Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 6, Number 2, 2021, Pages 309-323
https://doi.org/10.34198/ejms.6221.309323

Numerical Solutions of Mixed Integro-Differential Equations by Least-Squares Method and Laguerre Polynomial

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Abstract

The main objective of this article is to present a new technique for solving integro-differential equations (IDEs) subject to mixed conditions, based on the least-squares method (LSM) and Laguerre polynomial. To explain the effect of the proposed procedure will be discussed three examples of the first, second and three-order linear mixed IDEs. The numerical results used to demonstrate the validity and applicability of comparisons of this method with the exact solution shown that the competence and accuracy of the present method.

1. Introduction

Studied a sixth-kind Chebyshev collocation scheme will be considered for solving a category of (V-OFNQIDEs). The operational matrix of variable-order fractional derivative for sixth kind Chebyshev polynomials is derived and then, a clustering approach is employed to reduce the V-OFNQIDE to a system of nonlinear algebraic equations [1]. Developed a numerical method was implemented for solving IDEs with the weakly singular by using a new method depend on the cubic B-spline least-square method and a quadratic B-spline as a weight function. The numerical results are in suitable agreement with the exact solutions via calculating L^2 and L^∞ norms errors. Theoretically, discussed the stability evaluation of the current method using the Von-Neumann method, which explained that this technique is unconditionally stable [2]. A new method based on the Laplace Adomian decomposition with the Bernstein

Received: February 20, 2021; Accepted: April 13, 2021

2010 Mathematics Subject Classification: 65R20.

Keywords and phrases: mixed integro-differential equations, Laguerre polynomial, least-squares method.

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polynomial to solve the Voltaire integral equation and the integro- differential equation of the first and second types, and through examples and comparison accurate and approximate solutions the method is adopted [3]. Used waves and orthogonal polynomials to solve some of the system single Volterra integro-differential equations (SSVIDEs) presented the convergence analysis for the derivative of the approximation in terms of Legendre wavelets, to demonstrate the accuracy and efficiency of the proposed method through the given examples [4]. Developed is a new numerical scheme based on the Laguerre and Taylor polynomials, called matrix collocation scheme, solution nonlinear PIDEs, for the numerical solution of the mentioned nonlinear equations under the initial or boundary conditions [5]. The Euler polynomial was used to solve the (VIDEs) of the pantograph delay type in approximation of the solution. The method is discussed in detail and compared through numerical examples [6].

In this paper, we will be applying the least-squares method and Laguerre polynomial for solving linear mixed IDEs of the second type. The mixed IDEs in this method are transformed into a system of linear algebraic equations. Consider the following general forms of the mixed IDEs [7]:

$$u^{(k)}(x) + Du^{(k)}(x) + Bu(x)$$
$$= f(x) + \lambda_1 \int_0^x k_1(x,t)u(t)dt + \lambda_2 \int_c^d k_2(x,t)u(t)dt\,, \quad c \le x,t \le d, \qquad (1.1)$$

under the mixed conditions

$$\sum_{i=0}^{N-1} (c_{ji}u^{(i)}(c) + d_{ji}u^{(i)}(d)) = \beta_j, \qquad i = 0,1,\dots,N-1 \qquad (1.2)$$

where, $u(x)$ is an unknown function, the known functions $f(x)$, $k_1(x,t)$ and $k_2(x,t)$ are defined on the interval $[c,d]$, the function $k_1(x,t)$ and $k_2(x,t)$ can be expanded from the Maclaurin series, c_{ji}, d_{ji}, and β_j and are real or complex constants.

1.1. Laguerre polynomial and their properties

The Laguerre polynomials $L_i(x)$ are set of orthogonal polynomials over the interval $[0,\infty]$. The explicit formula for this polynomial is [8],

$$L_i(x) = \sum_{j=0}^{i} (-1)^j \frac{i!}{(i-j)!\,(j!)^2} x^j$$

where $L_0(x) = 1, L_1(x) = 1 - x$.

