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Approximate Solutions for Solving Time-Space Fractional Bioheat Equation Based on Fractional Shifted Legendre Polynomials

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Abstract. The aim of this article was employed a fractional-shifted Legendre polynomials (F-SLPs) in a matrix form to approximate the temporal and spatial derivatives of fractional orders for derived an approximate solutions for bioheat problem of a space-time fractional. The spatial-temporal fractional derivatives are described in the formula by the Riesz-Feller and the Caputo fractional derivatives of orders $v \in (1, 2]$ and $\gamma \in (0, 1]$, respectively. The proposed methodology applied for two examples for demonstrating its usefulness and effectiveness. The numerical results confirmed that the utilized technique is immensely effective, provides high accuracy and good convergence.

Key words. Collocation method, Time-space fractional bioheat equation, Fractional-shifted Legendre polynomials, Accuracy.

1. Introduction

Medical treatments like cryosurgery, cryopreservation, cancer hyperthermia, skin burns and thermal malady diagnostics, require an understanding of thermal phenomena and temperature behavior in living tissues. Therefore, bioheat transport in human tissues is a topic of high theoretical and applied benefit. Biothermal studies can assist the design of clinical thermal treatment equipment, evaluation of thermal treatment's effects on skin, and establishment of thermal protections for various purposes [1, 6, 18].

The physical phenomenon that explain heat transport in human tissue that includes the influence of blood flux on tissue temperature in a continuum was presented by Pennes [14], Furthermore it suggested a mathematical model to describe heat flux in biological tissue. The model known as the bioheat equation which that is still widely utilize [3].

Many researchers worked on the development Pennes bioheat model and fractional bioheat equation and gave very important analytic and computational solutions, and provided significant approximate solution, (for example, Ng et al. [13] in (2009), used the boundary element method; Lakhssassi et al. [12] in (2010), presented the analytic and numerical solutions by using the Jacobi elliptic functions and the Crank-Nicolson method; Singh et al.[17] in (2011), studied the numerical solutions by using finite difference and homotopy perturbation method; Jiang and Qi [10] in (2012), suggested a new fractional thermal wave model; Damor et al. [4] in (2013), used implicit finite difference method; Ezzat et al. [6] in (2014), suggested a new mathematical model; Ferrás et al.[7] in (2015), utilized an implicit finite difference; Qin and Wu [15] in (2016), applied a quadratic spline collocation technique; Kumar and Rai[11] in (2017), used finite element Legendre wavelet Galerkin methodology; Roohi et. al. [16] in (2018), determined the temperature distribution pattern during the hyperthermial therapy computationally by using Galerkin method; Hosseininia et al. [8] and Kumar et al. [19] in (2019); applied Legendre wavelet method, Kirchhoff's transformation, finite element Legendre wavelet and Galerkin method).

In this article, will introducing the approximate algorithm for solving one dimensional time-space fractional bioheat equation (T-SFBHE) based on F-SLPs.

2. Time-Space Fractional Bioheat Equation

The time-space fractional version of the one-dimensional unsteady state Pennes bioheat equation can be obtain by replacing the first order time derivative by Caputo fractional derivative of order $\gamma \in (0,1]$ and second order space derivative by Riesz-Feller derivative of fractional arbitrary positive real order $\nu \in (1,2]$. The T-SFBHE is given according to [17]

$$\rho c \frac{\partial^{\gamma} T(x,t)}{\partial t^{\gamma}} - k_* \frac{\partial^{\nu} T(x,t)}{\partial x^{\nu}} + W_b c_b (T(x,t) - T_a) = Q_{ext} + Q_{met}, \qquad t > 0, 0 \le x \le R,$$

where $\rho, c, k, T, t, x, T_a, W_b = \rho_b w_b, Q_{ext}$ and Q_{met} symbolizes density, specific heat, thermal conductivity, temperature, time, distance, artillery temperature ,blood perfusion rate, metabolic heat generation in skin tissue and external heat exporter in skin tissue respectively. The units and values of the symbols expressed in this equation are mentioned in Table1[5].

Table 1. The unit and value of the symbols expressed in the bioheat equation.

Symbol	T	T_a	t	x	ρ **and** ρ_b	c **and** c_b	K	ω_b	Q_{met}
Unit	°C	°C	s	m	kg/m^3	J/kg °C	W/m °C	m^3/s/ m^3	W/m^3
value		37			1000	4000	0.5	0.0005	420

with initial and boundary conditions

$$T(x,0) = T_c,$$
$$-k_* \frac{\partial T}{\partial x}\bigg|_{x=0} = q_0,$$
$$-k_* \frac{\partial T}{\partial x}\bigg|_{x=R} = 0.$$

where, q_0 is the heat flux on the skin surface.

