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The Numerical Solution of Time -Space Fractional Bioheat Equation By Using Fractional Quadratic Spline Methods

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Abstract. In this paper, a time-space fractional one-dimensional bioheat transfer model of temperature distribution in tissue has been solved Caputo fractional derivative for time fractional derivative with α order and fractional quadratic spline for space of fractional derivative with β order. The goal of this article is to make a comparison between exact and numerical solutions. Theoretically, the stability analysis uses Von Neumann method

INTRODUCTION

Penne's bioheat model [18] is widely used to study the heat transfer in skin tissue in our bodies. In the human body, skin is the largest living organ. Temperature distribution in skin tissue is very important for medical applications like skin cancer, skin burns etc. [4]. Fractional differential equations have been an important aspect of many studies owing to their appearance in many applications in medical science, fluid mechanics, viscoelasticity, and physics. Singh et al. (2011) presented the solution of fractional bioheat equation by taking the shifted Grunewald finite difference approximation for Riemann-Louville space fraction derivative finite difference method and HPM of fractional derivative of space and the Caputo fractional for the fractional time. It has been observed that the time taken to achieve hyperthermia in a position decreases as the order fractional derivative decreases.

Dehghan, M., Sabouri, M. discussed the one- and two-dimensional Penne's bioheat model, the implementation of triangular and quadrilateral elements method is completely explained. In the two-dimensional case, both triangular and quadrilateral elements are investigated. Through test problems, the discretization error generated from this method is reported. Damor et. al (2013) discussed the numerical solution of fractional bioheat equation with constant and Sinusoidal heat flux condition on skin by implicit finite difference method and the fractional time derivative is of Caputo form. Cui, et.al (2014) showed numerical solution for the time-fractional Penne's bioheat transfer equation on skin tissue and solved by Fourier Sine transform for second order derivative and the Caputo fractional derivative for the fractional time. Ezzat, et. al (2015) discussed the 2-D fractional bioheat equation by Laplace transforms of second order derivative, and investigated by the numerical calculations were performed to study the temperature transients in the skin exposed to instantaneous surface heating. Some comparisons were showed to estimate the effect the fractional order parameter an on the thermal wave. Mishra and Rai (2015) presented numerical solution of the fractional bioheat equation by finite difference of second order derivative and the Grunwald-Letnikov fractional derivative for the fractional time, and discussed analyze the stability and convergence. Luis Ferrás et. al (2015) studied the fractional bioheat transfer equation and solve it by approximate solution (numerically) by finite difference of second order derivative and the fractional time derivative by Caputo derivative, and discussed the stability and convergence by this scheme. Pandey (2015) discussed the 2-D fractional bioheat equation by using Galerkin FEM and he found the solution method in the cylindrical living tissue and noted the effects of thermal conductivities have the significant and more remarkable effects in temperature variation in living tissue. Damor et. al (2016) studied the fractional bioheat equation when the time fractional derivative and space fractional derivative in the form and solve it by Caputo fractional derivative of order $\alpha \in (0, 1]$ and Riesz–Feller fractional derivative of order $\beta \in (1, 2]$ respectively, and obtained solution in terms of Fox's H-function with

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some special cases, by using Fourier–Laplace transforms. Roohi R. et. al (2018) studied the fractional bioheat equation and solve it by of space-time fractional bioheat equation using fractional-order Legendre functions of fractional space order derivative and the fractional time derivative by Caputo derivative, and he note the magnitude of the temperature at the skin surface is a strong function of the space fractional order and conversely the effect of the time fractional order is almost negligible. Abdulhussein and Hameeda (2019) discussed the numerical solutions of fractional time bioheat equation of temperature distribution in tissue by nonpolynomial spline and exponential spline methods for second-order derivative for space derivative and Caputo fractional derivative for fractional time derivative by different values of fractional order α, and notes these methods are simple, easy to apply and effective, and discussed analyze the convergence and stability. Also, Abdulhussein and Hameeda (2019) studied the time-space fractional two-dimensional bioheat transfer model of temperature distribution in tissue was solved by Caputo fractional derivative for time fractional derivative with α order and fractional quadratic spline for space fractional derivative with β order. The stability analysis is theoretically using Von Neumann method.