2. Least-Square Method and Laguerre Polynomial

In this section, we will be implementing a new approach based on the LSM and Laguerre polynomial as a basis function to solve the equations (1.1) and (1.2).

We assume the approximate solution as

$$u_m(x) = \sum_{i=0}^{m} a_i L_i(x) \qquad c \leq x \leq d \tag{2.1}$$

where a_i are unknown constants and $L_i(x)$ are the Laguerre polynomial of degrees (i). Substituting equation (2.1) into equation (1.1), we get

$$\sum_{i=0}^{m} a_i L_i{}^{(k)}(x) + D\sum_{i=0}^{m} a_i L_i{}^{(k)}(x) + B\sum_{i=0}^{m} a_i L_i(x)$$

$$= f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i L_i(t)\,dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i L_i(t)\,dt, \tag{2.2}$$

2.1. Description of the method

The residual equation was given by

$$R(x, a_i) = R\big(x, u_m(x)\big)$$

$$= \sum_{i=0}^{m} a_i L_i{}^{(k)}(x) + D\sum_{i=0}^{m} a_i L_i{}^{(k)}(x) + B\sum_{i=0}^{m} a_i L_i(x)$$

$$-\left\{ f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i L_i(t)\,dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i L_i(t)\,dt \right\}. \tag{2.3}$$

Let

$$S(a_0, a_1, \dots, a_m) = \int_c^d [R(x, a_i)]^2 w(x)dx, \tag{2.4}$$

where the positive weight function specified on the interval $[c, d]$, is $w(x) = 1$, [9], thus,

$$S(a_0, a_1, \dots, a_m)$$

$$= \int_c^d \left[\sum_{i=0}^{m} a_i L_i^{(k)}(x) + D \sum_{i=0}^{m} a_i L_i^{(k)}(x) + B \sum_{i=0}^{m} a_i L_i(x) - f(x) \right.$$

$$\left. + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i L_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i L_i(t)\, dt \right]^2 dx, \qquad (2.5)$$

We can get the values of $a_i, i \geq 0$ by minimizing the value of S as follows [10]:

$$\frac{\partial S}{\partial a_i} = 0, \quad i = 0,1,\dots,m. \qquad (2.6)$$

Then from (2.5) by applying (2.6) get:

$$\frac{\partial S}{\partial a_i} = \int_c^d \left[\sum_{i=0}^{m} a_i L_i^{(k)}(x) + D \sum_{i=0}^{m} a_i L_i^{(k)}(x) + B \sum_{i=0}^{m} a_i L_i(x) - \left\{ f(x) \right.\right.$$

$$\left.\left. + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i L_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i L_i(t)\, dt \right\} \right] dx$$

$$\times \int_c^d \left[L_i^{(k)}(x) + A L_i^{(k)}(x) + B L_i(x) \right.$$

$$\left. - \left\{ \lambda_1 \int_0^x k_1(x,t) L_i(t) dt + \lambda_2 \int_c^d k_2(x,t) L_i(t) dt \right\} \right] dx = 0\,, \qquad (2.7)$$

thus, (2.7) are generated $(m+1)$ algebraic system of equations in $(m+1)$ unknown $a_i, i = 0, \cdots, m$, or in the matrix form as follows:

$$W = \begin{pmatrix} \int_c^d R(x,a_0) h_0 dx & \int_c^d R(x,a_1) h_0 dx & \dots & \int_c^d R(x,a_m) h_0 dx \\ \int_c^d R(x,a_0) h_1 dx & \int_c^d R(x,a_1) h_1 dx & \dots & \int_c^d R(x,a_m) h_1 dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_c^d R(x,a_0) h_m dx & \int_c^d R(x,a_1) h_m dx & \dots & \int_c^d R(x,a_m) h_m dx \end{pmatrix} \qquad (2.8)$$