3. Preliminaries and Notations

In this section, remind the principles essentials of the fractional calculus theory that will be used in this article.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ defined as [2]:

$$I^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \ \alpha > 0,$$
$$I^0 u(t) = u(t)$$

Definition 2. The Riemann-Liouville definition of fractional differential operator given as follows [9]:

$$D^{\alpha}u(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\dfrac{d^n}{dt^n}\displaystyle\int_0^t\dfrac{u(s)}{(t-s)^{\alpha-n+1}}\,ds, \alpha>0, n-1\le\alpha<n,\\ \dfrac{d^nu(t)}{dt^n}, \quad \alpha=n.\end{cases}$$

Definition 3. The Caputo definition of fractional differential operator is defined as [10]:

$$D^{\alpha}u(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t\dfrac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,ds, \ n-1\le\alpha<n,\\ \dfrac{d^nu(t)}{dt^n}, \quad \alpha=n.\end{cases}$$

The relation between the Riemann-Liouville and Caputo operators given by the expressions [17]:

$$D^{\alpha}I^{\alpha}\ u(t)=u(t),$$

$$I^{\alpha}D^{\alpha}\ u(t)=u(t)-\sum_{k=0}^{n-1}u^{(k)}(0^+)\frac{t^k}{k!}$$

For $\alpha\ge 0, v\ge -1$, and constant C, Caputo fractional derivative has some fundamental properties which are needed here as follows [9]:

$i)\ D^{\alpha}C=0,$

$$ii)\ D^{\alpha}t^{v}=\begin{cases}0 & \text{for } v\in N_0 \text{ and } v<\lceil\alpha\rceil\\ \dfrac{\Gamma(v+1)}{\Gamma(v+1-\alpha)}t^{v-\alpha}, & \text{for } v\in N_0 \text{ and } v\ge\lceil\alpha\rceil\end{cases}$$

$iii)\ D^{\alpha}(\sum_{i=0}^{n}c_i\,u_i(t))=\sum_{i=0}^{n}c_iD^{\alpha}u_i(t),\ where\ \{c_i\}_{i=0}^{n}\ are\ constant$

Definition 4. (generalized Taylor's formula). Suppose that $D^{i\alpha}u(t)\in C[0,1]$ for $i=0,1,\ldots,n-1$, then one has

$$u(t)=\sum_{i=0}^{n-1}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}u(0^+)+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}D^{n\alpha}u(\xi)$$

Where $0<\xi\le t, \forall t\in[0,k]$. Also, one has

$$\left|u(t)-\sum_{i=0}^{n-1}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}u(0^+)\right|\le K_{\alpha}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

where $K_{\alpha}\ge|D^{n\alpha}u(\xi)|$.

In case $\alpha=1$, the generalized Taylor's formula (10) is the classical Taylors formula [9].

4. Fractional Shifted Legendre Polynomials

Define the F-SLPs by introducing the change of variable $x=x^{\beta}$ and $N-1<\beta\le N$ on shifted Legendre polynomials. The F-SLPs $L_N(x^{\beta})$ is symbolized by $Fl_i^{\beta}(x)$. The F-SLPs are a particular solution of the normalized eigenfunctions of the Sturm-Liouville problem.

$$\left((x-x^{1+\beta})Fl_i^{\beta}(x)\right)'+\beta^2\,i(i+1)x^{\beta-1}Fl_i^{\beta}(x)=0, x\in[0,1].$$

Then $Fl_i^{\beta}(x)$ can be obtained as follows:

$$Fl_{i+1}^{\beta}(x)=\frac{(2i+1)(2x^{\beta}-1)}{i+1}Fl_i^{\beta}(x)-\frac{i}{i+1}Fl_{i-1}^{\beta}(x), i=1,2,\ldots$$

can derive the analytic form of $Fl_i^{\beta}(x)$ of degree $i\beta$ as follows:

$$Fl_i^{\beta}(x)=\sum_{s=0}^{i}b_{si}\ x^{s\beta},$$

where $b_{si}=(-1)^{i+s}(i+s)!/(i-s)!(s!)^2$ and $Fl_i^{\beta}(0)=(-1)^i, Fl_i^{\beta}(1)=1$.