In section 2, we introduce the time-space fractional Penne's bioheat transfer equation and all constants in it. In section 3, we present the mathematical background related to the fractional definitions. In section 4, we derive a fractional quadratic spline form for the fractional derivative with order β. In section 5, we derive a Caputo fractional derivative for the time fractional derivative. In section 6, we use Caputo fractional derivative with time fractional derivative and fractional quadratic spline with space fractional derivative in Pennes' bioheat transfer equation. In section 7, we study stability analysis for Pennes' bioheat transfer equation. In section 8, we apply and found numerical solutions to time-space fractional penne's bioheat equation by Caputo fractional derivative and fractional quadratic spline methods.

PENNE'S BIOHEAT TRANSFER EQUATION WITH TIME-SPACE FRACTIONAL DERIVATIVE

The time-space fractional Penne's bioheat transfer equation for modeling skin tissue heat transfer is expressed as [1-3], [7], [10], [12], [21]

$$\rho_t c_t \frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \mu \frac{\partial^\beta T(x,t)}{\partial x^\beta} + W_b c_b (T_a - T) + Q + q_m \tag{1}$$

$$T(x,0) = T^0 = f(x), \quad x \in (0,L), \tag{2}$$

$$\frac{\partial}{\partial x} T(x,0) = g(x), \quad x \in (0,L), \tag{3}$$

$$T(0,t) = h(t), \quad t > 0, \tag{4}$$

where $\alpha \in (0,1)$ is fractional order of time and $\beta \in (1,2)$ is fractional order of space, x is the distance from the skin surface, ρ_t, c_t are constants representing the density and the specific heat , respectively, and μ is the tissue thermal conductivity, W_b is the mass flow rate of blood per unit volume of tissue, c_b is the blood specific heat, q_m is the metabolic heat generation per unit volume, T_a represents the temperature of arterial blood, T is the temperature of tissue and the term $W_b c_b (T_a - T)$ represents the blood perfusion. It is worth mentioning that the W_b constant was experimentally obtained by Penne's for a human forearm, Q is the metabolic heat source.

DEFINITIONS

Definition (1): The Riemann-Liouville fractional derivative of order $\alpha \in (n-1, n)$, is $n \in N, t > a$, defined by [1-7], [9-11], [13-15], [22-23], [25]

$$ {}^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\psi^n} \int_a^t (t-\psi)^{n-\alpha-1} f(\psi)\, d\psi \quad , \quad \alpha \in (n-1, n)$$

Definition (2): The Caputo fractional derivative of order $\alpha \in (n-1, n), n \in N, t > a$ is defined by [1-7], [9-11], [13-15], [22-23], [25]

$$ {}^{C}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\psi)^{n-\alpha-1} \frac{d^n}{d\psi^n} f(\psi) d\psi \quad , \quad \alpha \in (n-1, n) $$

FRACTIONAL QUADRATIC SPLINE FORM FOR THE SPACE FRACTIONAL DERIVATIVE

Let us write the quadratic spline $Q_i(x)$ in the form:

$$ Q_i(x, t_j) = a_i(t_j) + b_i(t_j)(x - x_i) + c_i(t_j)(x - x_i)^2, \quad i = 0, 1, \ldots, n-1, \tag{5} $$

where $a_i(t_j)$, $b_i(t_j)$ and $c_i(t_j)$ are unknown coefficients, to derive expression for the coefficients of (5) in terms of S_i, S_{i+1} and F_{i+1}, at first we define [19], [20]

$$ Q_i(x_i) = S_i, \quad Q_i(x_{i+1}) = S_{i+1}, \quad \frac{\partial^{\beta} Q(x_{i+1}, t_j)}{\partial x^{\beta}} = F_{i+1} \tag{6} $$

