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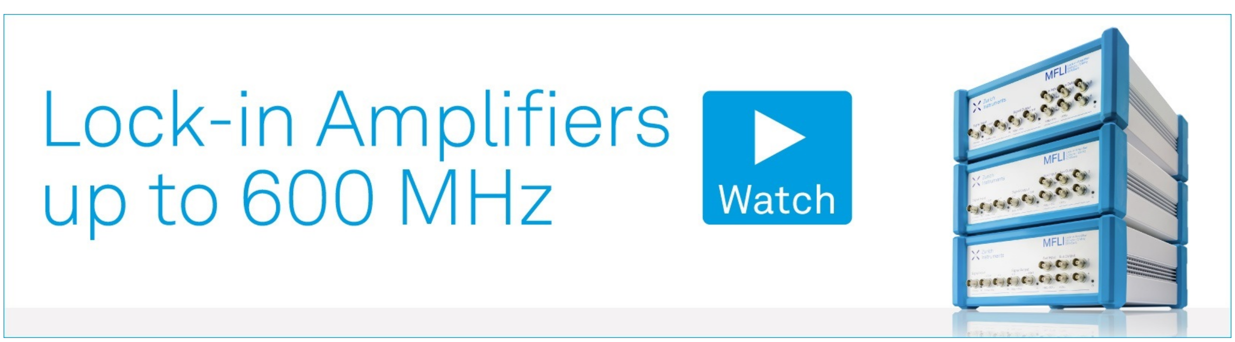

Analysis of Error Estimate for Expanded H^1- Galerkin MFEM of PIDEs with Nonlinear Memory

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Abstract. In this paper, the error estimate of expanded H^1-Galerkin mixed finite element methods (MFEMs) has been discussed with studied semi-discrete for parabolic integro-differential equations (PIDEs) with a nonlinear memory. In addition, we derived an error estimates for the unknown function, gradient function, and flux.

I. INTRODUCTION

Pani presented a new MFEM called H^1-Galerkin mixed finite element procedure [26] which splits given equations into a first-order system and can be viewed as a nonsymmetric version of least square method . Deepjyoti Goswami Amiya K. Pani and Sangita Yadav established optimal Error Estimates of two MFEMs for PIDEs with Nonsmooth Initial Data [19].\

Compared to standard mixed methods [13,15,17,18] , H^1-Galerkin MFEM has many good features. The first is choosing of finite element spaces that they are not subject into a Ladyzenskaja-Babuska-Brezzi (LBB) conditions. The second finite element spaces V_h (for approximating an unknown function) and W_h (for approximating the flux) may be of different polynomial degrees. Moreover, the L^2 and H^1 error estimates do not require a finite element mesh become quasi-uniform. Although we seek extra regularity in the solution, a best order of convergence to the flux in L^2 norm can be obtained. Up to now, H^1-Galerkin MFEMs have been widely used to solve some partial differential equations [28,30].

In [9], an H^1-Galerkin MFE procedure deals with a nonlinear parabolic equation in porous medium flow by combining the H^1-Galerkin formulation and the expanded MFEMs are suggested. The formulation has the advantages of H^1-Galerkin method and expanded MFEMs, it can solve the scalar unknown, its gradient and its flux directly. It is proper to the case anywhere the coefficient from the differential is a little tensor that do not need to be inverted. Furthermore, the formulation permits the use from standard continuous and piecewise (linear and tall-order) polynomials in contrast for continuously differentiable piecewise polynomials required by standard H^1-Galerkin methods, and is free of LBB condition as required by MFEMs. Certainly, this formulation has its hold disadvantages such as it needs to deal with the large size matrix.

The purpose of this paper is to extend the H^1-Galerkin MFEM developed in [11] to parabolic integro-differential equations with a nonlinear memory. Then, the paper will present the error estimates.

The rest of this article is orderly as follows: In Section 2, H^1-Galerkin MFEM combined with expanded MFE for nonlinear PIDEs with memory is established. In Section 3, optimal order error estimates for the semi-discrete scheme of the, H^1-Galerkin MFEM combined with expanded MFEM are proved. Throughout this research, c indicates a general positive constant which does not depend by h. in the alike time, we show a useful integral inequality.

$$\int_0^t \int_0^\tau |\emptyset(s)|^2 ds d\tau \leq \int_0^t |\emptyset(s)|^2 ds, \tag{1.1}$$

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where $\emptyset$ is a integrable function in $[0,t]$, $t \in [0,T]$. Also the kernel k is assumed to be positive definite ,i.e. , all $t \in (0,T], k \in L^1_{loc}(0,\infty)$, and

$$\int_0^t \left(\int_0^s k(t-s)v(\tau)d\tau\right) ds \geq 0 \qquad ,v \in C[0,T] \tag{1.2}$$

II. GOVERNING PROBLEM

Consider the following PIDE with nonlinear memory[33]:

$$u_t - \Delta u + \int_0^t k(t-s)(-\nabla\cdot\left(a(x,u)\nabla u + b(x,t)\right) + c(x,u)\cdot\nabla u + g(x,u)),$$
$$= f(x,t),(x,t)\,\Omega \times j, \tag{2.1}$$
$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times j,$$
$$u(x,0) = u_0(x), \qquad x \in \Omega,$$

Where, Ω be smooth bounded range in $\mathbb{R}^d (d = 1,2,3)$ with the Lipchitz continuous boundary $\partial\Omega$. $j = (o,T]$ is the time interval with $0 < T < \infty$, suppose the kernel k be positive definite as well as a smooth neither non smooth memory and f is a known function.

For clearly, we will not do the dependence of variable x in $a(x,u), b(x,u), c(x,u)$ and $g(x,u)$ we give the following hypotheses about a, b, c, f and u in the following show:

(1) The functions $a(u) \in \mathbb{R}^{d\times d}$ is a tensor function, $b(u)$ and $c(u) \in \mathbb{R}^d$ are vector functions and $g(u) \in \mathbb{R}^1$ is scalar function, respectively.

(2) All the functions $a(u), b(u), c(u)$ and $g(u)$ are continuously differentiable with respect to any variable also smooth and bounded.

Problem (2.1) and a nonlinear version thereof exist in many physical operations in which this is needful to take in tally the effects from memory due to a reduction of the usual diffusion equations, [20,23,29], to approximate the solution u of PIDEs. Both finite difference and finite element methods have been analysed widely in the past for both the linear and nonlinear problem [5,6,21,22,10,24,35,34]. Recently, many numerical methods like MFEM[16,30], finite volume element method (FVM)[32], and discontinuous Galerkin method (DGM) for space discretization or time discretization[25], have been proposed to solve PIDEs.

