2.1) case where by choosing Q_{ext}. So, the exact solution under initial and Neumann boundary conditions is:

$$T(x,y,t) = e^{-t}x^3y^{3.6} + 37. \qquad (65)$$

TABLE 2. Maximum errors obtained for Example 1 with $\gamma = 0.63$ and $v_1 = v_2 = 1.5$.

$N_1 = N_2 = M$	Maximum Error
2	3.922158384739305e-07
3	7.216142608257314e-05
4	6.375869214281238e-05
5	5.817127026190860e-04
6	5.138294572262225e-04
7	7.442164453124178e-04
8	6.344458501388317e-04
9	7.111922919520453e-04
10	5.847427290177620e-04

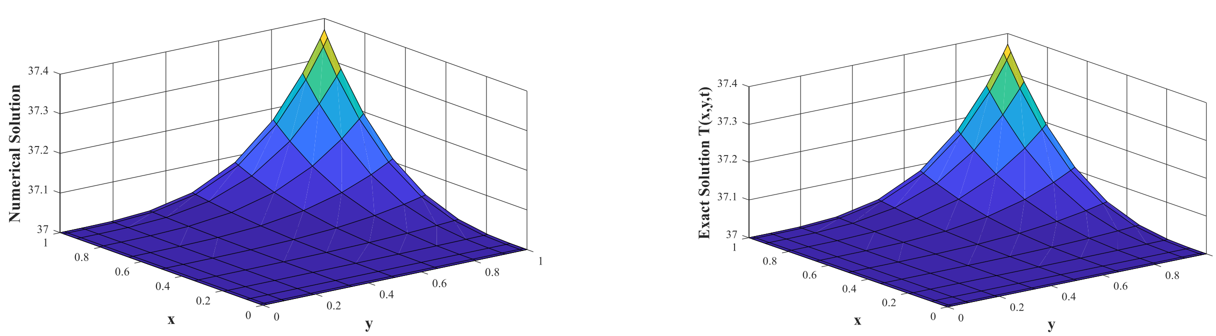

FIGURE1. Numerical and exact solutions for Example 1 at $\gamma = 0.63, v_1 = v_2 = 1.5$
$\mathcal{T} = \mathcal{R}_1 = \mathcal{R}_2 = 1,\ N_1 = N_2 = M = 10.$

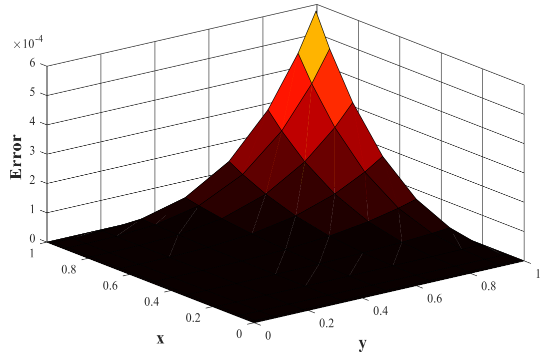

FIGURE 2: Maximum error for Example 1 at $\gamma = 0.63, v_1 = v_2 = 1.5$

$\mathcal{T} = \mathcal{R}_1 = \mathcal{R}_2 = 1, N_1 = N_2 = M = 10.$

Example 2: Consider the two dimensional T-SFBHE (2.1) case where by choosing Q_{ext}. So, the exact solution under initial and Neumann boundary conditions is:

$$T(x,y,t) = e^{-t}(1 - x^2 - y^3) + 37. \tag{66}$$

TABLE 3. Maximum errors obtained for Example 2 with $\gamma = 0.75, v_1 = 1.6, v_2 = 1.9$.