From (5), (6) and by Caputo fractional derivative we get

$$ a_i = S_i, \quad b_i = \frac{1}{h}(S_{i+1} - S_i) - \frac{h^{\beta-1}\Gamma(3-\beta)}{2} F_{i+1} \quad , \quad c_i = \frac{\Gamma(3-\beta)}{2h^{2-\beta}} F_{i+1} \tag{7} $$

Therefore, by (7) and the continuity conditions $Q_{i-1}^{(1)}(x_i) = Q_i^{(1)}(x_i)$, which gives the following relation

$$ F_{i+1} + F_i = \delta(S_{i+1} - 2S_i + S_{i-1}), \tag{8} $$

where $\delta = \frac{2}{h^{\beta}\Gamma(3-\beta)}$

CAPUTO FRACTIONAL DERIVATIVE FOR THE TIME FRACTIONAL DERIVATIVE

The Caputo time-fractional derivative at time point $t = t_{k+1}$, can be approximated, as

$$ \frac{\partial^{\alpha} T(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{r+1}} \frac{\partial T(x,\tau)}{\partial \tau} (t_{r+1} - \tau)^{-\alpha} d\tau $$

$$ = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{r} \int_{t_k}^{t_{k+1}} \frac{\partial T(x,\tau)}{\partial \tau} (t_{r+1} - \tau)^{-\alpha} d\tau $$

$$ = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{r} \frac{T(x, t_{k+1}) - T(x, t_k)}{\tau} \int_{t_k}^{t_{k+1}} (t_{r+1} - \tau)^{-\alpha} d\tau $$

To discretize the Caputo time-fractional derivative is used forward Euler scheme. Let $t_r = r\Delta t, r = 0, 1, \ldots, K$, in which $\tau = \frac{G}{K}$ is the time step size.

$$ \frac{\partial^{\alpha} T(x,t)}{\partial t^{\alpha}} = \frac{1}{\tau^{\alpha}\Gamma(2-\alpha)} \sum_{k=0}^{r} u_j [T(x, t_{r-k+1}) - T(x, t_{r-k})] $$

$$ = \frac{1}{\tau^{\alpha}\Gamma(2-\alpha)} [T^{r+1} + T^r(u_1 - 1) + \textstyle\sum_{k=1}^{r-1} T^{r-k}(u_{k+1} - u_k) - u_r T^0], \tag{9} $$

where $u_j = (j+1)^{1-\alpha} - (j)^{1-\alpha}$, for all $j = 0, \ldots, r$

Now, we define the semi discrete fractional differential operator $Y_t^{\alpha}(x, t_{r+1})$ as,

$$ Y_t^{\alpha}(x, t_{r+1}) = \frac{1}{\tau^{\alpha}\Gamma(2-\alpha)} [T^{r+1} + T^r(u_1 - 1) + \textstyle\sum_{j=1}^{r-1} T^{r-j}(u_{j+1} - u_j) - u_r T^0], \tag{10} $$

can be written equation (16) as,

$$ \frac{\partial^{\alpha} T(x, t_{r+1})}{\partial t^{\alpha}} = Y_t^{\alpha}(x, t_{r+1}) + E_T^{r+1} \tag{11} $$

where E_T^{r+1} is truncation error between $\frac{\partial^{\alpha} T(x, t_{r+1})}{\partial t^{\alpha}}$ and $Y_t^{\alpha}(x, t_{r+1})$, also is bounded

$$ |E_T^{r+1}| \le c\tau^2, \tag{12} $$

where the constant c depending on T.