III. EXPANDED H^1-GALERKIN MFEM FOR PIDES WITH NONLINEAR MEMORY

Weak Formulation

Re-write a equation (2.1) as following:

$$u_t - \nabla\cdot\left(\nabla u - \int_0^t k(t-s)\left(a(u)\nabla u + b(u)\right)ds\right)$$
$$+ \int_0^t k(t-s)\left(c(u)\cdot\nabla u + g(u)\right)ds$$
$$= f\,, \tag{3.1}$$

to definite the H^1-Galerkin MFEM combined with expanded mixed element method. We split the nonlinear PIDEs together memory (3.1) into first-order system as follows:

$$q = \nabla u - \int_0^t k(t-s)\left(a(u)\nabla u + b(u)\right)ds$$

And $\qquad \sigma = \nabla u$

as follows

$$u_t - \nabla\cdot q + \int_0^t k(t-s)\left(c(u)\cdot\sigma + g(u)\right)ds = f \tag{3.2a}$$
$$\sigma = \nabla u \tag{3.2b}$$

$$q = \sigma - \int_0^t k(t-s)\big(a(u)\sigma + b(u)\big)ds \quad (3.2c)$$

$u(x,0) = u_0(x)$ (3.2d)

A weak form into up equations is find $(u,\sigma,q) \in H_0^1(\Omega) \times W \times W$

Such that

$$(\sigma_t, p) + (\nabla\cdot q, \nabla\cdot p) - \left(\int_0^t k(t-s)\big(c(u).\sigma + g(u)\big)ds, \nabla\cdot p\right) = (f, \nabla\cdot p) \qquad \forall p \in W \quad (3.3a)$$

$$(\sigma, \nabla v) = (\nabla u, \nabla v) \qquad \forall v \in H_0^1(\Omega) \quad (3.3b)$$

$$(q, w) = (\sigma, w) - \left(\int_0^t k(t-s)\big(a(u)\sigma + b(u)\big)ds, w\right) \qquad \forall w \in W \quad (3.3c)$$

σ(0)=∇u_0 (x) (3.3d)

where

$$W = H(div,\Omega) = \left\{w \in \big(L^2(\Omega)\big)^d : \nabla\cdot w \in L^2(\Omega)\right\}$$

with norm

$$\| w \|_{H(div,\Omega)} = (\| w \|^2 + \| \nabla\cdot w \|^2)^{\frac{1}{2}}$$

and

$$V = H_0^1(\Omega) = \{v \in H^1(\Omega): v = 0 \quad 0n \quad \partial\Omega\}$$

with inner product $(.,.)$ also norm $\|.\|$.

To proof the equivalence between (3.2) and (3.3), we need the following lemmas:

Lemma 3.1 ([3]) Let Ω be a bounded range and the Lipschitz continuous boundary $\partial\Omega$. Then, every $q \in H(div,\Omega)$, there exists $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and a divergence free $\psi \in H(div,\Omega)$, $q = \nabla\emptyset + \psi$.

Lemma 3.2 ([12]) Let Ω be a bounded field also a Lipschitz continuous boundary $\partial\Omega$. Then, all $g \in L^2(\Omega)$, there exists $q \in \big(H^1(\Omega)\big)^d \subset H(div,\Omega)$, $\nabla\cdot q = g$.

Theorem 3.1 by using the conditions which explained in above Lemmas. $(u,\sigma,q) \in H_0^1(\Omega) \times W \times W$ is a solution of the system (3.2) if and only if it is a solution to the weak formulation (3.3).

Proof: A solution to the system (3.2) is a solution to the weak form(3.3). Then, we have to prove that a solution to the weak form (3.2) is a solution to the system (3.1).We choose $w = q - \sigma + \int_0^t k(t-s)\big(a(u)\sigma + b(u)\big)ds$ in (3.3c) to have

$$\left(q, q - \sigma + \int_0^t k(t-s)\big(a(u)\sigma + b(u)\big)ds\right) = \left(\sigma, q - \sigma + \int_0^t k(t-s)\big(a(u)\sigma + b(u)\big)ds\right) -$$

$$\left(\int_0^t k(t-s)\big(a(u)\sigma + b(u)\big)ds, q - \sigma + \int_0^t k(t-s)\big(a(u) + b(u)\big)ds\right) \text{ then}$$

$$\left(q - \sigma + \int_0^t k(t-s)\big(a(u)\sigma + b(u)\big)ds, q - \sigma + \int_0^t k(t-s)\big(a(u) + b(u)\big)ds\right) = 0 \text{ that means}$$

$$q = \sigma + \int_0^t k(t-s)(a(u)\sigma + b(u))ds. \quad (3.5)$$

Using Lemma 3.1, there exists a $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and a divergence free[4] $\psi \in H(div,\Omega)$ such that $\sigma = \nabla\emptyset + \psi$.

Putting $\sigma = \nabla\emptyset + \psi$ into (3.3b) exhibit

$$(\nabla\emptyset + \psi, \nabla v) = (\nabla u, \nabla v),$$

$$(\nabla\emptyset, \nabla v) + (\psi, \nabla v) = (\nabla u, \nabla v), \quad \forall v \in H_0^1(\Omega). \quad (3.6)$$

Divergence theorem indicate

$$(\psi, \nabla v) = -(\nabla\cdot\psi, v) = 0, \qquad \forall v \in H_0^1(\Omega). \quad (3.7)$$

From (3.6) and (3.7) that we get

$$(\nabla\emptyset, \nabla v) = (\nabla u, \nabla v), \qquad \forall v \in H_0^1(\Omega),$$

which implies

$$\nabla\emptyset = \nabla u.$$

Now inserting $\nabla\emptyset = \nabla u$ in $\sigma = \nabla\emptyset + \psi$ we obtain

$$\sigma = \nabla u + \psi. \quad (3.8)$$

To get $\psi = 0$, we substitute (3.5) and (3.8) in (3.2a) then by using the divergence theorem for the first term from the resulting equation to have:

$$(u_t, \nabla\cdot p) - (\psi_t, p) - (\nabla\cdot(\nabla u + \psi), \nabla\cdot p)$$

$$-\left(\nabla\cdot\left(\int_0^t k(t-s)\big(c(u)\cdot(\nabla u+\psi)+g(u)\big)ds\right),\nabla\cdot p\right)$$

$$=-(f,\nabla\cdot p),\qquad \forall p\in W. \tag{3.9}$$

putting $p=\psi$ in (3.9) and since $\psi\in H(div,\Omega)$ is divergence-free and $\nabla\cdot\psi=0$,we have

$$(\psi_t,\psi)=\frac{1}{2}\frac{d}{dt}(\psi,\psi)=0, \tag{3.10}$$

furthermore, using (3.8) for $t=0$ we get

$$\sigma(x,0)=\nabla u(x,0)+\psi(x,0)=\nabla u_0(x)+\psi(x,0),$$

which means $\psi(x,0)=0$.