Theorem1. The FLPs are orthogonal with the weight function $\omega_l^{\beta}(x)=x^{\beta-1}$ on the interval [0,1], then be orthogonally condition is

$$\int_0^1 Fl_N^{\beta}(x)\,Fl_M^{\beta}(x)\,\omega_l^{\beta}(x)\,dx=\frac{1}{(2N+1)\beta}\delta_{NM}$$

Proof. With $\int_0^1 L_N(x)\, L_M(x)\, \omega_l(x)\, dx\ = \frac{1}{(2N+1)}\delta_{NM}$, where δ_{NM} is the Kronecker function and the weight function $\omega_l(x) = 1$, let $x = x^{\beta}$ and then have

$\int_0^1 L_N(x)\, L_M(x)\ \omega_l(x)dx = \int_0^1 L_N\left(x^{\beta}\right) L_M\left(x^{\beta}\right) \beta x^{\beta-1}\, dx$

$= \frac{1}{(2N+1)}\delta_{NM},$

$\int_0^1 L_N\left(x^{\beta}\right) L_M\left(x^{\beta}\right) \beta x^{\beta-1}\, dx$

$= \int_0^1 Fl_N^{\beta}(x)\, Fl_M^{\beta}(x)\ \ \beta x^{\beta-1}\, dx = \frac{1}{(2N+1)}\delta_{NM},$

$\int_0^1 Fl_N^{\beta}(x)\, Fl_M^{\beta}(x)\ \ x^{\beta-1}\, dx = \frac{1}{(2N+1)\beta}\delta_{NM}.$

Then the theorem is proved.

□

A temperature function $T(x)$ square integrable on interval [0,1], may be expressed in order of F-SLPs as

$T(x) = \sum_{i=0}^{\infty} c_i\, Fl_i^{\beta}(x)$

where the coefficients c_i are obtained by

$c_i = \beta(2i+1) \int_0^1 Fl_i^{\beta}(x) T(x)\, \omega_l^{\beta}\ (x)dx,\ i = 0,1,2, \ldots$

If consider truncated series when $(N+1)$-term the F-SLPs in (17), obtain

$T(x) \approx T_N(x) = \sum_{i=0}^{N} c_i\, Fl_i^{\beta}(x) = C'\emptyset(x)$

where the fractional-shifted Legendre coefficient vectors C and $\emptyset(x)$ are given by

$C' = [c_0, c_1, \ldots, c_N],\ \emptyset(x) = \left[Fl_0^{\beta}(x), Fl_1^{\beta}(x), \ldots, Fl_N^{\beta}(x)\right]'.$

Theorem2. Suppose that $D^{i\beta}T(x) \in C[0,1]$ for $i = 0,1, \ldots, N.\ (2N+1)\beta \geq 1$ and $\boldsymbol{P}_N^{\beta} = span\left\{Fl_0^{\beta}(x), Fl_1^{\beta}(x), \ldots, Fl_N^{\beta}(x)\right\}$. If $T_N(x) = C'\emptyset(x)$ is the best approximation to $T(x)$ from $\boldsymbol{P}_N^{\beta}$ then the error bound is presented as follows:

$\|T(x) - T_N(x)\|_{\omega} \leq \frac{K_{\beta}}{\Gamma(N\beta+1)}\sqrt{\frac{1}{(2N+1)\beta}}$,

where $K_{\beta} \geq \left|D^{N\beta}T(x)\right|,\ \ x \in [0,1].$

Proof. Consider generalized Taylor's formula

$T\ (x) = \sum_{i=0}^{N} \frac{x^{i\beta}}{\Gamma\ (i\beta+1)} D^{i\alpha}T(0^{+}) + \frac{x^{N\beta}}{\Gamma(N\beta+1)} D^{N\beta}T\ (\xi),$

where $0\ <\ \xi\ \leq\ x, \forall x \in [0, 1]$. Also, one has

$\left|T\ (x) - \sum_{i=0}^{N} \frac{x^{i\beta}}{\Gamma\ (i\beta+1)} D^{i\beta}f(0^{+})\right| \leq M_{\beta}\ \frac{x^{N\beta}}{\Gamma(N\beta+1)}$

Since $T_N(x) = C'\emptyset(x)$ is the best approximation to $T(x)$ from $\boldsymbol{P}_N^{\beta}$ and

$\sum_{i=0}^{N} \left(\frac{x^{i\beta}}{\Gamma(i\alpha+1)}\right) D^{i\beta}f(0^{+}) \epsilon\, \boldsymbol{P}_N^{\beta},$

hence $\|T(x) - T_N(x)\|_{\omega}^2 \leq \left\|T(x) - \sum_{i=0}^{N} \frac{x^{i\beta}}{\Gamma\ (i\beta+1)} D^{i\beta}T(0^{+})\right\|_{\omega}^2 \leq \frac{K_{\beta}^2}{\left(\Gamma(N\beta+1)\right)^2} \int_0^1 x^{2N\beta}\, x^{\beta-1}dx,$

$\|T(x) - T_N(x)\|_{\omega}^2 \leq \frac{K_{\beta}^2}{\left(\Gamma(N\beta+1)\right)^2} \int_0^1 x^{(2N+1)\beta-1}\, dx$

$= \frac{K_{\beta}^2}{\left(\Gamma(N\beta+1)\right)^2 (2N+1)\beta}$

Now, take the square root both sides, then the theorem proved.