Theorem [11]: Let T be the exact solution of equation (1) and $\{T^k\}_{k=0}^{K}$ be the time discrete solution of equation (10) with initial condition $T^0 = f(x), x \in (0, L)$, then, it holds

$$\|T(t_k) - T^k\|_2 \leq cT^\alpha \tau^{2-\alpha}, \quad k = 1, 2, \dots, K \tag{13}$$

DERIVATION OF PENNE'S BIOHEAT EQUATION WITH TIME-SPACE FRACTIONAL DERIVATIVE BY CAPUTO TO FRACTIONAL DERIVATIVE AND FRACTIONAL QUADRATIC SPLINE FORM

The problem (1) - (4) of the time-space fractional Penne's bioheat transfer, can be write as

$$\frac{\partial^\beta T(x,t)}{\partial x^\beta} = \frac{\rho_t c_t}{\mu}\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} + \frac{W_b c_b}{\mu}(T - T_a) - \frac{Q}{\mu} - \frac{q_m}{\mu} \tag{14}$$

Using equation (6) in the equation (14), gives

$$F_i = \frac{\rho_t c_t}{\mu}\frac{1}{\tau^\alpha \Gamma(2-\alpha)}\left[T_i^{r+1} + T_i^r(u_1 - 1) + \sum_{k=1}^{r-1} T_i^{r-k}(u_{k+1} - u_k) - u_r T_i^0\right] + \frac{W_b c_b}{\mu}(T_i^r - T_a) - \frac{1}{\mu}Q_i^r - \frac{1}{\mu}q_m \tag{15}$$

Now, equation (15) can be written, as

$$F_{i+1} = \frac{\rho_t c_t}{\mu}\frac{1}{\tau^\alpha \Gamma(2-\alpha)}\left[T_{i+1}^{r+1} + T_{i+1}^r(u_1 - 1) + \sum_{k=1}^{r-1} T_{i+1}^{r-k}(u_{k+1} - u_k) - u_r T_{i+1}^0\right] + \frac{W_b c_b}{\mu}(T_{i+1}^r - T_a) - \frac{1}{\mu}Q_{i+1}^r - \frac{1}{\mu}q_m \tag{16}$$

Now, by adding the equations (15)-(16), and using (8) we obtain

$$A[T_{i+1}^{r+1} + T_i^{r+1}] + B(T_{i+1}^r + T_i^r) + A\sum_{k=1}^{r-1}(T_{i+1}^{r-k} + T_i^{r-k})w_k - Au_r(T_{i+1}^0 + T_i^0) - C(Q_{i+1}^r + Q_i^r) + 2D = \delta(T_{i+1}^r - 2T_i^r + T_{i-1}^r) \tag{17}$$

where $A = \frac{\rho_t c_t}{\mu}\frac{1}{\tau^\alpha \Gamma(2-\alpha)}$, $\quad B = \frac{\rho_t c_t}{\mu}\frac{1}{\tau^\alpha \Gamma(2-\alpha)}(u_1 - 1) + \frac{W_b c_b}{\mu}$, $C = \frac{1}{\mu}$,

$$D = -\frac{W_b c_b}{\mu}T_a - \frac{1}{\mu}q_m, \qquad w_k = (u_{k+1} - u_k)$$

Equation (17) contain of $(n-1)$ linear algebraic equations in the $(n+1)$ unknowns $T_i^r, i = 0, \dots, n$, so we need two boundary equation when $i = 0$ and $i = n$ can be use Taylor series, these two equations are

$$A[T_1^{r+1} + T_0^{r+1}] + B(T_1^r + T_0^r) + A\sum_{k=1}^{r-1}(T_1^{r-k} + T_0^{r-k})w_k - Au_r(T_1^0 + T_0^0) - C(Q_1^r + Q_0^r) + 2D = \delta(T_1^r - T_0^r - hT_{0(1)}^r) \tag{18}$$

$$A[2T_n^{r+1} + hT_{n(1)}^{r+1}] + B(2T_n^r + hT_{n(1)}^r) + A\sum_{k=1}^{r-1}(2T_n^{r-k} + hT_{n(1)}^{r-k})w_k - Au_r(2T_n^0 + hT_{n(1)}^0) - C(2Q_n^r + hQ_{n(1)}^r) + 2D = \delta(hT_{n(1)}^r - T_n^r + T_{n-1}^r) \tag{19}$$

where $T_{0(1)} = \left.\frac{\partial T}{\partial x}\right|_{x=0}$, and $T_{n(1)} = \left.\frac{\partial T}{\partial x}\right|_{x=n}$