Integrating (3.10) with respect time from 0 to t, we have

$$\psi(x,t)=0.$$

thus, we obtain $\sigma=\nabla u$.

Now, we can be rewritten (3.9) as follows:

$$(u_t,\nabla\cdot p)-\left(\nabla\cdot\left(\nabla u+\int_0^t k(t-s)\big(c(u)\cdot\nabla u+g(u)\big)ds\right),\nabla\cdot p\right)$$

$$=(f,\nabla\cdot p),\qquad \forall p\in H(div,\Omega). \tag{3.11}$$

Since $f,u_t\in L^2(\Omega)$, by Lemma (3.2) that there exists an $F\in H(div,\Omega)$, such that

$$\nabla\cdot F=u_t-f.$$

Thus, (3.11) becomes

$$(\nabla\cdot q,\nabla\cdot p)=(\nabla\cdot F,\nabla\cdot p),\qquad \forall p\in H(div,\Omega).$$

and we have that $\nabla\cdot F=\nabla\cdot q$, that is

$$u_t-\nabla\cdot p=f \tag{3.12}$$

by (3.5) with (3.12) we get

$$u_t-\nabla\cdot\left(\sigma+\int_0^t k(t-s)\ \big(c(u)\cdot\sigma+g(u)\big)ds\right)=f \tag{3.13}$$

then with (3.8) we conclude

$$u_t-\nabla\cdot\left((\nabla u+\psi)+\int_0^t k(t-s)\big(c(u)\cdot(\nabla u+\psi)+b(u)\big)ds\right)f \tag{3.14}$$

and $\psi=0$,we get

$$u_t-\nabla\cdot\left(\nabla u+\int_0^t k(t-s)\big(c(u)\cdot\nabla u+b(u)\big)ds\right)=f \tag{3.15}$$

this completes the proof.

Semi Discrete Scheme

To discuss the semi discrete H^1-Galerkin MFEM combined with procedure, we first give some definitions and some properties of projections. Let T_h be a partition of Ω to a finite number from elements, so that, $\overline{\Omega}=\bigcup_{k\in T_h}\overline{K}$ and element edges lying on the boundary may be curved $[27]$. Let h_k denote the triangle diameter of k. put $h=\max_{k\in T_h}h_k$. Let V_h be the finite dimensional subspaces of $H_0^1(\Omega)$ defined by

$$V_h=\{v_h\in H_O^1(\Omega):v_h|_k\in P_m(\mathrm{K})\},$$

where $P_m(K)$ denotes the spaces of polynomials from degree at most m on k. Moreover, we denote by W_h the vector spaces in MFE spaces with index k. It is well known to W_h and V_h satisfy the inverse property and the following approximation properties $[2]$,$[14]$:

$$\inf_{v_h\in V_h}\|v-v_h\|+h\|v-v_h\|_1\le ch^{m+1}\|v\|_{m+1},\qquad v\in H^{M+1}(\Omega),$$

$$\inf_{p_h\in W_h}\|p-p_h\|\le ch^{k+1}\|p\|_{k+1},\qquad p_h\in\left(H^{K+1}(\Omega)\right)^d,$$

$$\inf_{p_h\in W_h}\|\nabla.(p-p_h)\|\le ch^{k+1}\|p\|_{k+1},\qquad p_h\in\left(H^{K+1}(\Omega)\right)^d,$$

to analyze the error estimates, we require the following projection operators. Let $I_h:H_0^1(\Omega)\longrightarrow V_h$ be the Ritz projection [8] defined by:

$$(\nabla(u-I_hu_h),\nabla v_h)=0\qquad \forall v_h\in V_h \tag{3.16}$$

which the following results hold:

$$\|u-I_hu\|+h\|\nabla(u-I_hu)\|\le ch^{m+1}\|u\|_{m+1} \tag{3.17}$$

also we know that the Raviart–Thomas projection $R_h:H(div,\Omega)\longrightarrow W_h$ define by[7]

$$(\nabla\cdot(q-R_hq),\nabla\cdot p_h)=0,\qquad \forall p_h\in W_h \tag{3.18}$$

we have the following approximation features:

$$\| q - R_h q \| \leq c h^{k+1} \| q \|_{k+1}, \tag{3.19}$$

$$\| \nabla \cdot (q - R_h q) \| \leq c h^{k} \| q \|_{k+1}, \tag{3.20}$$

Now based on the above of preliminaries, we define H^1-Galerkin MFEM combined with expanded MFEM for the system (3.3) as follows : find$(u_h, \sigma_h, p_h) \in V_h \times W_h \times W_h$ such that

$$(\sigma_{ht}, p_h) + (\nabla \cdot q_h, \nabla \cdot p_h) - \left(\int_0^t k(t-s)(c(u_h) \cdot \sigma_h + g(u_h))ds, \nabla \cdot p_h\right) = (f, \nabla \cdot p_h) \tag{3.21a}$$

$$(\sigma_h, \nabla v_h) = (\nabla u_h, \nabla v_h) \tag{3.21b}$$

$$(q_h, w_h) = -\left(\int_0^t k(t-s)(a(u_h)\sigma_h + b(u_h))ds, w_h\right) \tag{3.21c}$$

$\forall p_h \in W_h$, $u_h \in V_h$ and $w_h \in W_h$.

IV. ERROR ESTIMATES TO THE SEMI-DISCRETE

In this section, we decompose error estimates for the H^1-Galerkin MFEM combined with expanded MFEM presented in Section 3.1 are:

$$u - u_h = u - I_h u + I_h u - u_h = (u - I_h u) + (I_h u - u_h) = \alpha + \beta,$$

$$q - q_h = q - R_h q + R_h q - q_h = (q - R_h q) + (R_h q - q_h) = \eta + \zeta,$$

and

$$\sigma - \sigma_h = \sigma - R_h \sigma + R_h \sigma - \sigma_h = (\sigma - R_h \sigma) + (R_h \sigma - \sigma_h) = \theta + \xi.$$

applying (3.3),(3.10), and auxiliary projections (3.5),$(3,7)$. We get the error equations in ξ, ζ and β as follows:

$$(\theta_t, p_h) + (\xi_t, p_h) = -(\nabla \cdot \eta, \nabla \cdot p_h) - (\nabla \cdot \zeta, \nabla \cdot p_h) + \left(\int_0^t k(t-s)\big(c(u) - c(u_h)\big) \cdot \sigma)\, ds, \nabla \cdot p_h\right) + \left(\int_0^t k(t-s) c(u_h)\theta ds, \nabla p_h\right) + \left(\int_0^t k(t-s) c(u_h)\xi ds, \nabla \cdot p_h\right) + \left(\int_0^t k(t-s)\big(g(u) - g(u_h)\big)ds, \nabla \cdot p_h\right) \tag{4.1}$$