□

For arbitrary a temperature function $T(x,t) \in L^2([0,1] \times [0,1])$, they can be expanded as the following formula:

$$T(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t)$$

where

$$c_{ij} = (2i+1)(2j+1)\beta\alpha \int_0^1 \int_0^1 T(x,t)\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t) \omega_l^{\beta}(x) \omega_l^{\alpha}(t) dxdt. \quad \text{i, j} = 0,1,\ldots$$

Theorem3. If the series $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t)$ converges uniformly to $T(x,t)$ on the square $[0,1] \times [0,1]$, then equation (25) can be proof as following

Proof. By multiplying $\omega_l^{\beta}(x) \omega_l^{\alpha}(t) Fl_N^{\beta}(x) Fl_M^{\alpha}(t)$ both sides of (24), where i and j are fixed and integrating term wise with regard to x and t on $[0,1] \times [0,1]$, then

$$\int_0^1 \int_0^1 T(x,t)\, Fl_N^{\beta}(x) Fl_M^{\alpha}(t) \omega_l^{\beta}(x) \omega_l^{\alpha}(t) dxdt =$$
$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \int_0^1 \int_0^1 Fl_i^{\beta}(x) Fl_j^{\alpha}(t) Fl_N^{\beta}(x) Fl_M^{\alpha}(t) \omega_l^{\beta}(x) \omega_l^{\alpha}(t) dxdt$$
$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \int_0^1 \omega_l^{\beta}(x)\, Fl_i^{\beta}(x) Fl_N^{\beta}(x) dx \int_0^1 \omega_l^{\alpha}(t)\, Fl_j^{\alpha}(t) Fl_M^{\alpha}(t) dt \square$$
$$= c_{ij} \int_0^1 \omega_l^{\beta}(x) \left[Fl_i^{\beta}(x)\right]^2 dx \int_0^1 \omega_l^{\alpha}(t) \left[Fl_j^{\alpha}(t)\right]^2 dt$$
$$= c_{ij} \frac{1}{(2i+1)\beta} \frac{1}{(2j+1)\alpha}$$

Theorem4. If the function $T(x,t)$ is a continuous function on $[0,1] \times [0,1]$ and the series $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t)$ converges uniformly to $T(x,t)$, then $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t)$ is the F-SLPs expansion of $T(x,t)$.

Proof. Using contradiction, let

$$\left.\begin{aligned} T(x,t) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t), \\ T(x,t) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t). \end{aligned}\right\}$$

Then there is at least one coefficient such that $c_{NM} \neq g_{NM}$ however

$$c_{NM} = (2N+1)(2M+1)\alpha\beta \int_0^1 \int_0^1 T(x,t)\, Fl_N^{\beta}(x) Fl_M^{\alpha}(t) \omega_l^{\beta}(x) \omega_l^{\alpha}(t) dxdt = g_{NM}$$

□

Theorem5. If two continuous functions defined on $[0,1] \times [0,1]$ have the identical F-SLPs expansions, then these two function are identical.

Proof. Suppose that $T(x,t)$ and $f(x,t)$ can be expended by F-SLPs as follows:

$$T(x,t) \approx \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t),$$
$$f(x,t) \approx \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t).$$

By subtracting equation (30) from (29), have

$$T(x,t) - f(x,t) \approx \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(c_{ij} - c_{ij}\right) Fl_i^{\beta}(x) Fl_j^{\alpha}(t)$$
$$= 0 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} 0\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t).$$

□

Theorem6. If the sum of the absolute value of the F-SLPs coefficients of a continuous function $T(x,t)$ forms a convergent series, then the F-SLPs expansion is absolutely uniformly convergent and converges to the function $T(x,t)$.

Proof. Consider

$$\left|\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t)\right| \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left|c_{ij}\right| \left|Fl_i^{\beta}(x)\right| \left|Fl_j^{\alpha}(t)\right| \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left|c_{ij}\right|$$

If consider truncated series in (24), satisfy

$T(x,t) \approx \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t) = \emptyset'(x) C \emptyset(t),$

where $C = \{c_{ij}\}_{i,j=0}^{N,M}$, $\emptyset(x) = \left[Fl_0^{\beta}(x), Fl_1^{\beta}(x), \dots, Fl_N^{\beta}(x)\right]'$ and $\emptyset(t) = [Fl_0^{\alpha}(t), Fl_1^{\alpha}(t), \dots, Fl_M^{\alpha}(t)]'$.