STABILITY ANALYSIS FOR FRACTIONAL QUADRATIC SPLINE METHOD

The stability of numerical schemes can be using von Neumann method. We consider

$$T_i^r = \xi_r e^{mi\theta} \tag{20}$$

where $m = \sqrt{-1}$, θ is real and ξ. We can rewrite (17) as

$$A(T_{i+1}^{r+1} + T_i^{r+1}) + B(T_{i+1}^r + T_i^r) + A\sum_{k=1}^{r-1}(T_{i+1}^{r-k} + T_i^{r-k})w_k + R = \delta(T_{i+1}^r - 2T_i^r + T_{i-1}^r) \tag{21}$$

where $R = -Au_r(T_{i+1}^0 + T_i^0) - C(Q_{i+1}^r + Q_i^r) + 2D$, Substituting (20) into (21), we obtain

$$A\left[\xi_{r+1}e^{m(i+1)\theta} + \xi_{r+1}e^{mi\theta}\right] + B\left(\xi_r e^{m(i+1)\theta} + \xi_r e^{mi\theta}\right) + A\sum_{k=1}^{r-1}\left(\xi_{r-k}e^{m(i+1)\theta} + \xi_{r-k}e^{mi\theta}\right)w_k = \delta\left(\xi_r e^{m(i+1)\theta} - 2\xi_r e^{mi\theta} + \xi_r e^{m(i-1)\theta}\right) \tag{22}$$

One can see that the component R is omitted because the constant value does not affect the stability of this scheme [16],[19].

By divided (21) by $\xi_r e^{mi\theta}$, we have

$$A\left[\xi_{r+1}e^{m\theta} + \xi_{r+1}\right] + B\left(\xi_r e^{m\theta} + \xi_r\right) + A\sum_{k=1}^{r-1}\left(\xi_{r-k}e^{m\theta} + \xi_{r-k}\right)w_k = \delta\left(\xi_r e^{m\theta} - 2\xi_r + \xi_r e^{-m\theta}\right) \tag{23}$$

It is easily seen that (23) can be rewritten as

$$|\xi_{r+1}| < \left|\frac{\delta}{A} + \frac{B}{A} + \sum_{k=1}^{r-1}\xi_{-k}w_k\right||\xi_r|, \tag{24}$$

Now, when $r = 0$, we get

$|\xi_1| < \left|\frac{\delta}{A} + \frac{B}{A}\right| |\xi_0|$, this led to $|\xi_1| < |\xi_0|$

where $\left|\frac{\delta}{A} + \frac{B}{A}\right| < 1$, and $\left|\frac{\delta}{A} + \frac{B}{A} + \sum_{k=1}^{r-1} w_k\right| < 1$ for all value of r

Therefore, $|\xi_{r+1}| < |\xi_r|$, hence, this scheme is stable.

Now, when the expanding equation (17) with Taylor series in terms of $T(x_i, t_r)$ is obtained:

$$E^{r+1} = A[T_{i+1}^{r+1} + T_i^{r+1}] + B(T_{i+1}^r + T_i^r) + A\sum_{k=1}^{r-1}(T_{i+1}^{r-k} + T_i^{r-k})\, w_k - Au_r(T_{i+1}^0 + T_i^0) - \delta(T_{i+1}^r - 2T_i^r + T_{i-1}^r) + 2D - C(Q_{i+1}^r + Q_i^r)$$