$$(\xi, \nabla v_h) = (\nabla \beta, \nabla v_h) - (\theta, \nabla v_h) \tag{4.2}$$

$$\begin{aligned}(\eta, w_h) + (\zeta, w_h) = (\theta, w_h) + (\xi, w_h) & \\ -\left(\int_0^t k(t-s)(a(u) - a(u_h))\sigma)ds, w_h)\right) & \\ -\left(\int_0^t k(t-s) a(u_h)\theta ds, w_h\right) & \\ +\left(\int_0^t k(t-s) a(u_h)\xi ds, w_h\right) & \\ -\left(\int_0^t k(t-s)\big(b(u) - b(u_h)\big)ds, w_h\right). &\end{aligned} \tag{4.3}$$

clearly, where

$$\begin{aligned} c(u) \cdot \sigma - c(u_h) \cdot \sigma_h &= (c(u) \cdot \sigma - c(u_h) \cdot \sigma + c(u_h) \cdot \sigma - c(u_h) \cdot \sigma_h \\ &= \big(c(u) - c(u_h)\big) \cdot \sigma + c(u_h) \cdot (\theta + \xi), \end{aligned}$$

We have $\theta = \sigma - R_h \sigma$ *and* $\xi = R_h \sigma - \sigma_h$, also the above equation is holds.

Similarly, we have

$$\begin{aligned} a(u)\sigma - a(u_h)\sigma_h &= (a(u) - a(u_h)\sigma + a(u_h)\sigma - a(u_h)\sigma_h \\ &= \big(a(u) - a(u_h)\big)\sigma + a(u_h)(\theta + \xi). \end{aligned}$$

Now, we will prove the error estimates for $u - u_h, \sigma - \sigma_h$, and $q - q_h$.

Theorem 4.1 Assume that $\sigma_h(0) = R_h \nabla u_0(x)$ and let (u, σ, q) and (u_h, σ_h, q_h)be the solution of (3.3) and (3.10), respectively. Then, the optimal error estimates hold:

(a) $\| u - u_h \|_{H^1} \leq C h^{\min(k+1, m)}$

(b) $\| \nabla . (q - q_h) \| \leq C h^{\min(k, m+1)}$

(c) $\| u - u_h \| + \| \sigma - \sigma_h \| + \| q - q_h \| \leq C h^{\min(k+1, m+1)}$

where $k \geq 1$ and $m \geq 1$ for $d = 2,3$. The index k can be relaxed to include the case of $k = 0$ for $d = 1$.

Proof . For prove (a) we use the triangle inequality, we have

$$\begin{aligned} \| u - u_h \|_{H^1} &= \|u - I_h u + I_h u - u_h\|_{H^1} \qquad (\pm I_h u) \\ &\leq \|u - I_h u\|_{H^1} + \|I_h u - u_h\|_{H^1} \\ &\leq \|\alpha\|_{H^1} + \|\beta\|_{H^1} \end{aligned} \tag{4.4}$$

where $u - I_h u = \alpha$ and $I_h u - u_h = \beta$.

Since estimate of α is given in (3.17) it only we need to estimate β, we choose $v_h = \beta$ in (4.2), to have

$$(\xi, \nabla\beta) = (\nabla\beta, \nabla\beta) - (\theta, \nabla\beta) \tag{4.5}$$

applying Young's inequalities for each term of the above equation

$$(\xi,\nabla\beta) \le \epsilon\|\xi\|^2 + \frac{1}{4\epsilon}\|\nabla\beta\|^2$$

$$(\nabla\beta,\nabla\beta) \le \epsilon\|\nabla\beta\|^2 + \frac{1}{4\epsilon}\|\nabla\beta\|^2$$

$$(\theta,\nabla\beta) \le \epsilon\|\theta\|^2 + \frac{1}{4\epsilon}\|\nabla\beta\|^2$$

where $\left(ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2\right)$

Then, substituting the above inequalities into (4.5) we obtain

$$\epsilon\|\nabla\beta\|^2 + \frac{1}{4\epsilon}\|\nabla\beta\|^2 \le \epsilon\|\xi\|^2 + \frac{1}{4\epsilon}\|\nabla\beta\|^2 + \epsilon\|\theta\|^2 + \frac{1}{4\epsilon}\|\nabla\beta\|^2$$

$$\epsilon\|\nabla\beta\|^2 + \frac{1}{4\epsilon}\|\nabla\beta\|^2 - \frac{1}{2\epsilon}\|\nabla\beta\|^2 \le \epsilon(\|\xi\|^2 + \|\theta\|^2)$$

$$\left(\epsilon + \frac{1}{4\epsilon} - \frac{1}{2\epsilon}\right)\|\nabla\beta\|^2 \le \epsilon(\|\xi\|^2 + \|\theta\|^2)$$

Where $c = \epsilon + \frac{1}{4\epsilon} - \frac{1}{2\epsilon}$ and $c_1 = \frac{\epsilon}{c}$

since $\beta \in V_h \subset H_0^1(\Omega)$,then $\|\beta\| \le c_0\|\nabla\beta\|$,thus ,we have the estimate $\|\beta\|$,

$$\|\nabla\beta\|^2 \le c_1(\|\xi\|^2 + \|\theta\|^2) \tag{4.6}$$

$$\|\beta\|^2 \le cc_0(\|\xi\|^2 + \|\theta\|^2) \tag{4.7}$$

note that into the above inequality, we need to estimate ξ , we choose $w_h = \xi$ in (4.3) to get

$$\begin{aligned}(\xi,\xi) = (\eta,\xi) + (\zeta,\xi) - (\theta,\xi) &+ \left(\int_0^t k(t-s)(a(u) - a(u_h))\sigma)ds,\xi\right)\\ &+\left(\int_0^t k(t-s)a(u_h)\theta ds,\xi\right)\\ &-\left(\int_0^t k(t-s)a(u_h)\xi ds,\xi\right)\\ &+\left(\int_0^t k(t-s)(b(u) - b(u_h))ds,\xi\right)\\ &= \textstyle\sum_{i=1}^{7} L_i\end{aligned} \tag{4.8}$$

We apply Young's inequalities to estimate the terms on right side with appropriately small ε,

$$|L_1| = |-(\theta,\xi)| \le c\|\theta\|^2 + \varepsilon\|\xi\|^2 \tag{4.9}$$

$$|L_2| = |(\eta,,\xi)| \le c\|\eta\|^2 + \varepsilon\|\xi\|^2 \tag{4.10}$$

$$|L_3| = |(\zeta,\xi)| \le c\|\zeta\|^2 + \varepsilon\|\xi\|^2 \tag{4.11}$$

$$|L_4| = \left|-\left(\int_0^t k(t-s)(a(u) - a(u_h))\sigma ds,\zeta\right)\right|$$

$$\le cc_1c_2\int_0^t(\|\alpha\|^2 + \|\beta\|^2) + \varepsilon\|\xi\|^2 \tag{4.12}$$

where c_1 depends on $K(t-s)$,and c_2 depends on $\|\sigma\|_{W_\infty^1(L^\infty)}$.