□

5. Two Dimensional Fractional-Shifted Legendre Operational Matrix of Fractional Differentiation

The derivative of the $\emptyset(x)$ can be approximated as follows

$\emptyset^{(v)}(x) \approx D^{v} \emptyset(x),$

where D^{v} and D^{γ} are called the F-SLPs operational matrix of space and time derivatives

Theorem7. Suppose D^{v} is $(N+1)\times(N+1)$ operational matrix of Caputo fractional derivatives of order $v > 0, \beta > \frac{v}{2}$, when $\beta \in \mathbb{N}$;then the elements of $D^{v}$are obtained as

$$\{d_{ij}\}_{i,j=0}^{N,N} = (2j+1)\beta \sum_{s=0}^{i}\sum_{r=0}^{j} b_{rj}\, b'_{si} \frac{\Gamma(s\beta+1)}{\Gamma(s\beta-v+1)} \frac{1}{(s+r+1)\beta - v}.$$

where

$$b'_{si} = \begin{cases} 0, & s\beta \in N_0,\ s\beta < v, \\ b'_{si} = b_{si} & s\beta \notin N_0, s\beta \geq \lceil v \rceil \ or\ s\beta \in N_0, s\beta \geq v. \end{cases}$$

Proof. With the properties of the derivative (ii) and the orthogonally of FLPs, have

$D^{v} Fl_i^{\beta}(x) = \sum_{s=0}^{j} b'_{sj} \frac{\Gamma(s\beta+1)}{\Gamma(s\beta-v+1)} x^{s\beta-v}.$

Let

$x^{s\beta-v} = \sum_{j=0}^{N} d_j\, Fl_j^{\beta}(x)$

multiplying both sides of the equation (38) by $\omega_l^{\beta}(x) Fl_j^{\beta}(x)$, get

$d_j = (2j+1)\beta \sum_{r=0}^{N} b_{rj} \frac{1}{(s+r+1)\beta - v},$

substituting the equations (38) and (39) into equation (37), yields

$$D^{v} Fl_i^{\beta}(x) = (2j+1)\beta \sum_{j=0}^{N}\sum_{s=0}^{i}\sum_{r=0}^{j} b_{rj} b'_{si} \frac{\Gamma(s\beta+1)}{\Gamma(s\beta-v+1)} \frac{1}{(s+r+1)\beta - v} Fl_j^{\beta}(x),$$

Hence, $d_{ij} = (2j+1)\beta \sum_{s=0}^{i}\sum_{r=0}^{j} b_{rj}\, b'_{si} \frac{\Gamma(s\beta+1)}{\Gamma(s\beta-v+1)} \frac{1}{(s+r+1)\beta - v}.\ i,j = 0,1,\dots,N$

□

6. Method for Solution

Now will structure the approximate solution of equation (1), under given conditions, as the following series form

$T(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t_{ij}\, Fl_i^{\beta}(x) Fl_j^{\alpha}(t),$

which equivalent the matrix form

$T(x,t) = \emptyset(x)' T \emptyset(t),$

where $T = \{t_{ij}\}_{i,j=0}^{N,M}$, $\emptyset(t) = [Fl_0^{\alpha}(t), Fl_1^{\alpha}(t), \dots, Fl_M^{\alpha}(t)]'$ and

$\emptyset(x) = \left[Fl_0^{\beta}(x), Fl_1^{\beta}(x), \dots, Fl_N^{\beta}(x)\right]'.$

The approximate of the first spatial derivative as

$\frac{\partial T(x,t)}{\partial x} = \emptyset'(x)(D_x)'\, T \emptyset(t),$

and the fractional temporal and spatial derivatives as

$\frac{\partial^{\gamma} T(x,t)}{\partial t^{\gamma}} = \emptyset'(x) T D_t^{\gamma} \emptyset(t),$

$\frac{\partial^v T(x,t)}{\partial x^v} = \emptyset'(x)(D_x^v)'\, T\emptyset(t),$

applying the solution method for T-SFBHE in (1), have

$\rho c\, \emptyset'(x)TD_t^\gamma \emptyset(t) - k_* \emptyset'(x)(D_x^v)'T\emptyset(t) + W_b c_b \emptyset'(x)T\emptyset(t) = \emptyset'(x)Q_{ext}\emptyset(t) + \emptyset'(x)Q_{met}\emptyset(t) + \emptyset'(x)W_b c_b T_a I\emptyset(t),$

where $g(x,t) = Q_{ext} + Q_{met} + W_b c_b T_a$

$\emptyset'(x)G\, \emptyset(t) = \emptyset'(x)Q_1\emptyset(t) + \emptyset'(x)Q_2\emptyset(t) + \emptyset'(x)Q_3\emptyset(t)$

where

$G = \{g_{ij}\}_{i,j=0}^{N,M},$

$$g_{ij} = \alpha\beta(2i+1)(2j+1)\int_0^1\int_0^1 g(x,t)\, Fl_i^\beta(x)Fl_j^\alpha(t)\omega_l^\beta(x)\omega_l^\alpha(t)dxdt.$$