$$= A\left(\left(1 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)T_i^{r+1} + T_i^{r+1}\right)$$
$$+ B\left(1 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)T_i^r + BT_i^r$$
$$+ A\sum_{k=1}^{r-1}\left(\left(1 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)T_i^{r-\mathrm{k}} + T_i^{r-\mathrm{k}}\right)w_k$$
$$- Au_r\left(\left(1 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)T_i^0 + T_i^0\right)$$
$$- \delta\left(\frac{h^2}{2}D_x^2 + \frac{h^4}{24}D_x^4 + \frac{h^6}{720}D_x^6 \dots\right)$$
$$+ \left(1 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)Q_i^r + Q_i^r - 2D$$

$$= A\left(2 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)T_i^{r+1}$$
$$+ B\left(2 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)T_i^r$$
$$+ A\sum_{k=1}^{r-1}\left(2 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)T_i^{r-\mathrm{k}}\, w_k$$
$$- Au_r\left(2 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)T_i^0$$
$$- \delta\left(\frac{h^2}{2}D_x^2 + \frac{h^4}{24}D_x^4 + \frac{h^6}{720}D_x^6 \dots\right)$$
$$+ \left(2 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)Q_i^r - 2D$$

$$= \left(2 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)\left(AT_i^{r+1} + BT_i^r + A\sum_{k=1}^{r-1}T_i^{r-\mathrm{k}}w_k - Au_rT_i^0 + Q_i^r\right)$$
$$- \delta\left(\frac{h^2}{2}D_x^2 + \frac{h^4}{24}D_x^4 + \frac{h^6}{720}D_x^6 \dots\right) - 2D$$

$$= \left(2 + hD_x + \frac{h^2}{2}D_x^2 + \frac{h^3}{6}D_x^3 + \frac{h^4}{24}D_x^4 + \frac{h^5}{120}D_x^5 + \frac{h^6}{720}D_x^6 \dots\right)\left(AT_i^{r+1} + BT_i^r + A\sum_{k=1}^{r-1}T_i^{r-k}\, w_k - Au_rT_i^0\right.$$
$$\left. + \left(\boldsymbol{\rho_t c_t D_t^{2-\alpha}} - \boldsymbol{\mu D_x^{4-\beta}} + W_b c_b\right)T_i^r\right) - \delta\left(\frac{h^2}{2}D_x^2 + \frac{h^4}{24}D_x^4 + \frac{h^6}{720}D_x^6 \dots\right) - 2D$$

With above discussion and theorem (13), the scheme obtained is of $\mathcal{O}\left(D_t^{2-\alpha} + D_x^{4-\beta}\right)$

NUMERICAL EXPERIMENT

In this section, we will apply the Caputo fractional derivative and fractional quadratic spline method for the following two examples of Pennes' bioheat problem to check the efficiency of this method. All calculations are implemented with MAPLE software.

Example 1: Consider Pennes' bioheat equation (1) with the conditions:

$$T(x,0) = 37 - x^3\,, \qquad x \in (0, L),$$

$$\frac{\partial}{\partial x} T(x,0) = -3x^2\,, \qquad x \in (0, L)\,,$$

$$T(0,t) = 37\,, \qquad t > 0\,,$$

so, the exact solution given as $T(x,t) = xt^2 - x^3 + 37$,

Table 1: The Errors of Numerical Solutions for various values of for example 1 at

α	β	$L_2 - error$	$L_\infty - error$
0.1	1.1	0.7870625784158495667478956e-3	0.47214706254818467088e-3
	1.5	0.8837127051754626343607446e-3	0.49078994712333397144e-3
	1.9	0.9675164600794557606047586e-3	0.47196788202336346865e-3
0.5	1.1	0.4758540945945874685078846e-4	0.28392526115704159920e-4
	1.5	0.5321853461100220415565954e-4	0.29476424845401977290e-4
	1.9	0.5803595029161770846755660e-4	0.28382108546338356240e-4
0.9	1.1	0.5038613203544799384251739e-5	0.28553584121042966100e-5
	1.5	0.5428367423333256189800331e-5	0.29287733508185865200e-5
	1.9	0.5718385938044194771110329e-5	0.28546528063169777500e-5
1	2	0.3953101254607893293748461e-5	0.19525000000000000000e-5