$$|L_5| = \left|-\left(\int_0^t k(t-s)a(u_h)\theta ds,\xi\right)\right|$$

$$\le cc_1c_3\int_0^t\|\theta\|^2 ds + \varepsilon\|\xi\|^2 \tag{4.13}$$

where c_3 depends on $a(u_h)$

$$|L_6| = \left|\left(\int_0^t k(t-s)a(u_h)\xi ds,\xi\right)\right| = 0 \tag{4.14}$$

$$|L_7| = \left|-\left(\int_0^t k(t-s)(b(u) - b(u_h))ds,\xi\right)\right|$$

$$\le cc_1\int_0^t(\|\alpha\|^2 + \|\beta\|^2) + \varepsilon\|\xi\|^2 \tag{4.15}$$

combining the above inequalities from (4.9) to (4.15), we obtain

$$\|\xi\|^2 \le C_1(\|\theta\|^2 + \|\eta\|^2 + \|\zeta\|^2) + C_1\int_0^t(\|\alpha\|^2 + \|\beta\|^2 + \|\theta\|^2)ds \tag{4.16}$$

where $C_1 = C_1(c, c_1, c_2, c_3)$.

Also here, we need to estimate ζ, taking $w_h = \zeta$ in (4.3) yields

$$\begin{aligned}(\zeta,\zeta) = (\theta,\zeta) + (\xi,\zeta) - (\eta,\zeta)&\\ &-\left(\int_0^t k(t-s)(a(u) - a(u_h))\sigma)ds,\zeta\right)\\ &-\left(\int_0^t k(t-s)a(u_h)\theta ds,\zeta\right)\end{aligned}$$

$$+\left(\int_0^t k(t-s)a(u_h)\xi ds,\zeta\right)$$
$$-\left(\int_0^t k(t-s)\big(b(u)-b(u_h)\big)ds,\zeta\right)$$
$$=\sum_{i=1}^{7} E_i \tag{4.17}$$
We use Young's inequalities to estimate the right side term by the term.
$$|E_1| = |(\theta,\zeta)| \le c\|\theta\|^2 + \varepsilon\|\zeta\|^2 \tag{4.18}$$
$$|E_2| = |(\xi,\zeta)| \le c\|\xi\|^2 + \varepsilon\|\zeta\|^2 \tag{4.19}$$
$$|E_3| = |-(\eta,\zeta)| \le c\|\eta\|^2 + \varepsilon\|\zeta\|^2 \tag{4.20}$$
$$|E_4| = \left|\left(\int_0^t k(t-s)(a(u)-a(u_h))\sigma)ds,\zeta\right)\right|$$
$$\le cc_1c_2\int_0^t(\|\alpha\|^2+\|\beta\|^2)ds + \varepsilon\|\zeta\|^2 \tag{4.21}$$
$$|E_5| = \left|\left(\int_0^t k(t-s)a(u_h)\theta ds,\zeta\right)\right|$$
$$\le cc_1c_3\int_0^t\|\theta\|^2 ds + \varepsilon\|\zeta\|^2 \tag{4.22}$$
$$|E_6| = \left|\left(\int_0^t k(t-s)a(u_h)\xi ds,\zeta\right)\right| \tag{4.23}$$
$$|E_7| = \left|\left(\int_0^t k(t-s)\big(b(u)-b(u_h)\big)ds,\zeta\right)\right|$$
$$\le cc_1\int_o^t(\|\alpha\|^2+\|\beta\|^2) + \varepsilon\|\zeta\|^2 \tag{4.24}$$
Thus, by appropriately small ε, setting (4.18) – (4.24) into (4.17) to yield.
$$\|\zeta\|^2 \le C_2(\|\theta\|^2+\|\xi\|^2+\|\eta\|^2) + C_2\int_0^t(\|\alpha\|^2+\|\beta\|^2+\|\theta\|^2)ds \tag{4.25}$$
Here , $C_2 = C_2(c,c_1,c_2,c_3)$

Now, putting (4.25) into (4.16) we get,
$$\|\xi\|^2 \le (\|\theta\|^2+\|\eta\|^2+C_2(\|\theta\|^2+\|\xi\|^2+\|\eta\|^2)$$
$$+C_2\int_0^t(\|\alpha\|^2+\|\beta\|^2+\|\theta\|^2)ds$$
$$+C_1\int_0^t(\|\alpha\|^2+\|\beta\|^2+\|\theta\|^2)ds$$
$$\|\xi\|^2 \le C_3(\|\theta\|^2+\|\eta\|^2+\|\xi\|^2) + C_3\int_0^t(\|\alpha\|^2+\|\beta\|^2+\|\theta\|^2)ds \tag{4.26}$$
where $C_3 = C_3(C_1,C_2)$.

combining (4.7) with (4.26) we obtain,
$$\|\xi\|^2 \le C_3(\|\theta\|^2+\|\eta\|^2+\|\xi\|^2) + C_3\int_0^t(\|\alpha\|^2+\|\theta\|^2+\|\xi\|^2)ds \tag{4.27}$$
applying Gronwall inequalities [1] to the above equation, one has
$$\|\xi\|^2 \le C_3(\|\theta\|^2+\|\eta\|^2) + C_3\int_0^t(\|\alpha\|^2+\|\theta\|^2)ds \tag{4.28}$$
from (3.17) , (3.19) and (4.28) we have
$$\|\xi\|^2 \le C_3\left(ch^{2(k+1)}\|\sigma\|^2_{L^\infty(H^{k+1})} + ch^{2(k+1)}\|q\|^2_{L^\infty(H^{k+1})}\right)$$
$$+C_3\int_0^t(ch^{2(m+1)}\|u\|^2_{L^\infty(H^{m+1})} + ch^{2(k+1)}\|\sigma\|^2_{L^\infty(H^{K+1})})$$
$$\|\xi\|^2 \le C_5h^{2\min(k+1,m+1)}\left(\|\sigma\|^2_{L^\infty(H^{k+1})} + \|q\|^2_{L^\infty(H^{k+1})} + \|u\|^2_{L^\infty(H^{m+1})}\right) \tag{4.29}$$