This is generate $NM + N + M + 1$ algebraic equations by multiplying $\omega_l^\beta(x)\omega_l^\alpha(t)Fl_i^\beta(x)Fl_i^\alpha(t)$ for $i = 0,1,2,\dots,N; j = 0,1,2,\dots,M$, integrating from 0 to 1 and using the orthogonal property, to get

$$T(\rho c\, D_t^\gamma - \omega_b\rho_b c_b I) - k_*(D_x^v)'T = G,$$

with the initial condition from equation (2) in matrix form

$T\emptyset(0) \approx F$

where $F = [f_0, f_1, \dots, f_M]'$

$f_j = \beta(2j+1)\int_0^1 T(x,0)\, Fl_j^\beta(x)\omega_l^\beta(x)dx$

and boundary conditions respectively from equations (3) and (4) in matrix form, have

$-k_*\emptyset'(0)D_x{}'\, T \approx K'$

$-k_*\emptyset'(R)D_x{}'\, T \approx H'$

where $K = [k_0 \quad k_1 \quad . \quad . \quad . \quad k_N]'$ and $H = [h_0 \quad h_1 \quad . \quad . \quad . \quad h_N]'$

$k_i = \alpha(2i+1)\int_0^1 T_x(0,t)\, Fl_i^\alpha(t)\omega_l^\alpha(t)dt$

$h_i = \alpha(2i+1)\int_0^1 T_x(R,t)\, Fl_i^\alpha(t)\omega_l^\alpha(t)dt$

which generate $NM + N + M + 1$ linear algebraic equations by equation (49) together with equations (50),(52) and (53). These unknown coefficients T can be solve by solving Sylvester system.

7. Error Analysis

Consider $e(x,t) = T(x,t) - T_{NM}(x,t)$ as the error function where $T_{NM}(x,t)$ and $T(x,t)$ are the approximate and exact solutions of equation (1).

Therefore, $T_{NM}(x,t)$ satisfies the following problem

$\rho c \frac{\partial^\gamma T_{NM}(x,t)}{\partial t^\gamma} - k_* \frac{\partial^v T_{NM}(x,t)}{\partial x^v} + W_b c_b T_{NM}(x,t) + R_{NM}(x,t) = g(x,t),$

where $R_{NM}(x,t)$ is the residual function,

$R_{NM}(x,t) = \rho c \frac{\partial^\gamma T_{NM}(x,t)}{\partial t^\gamma} - k_* \frac{\partial^v T_{NM}(x,t)}{\partial x^v} + W_b c_b T_{NM}(x,t) - g(x,t).$

find an approximation $\tilde{e}_{nm}(x,t)$ to the error function $e_{nm}(x,t)$ in the same previous procedure, so the solution of the problem, the error function satisfies the problem

$\rho c \frac{\partial^\gamma e_{NM}(x,t)}{\partial t^\gamma} - k_* \frac{\partial^v e_{NM}(x,t)}{\partial x^v} + W_b c_b e_{NM}(x,t) = R_{NM}(x,t)$

should note that in order to construct the approximate $\tilde{e}_{nm}(x,t)$ to the error function $e_{nm}(x,t)$, only equation (58) needs to be recomputed in the same procedure as doing before for the solution of equation (1).

8. Numerical Examples

In this section, apply the algorithm, which presented in section 6 for solving the T-SFBHE in the two examples based on F-SLPs. In order to showing a capability of the collocation method for achieving the high accuracy. In these examples, the solution obtained from the approximate technique is synonymous with the accurate solution.

Where the parameters $\rho, c, k_*, T, t, x, T_a, W_b = \rho_b w_b$ and Q_{met} are obtained from Table 1

Example1: Consider the T-SFBHE (1) where by choosing Q_{ext} so the exact solution is:

$T(x,t) = xt^2(2-x) + 37$

with the initial condition

$T(x,0) = 37, \;\; x \in [0,R]$

and boundary conditions

$-k_* \frac{\partial T(0,t)}{\partial x} = 2t^2 \;, t > 0$

$-k_* \frac{\partial T(R,t)}{\partial x} = 2t^2(1-R) \;, t > 0$

Table2. Absolut errors obtained for Example 1 with $R = 1$ and $N = M = 12$.