$N_1 = N_2 = M$	**Maximum Error**
2	2.840849002247126e-03
3	1.344468016142741e-03
4	1.125036383839984e-03
5	6.377698966986145e-05
6	5.738165020829911e-05
7	3.158523192325902e-04
8	2.720642876425927e-04
9	4.160982083121212e-04
10	4.160982083121212e-04

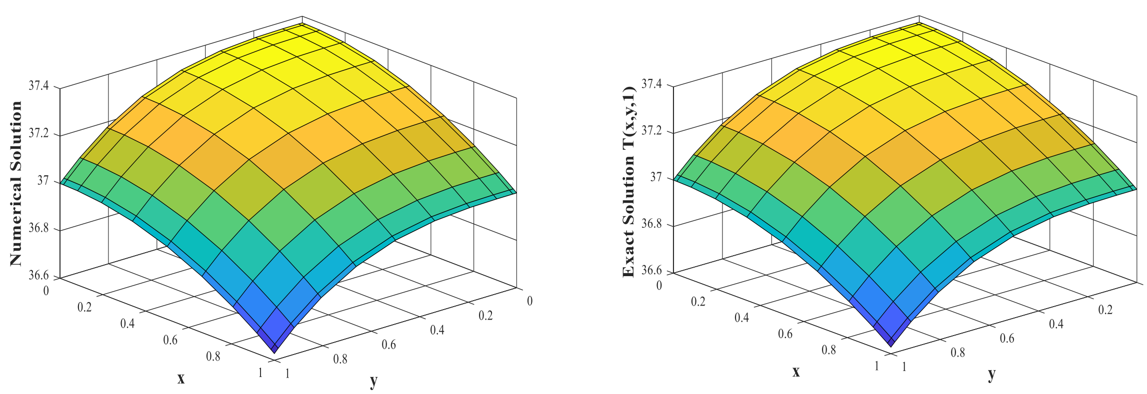

FIGURE 3. Numerical and exact solutions for Example 1 at $\gamma = 0.75, v_1 = 1.6, v_2 = 1.9$

$\mathcal{T} = \mathcal{R}_1 = \mathcal{R}_2 = 1,\ N_1 = N_2 = M = 10$.

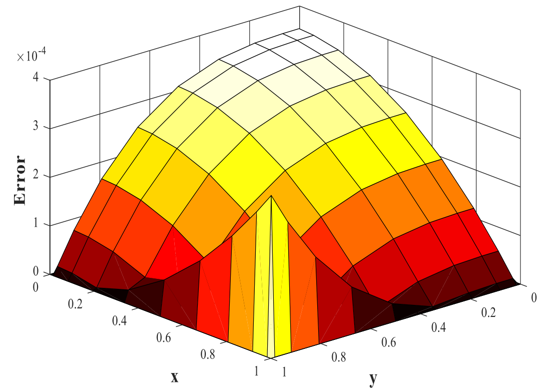

FIGURE 4: Maximum error for Example 1 at $\gamma = 0.75, v_1 = 1.6, v_2 = 1.9$ $\mathcal{T} = \mathcal{R}_1 = \mathcal{R}_2 = 1, N_1 = N_2 = M = 10$.

CONCLUSION

In this article, an approximate approach for solving two-dimensional T-SFBHE has been introduced. The fractional derivatives are described in the Caputo form. The proposed technique depend on the spectral collocation method of operational matrix formula for the shifted Jacobi-Gauss-Lobatto polynomials. From Figures 1 and 3 clarified a comparison between then numerical and exact solutions of examples 1 and 2 respectively. The Figures 2 an 4 indicated the maximum error values observed that the low error for all sample size, with the best performance occurring for $N_1 = N_2 = M = 2(N_1 = N_2 = M = 7)$ at just under $3.9 \times 10^{-7}(7.4 \times 10^{-4})$ and for $N_1 = N_2 = M = 4(N_1 = N_2 = M = 2)$ at just under $5.7 \times 10^{-5}(2.8 \times 10^{-3})$ respectively.

The error of the numerical solution was estimated theoretically and the exponential convergence rate of the proposed method in both temporal and spatial discretization was graphically investigated analyzed. The numerical results show that the present technique has higher accuracy, good convergence, and reasonable stability (depending on Figures 2 and 4) by using few grid points.

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