$$\|\xi\|^2 \le C_5h^{2\min(k+1,m+1)} \tag{4.30}$$
$$\|\xi\| \le C_5h^{\min(k+1,m+1)} \tag{4.31}$$
Now substituting (4.29) into (4.7) with (3.19) we obtain
$$\|\beta\|^2 \le cc_0(C_5h^{2\min(k+1,m+1)}\left(\|\sigma\|^2_{L^\infty(H^{k+1})} + \|q\|^2_{L^\infty(H^{k+1})} + \|u\|^2_{L^\infty(H^{m+1})}\right)$$
$$+Ch^{2(k+1)}\|q\|^2_{L^\infty(H^{k+1})})$$
$$\|\beta\|^2 \le C_6h^{2\min(k+1,m+1)}\left(\|\sigma\|^2_{L^\infty(H^{k+1})} + \|q\|^2_{L^\infty(H^{K+1})} + \|u\|^2_{L^\infty(H^{m+1})}\right) \tag{4.32}$$
$$\|\beta\|^2 \le C_6h^{2\min(k+1,m+1)}$$
$$\|\beta\| \le C_6h^{\min(k+1,m+1)} \tag{4.33}$$
where from the given in the theorem we have
$$\|\sigma\|^2_{L^\infty(H^{k+1})} + \|q\|^2_{L^\infty(H^{k+1})} + \|u\|^2_{L^\infty(H^{m+1})} = 1$$

Now substituting (4.32) into (4.4) with (3.17) we get

$$\|u-u_h\|_{H^1} \le Ch^m\|u\|_{m+1} + C_6 h^{\min(k+1,m+1)} \le Ch^{\min(k+1,m)} \tag{4.34}$$

where $C = C(C, C_6)$

and

$$\|u-u_h\|_{H^1} \le Ch^m\|u\|_{m+1}$$

but

$$\|u-u_h\| \le ch^{m+1}\|u\|_{m+1},$$

similarly, we prove (b), namely

$$\begin{aligned}\|\nabla\cdot(q-q_h)\| &= \|\nabla\cdot(q-R_hq+R_hq-q_h)\| \\ &\le \|\nabla\cdot(q-R_hq)\| + \|\nabla\cdot(R_hq-q_h)\| \\ &\le \|\nabla\cdot\eta\| + \|\nabla\cdot\zeta\|.\end{aligned} \tag{4.35}$$

Since, estimate of η is given in (3.19), it sufficient to estimate $\|\nabla\cdot\zeta\|$. We choose $p_h = \zeta$

$$\begin{aligned}(\nabla\cdot\zeta,\nabla\cdot\zeta) &= -(\theta_t,\zeta) - (\xi_t,\zeta) - (\nabla\cdot\eta,\nabla\cdot\zeta) \\ &+ \left(\int_0^t k(t-s)\big(c(u)-c(u_h)\big)\cdot\sigma)ds, \nabla\cdot\zeta\right) \\ &+ \left(\int_0^t k(t-s)c(u_h)\theta ds, \nabla\cdot\zeta\right) \\ &+ \left(\int_0^t k(t-s)c(u_h)\xi ds, \nabla\cdot\zeta\right) \\ &+ \left(\int_0^t k(t-s)\big(g(u)-g(u_h)\big)ds, \nabla\cdot\zeta\right) \\ &= \sum_{i=1}^{7} D_i\end{aligned} \tag{4.36}$$

Using Young's inequalities to bound all the terms on the right side, we get

$$|D_1| = |-(\theta_t,\zeta)| \le \tfrac{2C_2}{a_1}\|\theta_t\|^2 + \tfrac{a_1}{2C_2}\|\zeta\|^2 \tag{4.37}$$

$$|D_2| = |-(\xi_t,\zeta)| \le \tfrac{2C_2}{a_1}\|\xi_t\|^2 + \tfrac{a_1}{2C_2}\|\zeta\|^2 \tag{4.38}$$

$$|D_3| = |(\nabla\cdot\eta,\nabla\cdot\zeta)| \le c\|\nabla\cdot\eta\|^2 + \varepsilon\|\nabla\cdot\zeta\|^2 \tag{4.39}$$

$$\begin{aligned}|D_4| &= \left|\left(\int_0^t k(t-s)\big(c(u)-c(u_h)\big)\cdot\sigma ds, \nabla\cdot\zeta\right)\right| \\ &\le cc_1c_2\int_0^t(\|\alpha\|^2+\|\beta\|^2)\,ds + \varepsilon\|\nabla\cdot\zeta\|^2\end{aligned} \tag{4.40}$$

$$\begin{aligned}|D_5| &= \left|\left(\int_0^t k(t-s)c(u_h)\theta ds, \nabla\cdot\zeta\right)\right| \\ &\le cc_1c_4\int_0^t\|\theta\|^2\,ds + \varepsilon\|\nabla\cdot\zeta\|^2\end{aligned} \tag{4.41}$$

where c_4 depends on $c(u_h)$.

$$\begin{aligned}|D_6| &= \left|\left(\int_0^t k(t-s)c(u_h)\xi ds, \nabla\cdot\zeta\right)\right| \\ &\le cc_1c_4\int_0^t\|\xi\|^2 ds + \varepsilon\|\nabla\cdot\zeta\|^2\end{aligned} \tag{4.42}$$

$$\begin{aligned}|D_7| &= \left|\left(\int_0^t k(t-s)(g(u)-g(u_h)ds, \nabla\cdot\zeta\right)\right| \\ &\le c\int_0^t(\|\alpha\|^2+\|\beta\|^2)\,ds + \varepsilon\|\nabla\cdot\zeta\|^2\end{aligned} \tag{4.43}$$

Thus, by appropriately small ε, setting (4.37) – (4.43) into (4.36) yields

$$\begin{aligned}\|\nabla\cdot\zeta\|^2 &\le C_4(\|\theta_t\|^2 + \|\xi_t\|^2 + \|\nabla\cdot\eta\|^2 + \|\zeta\|^2) \\ &+ C_4\int_0^t(\|\alpha\|^2+\|\beta\|^2+\|\theta\|^2+\|\xi\|^2)ds\end{aligned} \tag{4.44}$$

where $C_4 = C_4(c_4, a_1)$,

Here, we need to estimate $\|\xi_t\|$, we choose $p_h = \xi_t$ in (4.1) and obtain

$$\begin{aligned}(\xi_t,\xi_t) &= -(\theta_t,\xi_t) - (\nabla\cdot\eta,\nabla\cdot\xi_t) - (\nabla\cdot\zeta,\nabla\cdot\xi_t) \\ &+ \left(\int_0^t k(t-s)\big(c(u)-c(u_h)\big)\cdot\sigma)ds, \nabla\cdot\xi_t\right)\end{aligned}$$

$$+\left(\int_0^t k(t-s)c(u_h)\theta ds, \nabla\cdot\xi_t\right)$$
$$+\left(\int_0^t k(t-s)c(u_h)\xi ds, \nabla\cdot\xi_t\right)$$
$$+\left(\int_0^t k(t-s)\big(g(u)-g(u_h)\big)ds, \nabla\cdot\xi_t\right) = \sum_{i=0}^{7} T_i \quad (4.45)$$