$(\boldsymbol{x}, \boldsymbol{t})$	Absolute error $\boldsymbol{\alpha = 0.5, \beta = 1.5}$	Absolute error $\boldsymbol{\alpha = 0.75, \beta = 1.75}$	Absolute error $\boldsymbol{\alpha = 0.95, \beta = 1.95}$
(0,0)	1.1373e-08	1.9178e-07	1.1509e-07
(0.1,0.1)	1.6373e-05	7.5360e-05	9.8106e-05
(0.2,0.2)	4.5590e-05	1.3012e-04	5.5986e-05
(0.3,0.3)	9.7485e-05	9.4045e-05	1.5015e-04
(0.4,0.4)	1.2577e-04	5.6854e-05	3.7275e-04
(0.5,0.5)	8.1411e-05	2.4769e-04	4.6486e-04
(0.6,0.6)	2.7426e-05	3.8733e-04	4.2448e-04
(0.7,0.7)	1.4291e-04	4.6677e-04	3.8853e-04
(0.8,0.8)	2.0478e-04	5.6629e-04	5.7097e-04
(0.9,0.9)	1.4727e-05	4.7077e-04	8.9061e-04
(1,1)	1.7356e-03	2.9567e-03	3.9126e-03

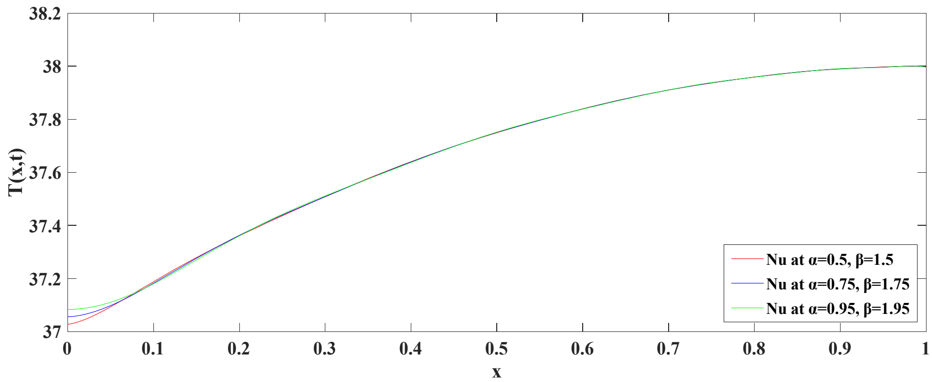

Figure1. Comparison between the numerical solutions for Example 1 at $t = 1$, $R = 1, N = M = 12$.

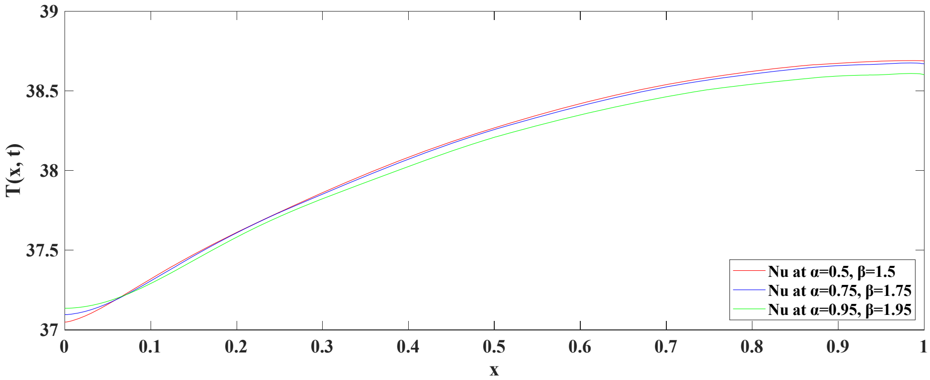

Figure 2. Comparison between the numerical solutions for Example 1 at $t = 1.3$, $R = 1, N = M = 12$

Example 2: Consider the T-SFBHE (1) where by choosing Q_{ext} so the exact solution is:

$$T(x,t) = x^{\frac{3}{2}} e^{-t} + 37$$

with the initial condition

$$T(x,0) = x^{\frac{3}{2}} + 37, \quad x \in [0,R]$$

and boundary conditions

$$-k_* \frac{\partial T(0,t)}{\partial x} = 0 \ , t > 0$$

$$-k_* \frac{\partial T(R,t)}{\partial x} = \frac{3}{2} (\mathrm{R})^{\frac{1}{2}} e^{-t} \, , t > 0$$

Table 3. Absolut errors obtained for Example 2 with $R = 1$ and $N = M = 12$.