$\alpha = 0.1, 0.5, 0.9, 1$, $\beta = 1.1, 1.5, 1.9, 2$ and $\tau = 0.001$

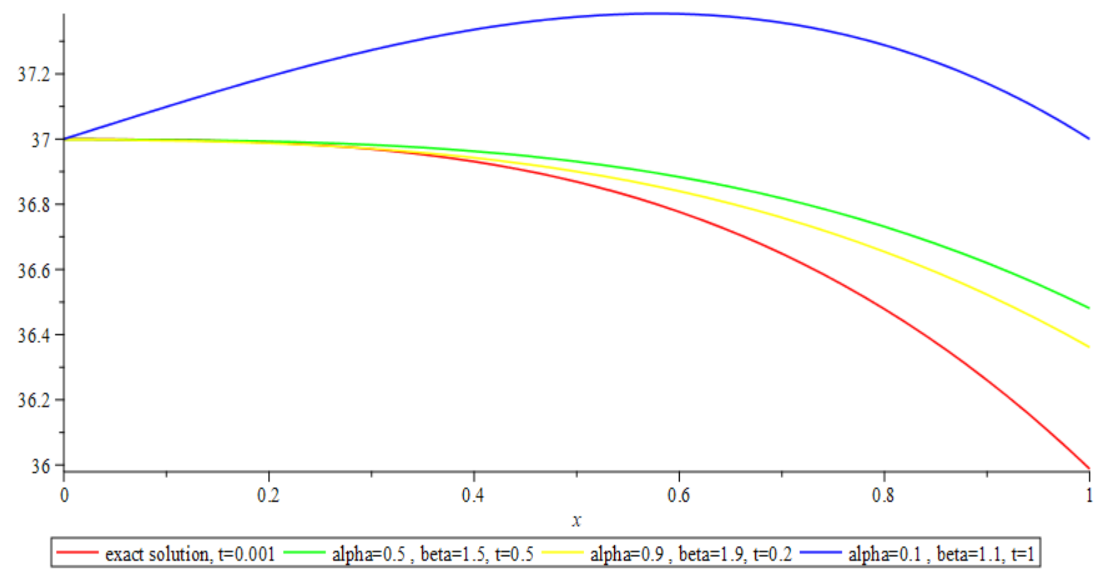

Fig. 1: Comparison between exact and numerical solutions for example1, for various values of α, β, and τ

Example 2: The penne's bioheat equation (1) and initial and boundary conditions
$T(x,0) = \sin x^2 - x^{2+\alpha} + 37$, $T(0,t) = \sin x^2 - x^{2+\alpha} + 37$,

$$\frac{\partial}{\partial x} T(x,0) = 2x \cos x^2 - (2+\alpha)x^{1+\alpha}\,, \qquad x \in (0, L)\,,$$

so, the exact solution given as $T(x,t) = \sin x^2 - (x^{2+\alpha} + t^{1+\beta}) + 37$,

α	β	$L_2 - error$	$L_\infty - error$
0.3	1.3	0.1271172634489825954903830e-3	0.6716312943412288574e-4
	1.6	0.1465753232580572748871121e-3	0.6863996494504144620e-4
	1.9	0.1676134726919479957240147e-3	0.6456976056569750501e-4
0.6	1.3	0.1479995510802717639209199e-4	0.8151114674172323840e-5
	1.6	0.1744515252544722000135068e-4	0.8429218519397601080e-5
	1.9	0.2008191846122404331312605e-4	0.7939837536197649720e-5
0.9	1.3	0.1640133837746988905079603e-5	0.9919780996006275400e-6
	1.6	0.2251575022159511510504109e-5	0.1124724186392396550e-5
	1.9	0.2646548054348782868132605e-5	0.1070518280442583650e-5
1	2	0.1462641134596733271169952e-5	0.5400672147179471400e-6