We analyse the right-hand side terms of (4.45) by using Young's inequalities with appropriately small ε, we obtain

$$|T_1| = |-(\theta_t, \xi_t)| \le c\|\theta_t\|^2 + \varepsilon\|\xi_t\|^2 \quad (4.46)$$
$$|T_2| = |-(\nabla\cdot\eta, \nabla\cdot\xi_t| \le c\|\nabla\cdot\eta\|^2 + \varepsilon\|\nabla\cdot\xi_t\|^2 \quad (4.47)$$
$$|T_3| = |-(\nabla\cdot\zeta, \nabla\cdot\xi_t)| \le c\|\nabla\cdot\zeta\|^2 + \varepsilon\|\nabla\cdot\xi_t\|^2 \quad (4.48)$$
$$|T_4| = \left|\left(\int_0^t k(t-s)(c(u)-c(u_h))\cdot\sigma)ds, \nabla\cdot\xi_t\right)\right|$$
$$\le cc_1c_2\int_0^t(\|\alpha\|^2 + \|\beta\|^2)\,ds + \varepsilon\|\nabla\cdot\xi_t\|^2 \quad (4.49)$$
$$|T_5| = \left|\left(\int_0^t k(t-s)c(u_h)\theta, \nabla\cdot\xi_t\right)\right|$$
$$\le cc_1c_4\int_0^t\|\theta\|^2\,ds + \varepsilon\|\nabla\cdot\xi_t\|^2 \quad (4.50)$$
$$|T_6| = \left|\left(\int_0^t k(t-s)c(u_h)\xi ds, \nabla\cdot\xi_t\right)\right|$$
$$\le cc_1c_4\int_0^t\|\xi\|^2 ds + \varepsilon\|\nabla\cdot\xi_t\|^2 \quad (4.51)$$
$$|T_7| = \left|\left(\int_0^t k(t-s)\big(g(u)-g(u_h)\big)ds, \nabla\cdot\xi_t\right)\right|$$
$$\le cc_1\int_0^t(\|\alpha\|^2 + \|\beta\|^2)\,ds + \varepsilon\|\nabla\cdot\xi_t\|^2, \quad (4.52)$$

Thus, combining the above inequalities from (4.46) to (4.52), we get

$$\|\xi_t\|^2 + \|\nabla\cdot\xi_t\|^2 \le C_7(\|\theta_t\|^2 + \|\nabla\cdot\eta\|^2 + \|\nabla\cdot\zeta\|^2)$$
$$+C_7\int_0^t(\|\alpha\|^2 + \|\beta\|^2 + \|\theta\|^2 + \|\xi\|^2)ds \quad (4.53)$$

then, $\|\xi_t\|^2 \le \|\xi_t\|^2 + \|\nabla\cdot\xi_t\|^2$,

thus, (4.53) becomes

$$\|\xi_t\|^2 \le C_7(\|\theta_t\|^2 + \|\nabla\cdot\eta\|^2 + \|\nabla\cdot\zeta\|^2)$$
$$+C_7\int_0^t(\|\alpha\|^2 + \|\beta\|^2 + \|\theta\|^2 + \|\xi\|^2)ds \quad (4.54)$$

from (4.54) into (4.44) we have

$$\|\nabla\cdot\zeta\|^2 \le C_4(\|\theta_t\|^2 + C_7(\|\theta_t\|^2 + \|\nabla\cdot\eta\|^2 + \|\nabla\cdot\zeta\|^2)$$
$$+C_7\int_0^t(\|\alpha\|^2 + \|\beta\|^2 + \|\theta\|^2 + \|\xi\|^2)ds + \|\nabla\cdot\eta\|^2$$
$$+\|\xi\|^2 + C_4\int_0^t(\|\alpha\|^2 + \|\beta\|^2 + \|\theta\|^2 + \|\xi\|^2)ds$$

after simplify we get

$$\|\nabla\cdot\zeta\|^2 \le C_8(\|\theta_t\|^2 + \|\nabla\cdot\eta\|^2 + \|\xi\|^2)$$
$$+C_8\int_0^t(\|\alpha\|^2 + \|\beta\|^2 + \|\theta\|^2 + \|\xi\|^2)ds \quad (4.55)$$

where $C_8 = C_8(C_4, C_7)$

now substituting (4.29) and (4.32) into (4.55) we get

$$\|\nabla\cdot\zeta\| \le C_8 h^{\min(k+1,m+1)} \quad (4.56)$$

where $\|\sigma_t\|^2_{L^\infty(H^{k+1})} + \|q\|^2_{L^\infty(H^{k+1})} + \|u\|^2_{L^\infty(H^{m+1})} + \|\sigma\|^2_{L^\infty(H^{k+1})} = 1$

then putting (4.55) and (3.20) in (4,34) we have

$\|\nabla.(q-q_h)\| \le Ch^k\|q\|_{k+1} + C_8 h^{\min(k+1,m+1)}$

$$\leq C h^{\min(k,m+1)} \tag{4.57}$$

where $\quad C = C(C, C_8)$

for (c) we have

$$\|u-u_h\| + \|\sigma-\sigma_h\| + \|q-q_h\|$$
$$\leq \|u - I_h u + I_h u - u_h\| + \|\sigma - R_h\sigma + R_h\sigma - \sigma_h\|$$
$$+\|q - R_h q + R_h q - q_h\|$$
$$\leq \|u - I_h u\| + \|I_h u - u_h\| + \|\sigma - R_h\sigma\|$$
$$+\|R_h\sigma - \sigma_h\| + \|q - R_h q\| + \|R_h q - q_h\|$$
$$\leq \|\alpha\| + \|\beta\| + \|\theta\| + \|\xi\| + \|\eta\| + \|\zeta\| \tag{4.58}$$