$(\boldsymbol{x}, \boldsymbol{t})$	**Absolute error** $\boldsymbol{\alpha = 0.5, \beta = 1.5}$	**Absolute error** $\boldsymbol{\alpha = 0.75, \beta = 1.75}$	Absolute error $\boldsymbol{\alpha = 0.95, \beta = 1.95}$
(0,0)	1.1620e-08	1.7895e-03	4.5399e-03
(0.1,0.1)	8.5099e-08	2.0225e-04	6.9776e-04
(0.2,0.2)	1.8188e-08	7.8602e-05	3.1626e-04
(0.3,0.3)	1.2060e-07	4.7694e-05	1.8943e-04
(0.4,0.4)	2.9296e-08	3.9389e-05	1.7286e-04
(0.5,0.5)	1.2435e-07	4.9954e-05	1.2595e-04
(0.6,0.6)	5.2647e-08	1.0175e-04	9.7641e-05
(0.7,0.7)	1.1404e-07	1.3178e-04	7.7762e-05
(0.8,0.8)	4.4359e-08	1.3219e-04	1.1677e-04
(0.9,0.9)	1.4109e-07	6.2032e-05	3.8348e-05
(1,1)	1.4402e-07	3.5154e-05	2.7918e-07

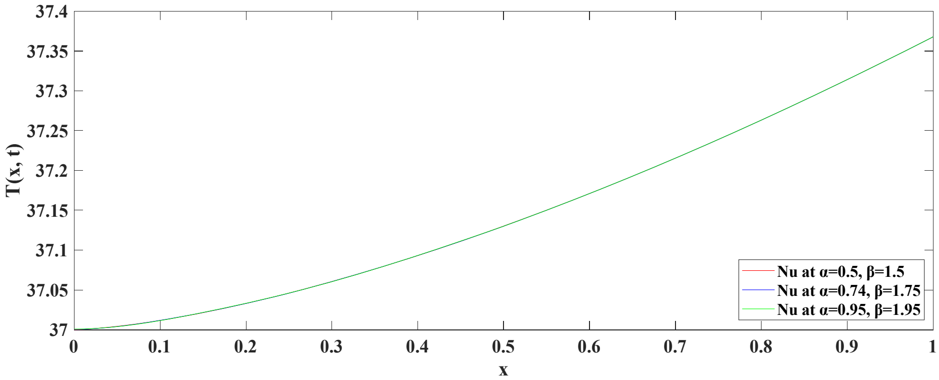

Figure 3. Comparison between the numerical solutions for Example 2 at $t = 1$, $R = 1, N = M = 12$.

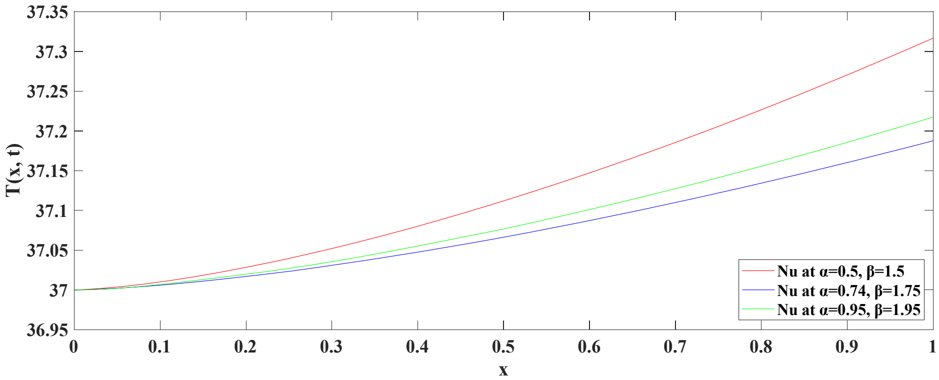

Figure 4. Comparison between the numerical solutions for Example 2 at $t = 1.15$, $R = 1, N = M = 12$.

9. Conclusions

In this work, the approximate algorithm structured on the F-SLPs in the matrix form to estimate the fractional derivatives to found the numerical solutions of the T-SFBHE. The Caputo formula utilized into approximate the fractional derivatives. Figs. 1-4 and Tables 2-3 indicated that the numerical results for Example 1 and 2 of a present technicality has a higher accuracy, good convergence, reasonable stability as well as a minimal computational effort by utilizing a few mesh grid. Concluded that the target numerical approach can be solve a various kinds of models of any fractional orders. In addition expected that the present methodology may present a more exact estimate by employing some other families based on orthogonal polynomials.

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