Table 2: The Errors of Numerical Solutions for example 2 and various values of $\alpha = 0.3, 0.6, 0.9, 1$, $\beta = 1.3, 1.6, 1.9, 2$ and $\tau = 0.001$

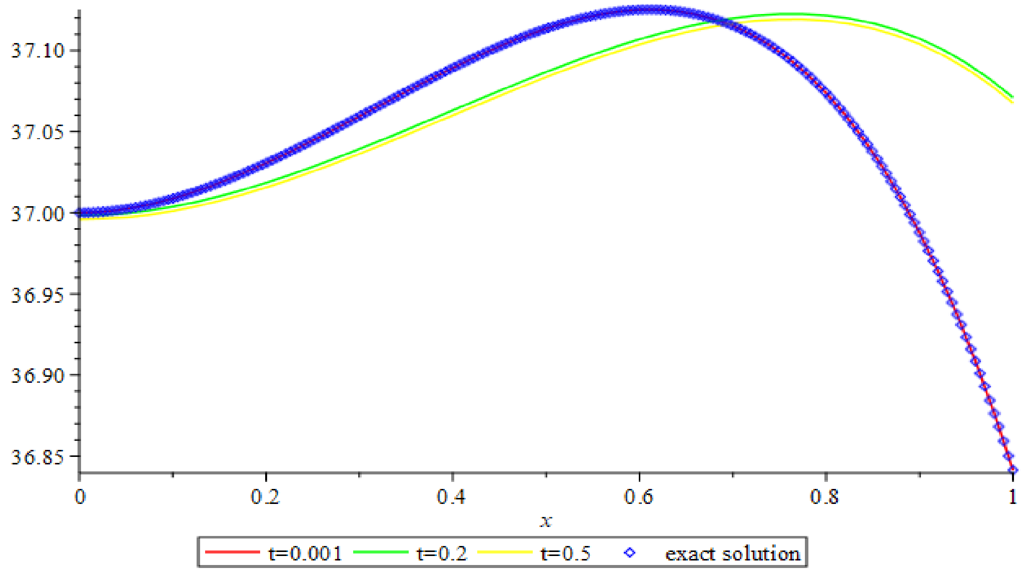

Fig. 2: Comparison between exact and numerical solutions for example2 at $\alpha = 0.9$, $\beta = 1.9$ and various values of $\tau = 0.001, 0.2, 0.5$

The initial and boundary conditions are derived from the exact solution. We applied fractional quadratic spline method to solved the problem (19-21) for $h = 0.02$, with $\rho_t = 1000$, $c_t = c_b = 4000$, $\mu = 0.5$, $\omega_b = 0.0005$, $W_b = 0.0005$, $T_a = 37, Q_m = 420$ and $\alpha \in (0,1)$, $\beta \in (1,2)$, the computed solutions are compared with exact solutions at mish points. The L_2 and L_∞- errors in solutions at $\tau = 0.001$, 0.2, $0.5, 1$ are tabulated in table 1, 2.

CONCLUSION

The aim of this paper is to compare the performance of the model approach based on the Caputo fractional derivative and fractional quadratic spline methods, which have been consider for solving the fractional time-space bioheat equation numerically for different values of fractional order α and β for time and space derivatives respectively. In general, we conclude that this scheme is powerful, effective, highly accurate and need a small number of iterations. Furthermore, the present algorithm is simple, easy to apply, and the results clarify the effectiveness of the proposed method. In addition, we note that L_2error decreasing when α and β are increasing. The von Neumann stability analysis of this method discussed to illustrate that this scheme is unconditionally stable and, it can be seen that the numerical solution converges to exact solution with order $O\left(\tau^{2-\alpha} + h^{4-\beta}\right)$.

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