Now, substituting (3.17),(3.19),(4.31) and (4.33) into (4.25), we get

$$\|\zeta\| \leq Ch^{\min(k+1,m+1)} \tag{4.59}$$

Therefore, setting (3.17), (3.19) ,(4.31) , (4,33) and (4.59) into (4.58) we obtain

$$\|u-u_h\| \leq C h^{\min(k+1,m+1)} \tag{4.60}$$

REFERENCES

1. Beesack, P.R., Gronwall inequalities, *Carleton Mathematical Lecture Notes*,1975.
2. F. Breezi, *On the existence, Uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers,* RAIRO Anal Numer 8 (1974), 129–151.
3. S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods,* Springer-Verlag , New York, 1991.
4. F. Brezzi, M. Fortin, *Mixed and Hybrid Finite Element Methods*. Springe (1991).
5. J. R. Cannon and Y. Lin, *Non-classical H^1 projection and Galerkin methods for nonlinear parabolic integro-differential equation,* Calcolo 25 (1988) 187-201.
6. J. R. Cannon and Y. Lin, *A priori L^2 error estimates for finite element methods for nonlinear diffusion equations with memory,* SIAM J. Numer. Anal. 27 (1990) 595-607.
7. J. R. Cannon and Y. Lin, *Non-classical H^1 projections and Galerkin methods for nonlinear parabolic integro-differential equations*, Calcolo 25 (1988),187-201.
8. H. Che, Z. Zhou, Z. Jiang, Y. Wang, *H^1-Galerkin Expanded Mixed Finite Element Methods for Nonlinear Pseudo-Parabolic Integro-Differential Equations,* Wiley Periodicals, Inc.29(2012),799-817.
9. Z. Chen, *Expanded mixed finite element methods for linear second-order elliptic problems,* RAIRO Mat h Model Numer Anal 32 (1998), 479–499.
10. C. Chen and T. Shih, *Finite element methods for integro-differential equations, Singapore:* World Scientific Pub. Co., 1998.
11. H. Chen and H. Wang, *An optimal-order error estimate on an H^1-Galerkin for a nonlinear parabolic equation in porous medium flow*, Numer Meth Part Diff Equat 26 (2010), 188–205.
12. P. G. Ciarlet , *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978.
13. J. Douglas, R. Ewing, and M. Wheeler, *The approximation of the pressure by a mixed method in the simulation of miscible displacement,* RAIRO Anal Numer 17 (1983), 17–33.
14. J. Douglas, *Super convergence in the pressure in the simulation of miscible displacement*, SIAM J Numer Anal 22 (1985), 962–969.
15. J. Droniou , M. Ilyas, B. P. Lamichhane, and G. E. *Wheeler. A mixed finite element method for a sixth-order elliptic problem.* IMA Journal of Numerical Analysis,39(2017):374–397.
16. R. E. Ewing, Y. Lin, T. Sun, J. Wang, and S. Zhang, *Sharp L^2-error estimates and super convergence of mixed finite element methods for non-Fickian flows in porous media,* SIAM J. Numer. Anal., 40 (2002), 1538-1560.
17. B. Fraeijs de Veubeke, *Displacement and equilibrium models in the finite element method,* Wiley, London, 1965, 145–197.
18. G. N. Gatica, R. R. Baier, and G. *Tierra. A mixed finite element method for Darcy's equations with pressure dependent porosity.* Math. Comput., 85(2016):1–33.
19. D. Go swami, A. K. Pani and S. Yadav, *Optimal Error Estimates of Two Mixed Finite Element Methods for Parabolic Integro-Differential Equations with Nonsmooth Initial Data*, J Sci Comput 56 (2013) ,131–164 .

20. M. Gurtin and A. Pipkin, *A general theory of heat conduction with finite wave speeds,* Arch. Rational Mech. Anal., 31 (1968), 113-126.
21. Y. Lin, *Galerkin methods for nonlinear parabolic integrodifferential equations with nonlinear boundary conditions,* SIAM J. Numer. Anal., 27 (1990), 608-621.
22. J. C. Lopez-Marcos, *A difference scheme for a nonlinear partial integrodifferential equation,* SIAM J. Numer. Anal. 27 (1990), 20-31.
23. Smys, S., Joy Chen, and Subarna Shakya, eds. "Preface: 2nd International Conference on Inventive Research in Material Science and Technology (ICIRMCT 2019)." In *AIP Conference Proceedings*, vol. 2087, no. 1, p. 010001. AIP Publishing LLC, 2019.
24. R. K. Miller, *An integro-differential equation for rigid heat conductions with memory,* J. Math. Anal. Appl., 66 (1978), 313-332.
25. K. Mustapha, H. Brunner, H. Mustapha, and D. Schotzau, *An hp-version discontinuous Galerkin method for integro-differential equations of parabolic type,* SIAM J. Numer. Anal., 49 (2011), 1369-1396.
26. A. K. Pani and T. E. Peterson, *Finite element methods with numerical quadrature for parabolic integrodifferential equations,* SIAM J.Numer. Anal., 33 (1996), 1084-1105.
27. A. K. Pani, an H^1*-Galerkin mixed finite element method for parabolic differential equations,* SIAM J. Numer Anal 35 (1998), 712–727.
28. Sahulhameedu, S., Chen, J., & Shakya, S. (Eds.). (2018, May). Preface: International Conference on Inventive Research in Material Science and Technology (ICIRMCT 2018). In AIP Conference Proceedings (Vol. 1966, No. 1, p. 010001). AIP Publishing LLC.
29. A. K. Pani and G. Fair weather, H^1*-Galerkin mixed finite element methods for parabolic partial integro-differential equations,* IMA J Numer Anal 22 (2002), 231–252.
30. A. K. Pani, R. K. Sinha, and A. K. Otta*, An* H^1*-Galerkin mixed finite element method for second order hyperbolic equations*, Int J Numer Anal Mode 2 (2004), 111–129.
31. M. Raynal, *On some nonlinear problems of diffusion, in Volterra Equations*, S. London and 0. Stiffens, eds., Lecture Notes in Math., 737, Springer-Verlag, Berlin, New York, 737 (1979), 251-266.
32. Z. Shi, Y. Zhao, Y. Tang, F.Wang and Y. Shi, Super convergence analysis of an H^1-Galerkin mixed finite element method for two-dimensional multi-term time fractional diffusion equations, International Journal of Computer Mathematics,95(2017),1845-1857.
33. R. K. Sinha, R. E. Ewing, and R. D. Lazarov, *Mixed finite element approximations of parabolic integro-differential equations with nonsmooth initial data,* SIAM J. Numer. Anal., 47 (2009), 3269-3292.
34. R. K. Sinha, R. E. Ewing, and R. D. Lazarov*, Some new error estimates of a semi discrete finite volume element method for a parabolic integro-differential equation with nonsmooth initial data,* SIAM J. Numer. Anal., 43 (2006), 2320-2343.
35. W. Wanga and Q. Hong, *Two-grid economical algorithms for parabolic integro- differential equations with nonlinear memory Applied Numerical Mathematics,* 142(2019), 28-46.
36. D. Xu, *Analytical and numerical solutions of a class of nonlinear integro-differential equations with* L^1 *kernels . Nonlinear Analysis:* Real World Applications 51 (2019) 103002.
37. T. Zhang*, Finite element methods for partial differential-integral equations*, Beijing: Science Press, 2009 (in Chinese).