$$G=\begin{pmatrix} \int_c^d \{f(x)\}h_0 dx \\ \int_c^d \{f(x)\}h_1 dx \\ \vdots \\ \int_c^d \{f(x)\}h_m dx \end{pmatrix} \tag{2.9}$$

where,

$$h_i = L_i^{(k)} + DL_i{}^{(k)}(x) + BL_i(x) - \left\{\lambda_1 \int_0^x k_1(x,t)L_i dt + \lambda_2 \int_c^d k_2(x,t)L_i dt\, dx\right\} \tag{2.10}$$

$$R(x,a_i) = \sum_{i=0}^{m} a_i L_i{}^{(k)}(x) + D\sum_{i=0}^{m} a_i L_i{}^{(k)}(x) + B\sum_{i=0}^{m} a_i L_i(x)$$
$$-\left\{\lambda_1 \int_0^x k_1(x,t)\sum_{i=0}^{m} a_i L_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t)\sum_{i=0}^{m} a_i L_i(t)\, dt\right\}, \tag{2.11}$$

$$WA = G \text{ or } A = [W:G]. \tag{2.12}$$

Property [11]. $\forall x \in \overline{\Omega}$ the matrix W defined in (2.12) is non-singular. □

The equation (1.1) compatible with the system of linear algebraic equations (m + 1) with the unknown Laguerre coefficients of $a_i, i = 0,1, \dots, m$.

It is possible to describe another type of (2.12) by applying the conditions as.

$$[U_i : \beta_i], \quad i = 0,1, \dots, N-1$$

where,

$$U_i = [u_{i0} u_{i1} u_{i2} \; \dots \; u_{iN}], \qquad i = 0,1,2,\dots,N-1. \tag{2.13}$$

The solution of (1.1) under conditions (1.2) can be get by changing the row matrices (2.13) by the last (m) rows of the matrix form (2.12) we get the new augmented matrix [12, 13, 14].

$$[\widetilde{W}:\tilde{G}] = \begin{pmatrix} \int_c^d R(x,a_0)h_0dx & \int_c^d R(x,a_1)h_0dx & \cdots & \int_c^d R(x,a_m)h_0dx & ; & G_0 \\ \int_c^d R(x,a_0)h_1dx & \int_c^d R(x,a_1)h_1dx & \cdots & \int_c^d R(x,a_m)h_1dx & ; & G_1 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ \int_c^d R(x,a_0)h_{m_{N_0}}dx & \int_c^d R(x,a_1)h_{m_{N_1}}dx & \cdots & \int_c^d R(x,a_m)h_{m_{N_m}}dx & ; & G_{m-N} \\ & u_{00}u_{01}u_{0N} & & & ; & \beta_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ u_{(m-1)0}u_{(m-1)1} & \cdots & & u_{(m-1)N} & ; & \beta_{N-1} \end{pmatrix}$$

$$A = \widetilde{W}^{-1}\widetilde{G},$$

therefore, the matrix A is uniquely determined. Also, the equation (1.1) with conditions (1.2) has a unique solution [1].

3. Convergence Analysis

Based on the numerical methods introduced in the second section, we will review an approximation of the above errors to prove that the approximate solution $u_m(x)$ will converge to the exact solution $u(x)$ of (1.1).

Lemma [15]. *Let $u(x) \in H^k(\mathbb{R}_+)$, such that*

$$\varphi_m u(x) = \sum_{i=0}^{m} a_i L_i(x)$$

is the best approximation polynomial of $u(x)$ for any $m \geq 0$, and $0 \leq k \geq m$, we have

$$||\, u - \varphi_m u||_{H^k(\mathbb{R}_+)} \leq C m^{k-\frac{m}{2}} ||\, u||_{H^m_{w,m}(\mathbb{R}_+)} \tag{3.1}$$

where C is a positive constant, which depends on m.

Proof. Let φ_m denote the symmetric truncation to the grade, i.e.,

$$\varphi_m^*(\sum_{j=-\infty}^{\infty} a_j L(x\,)) = \sum_{j=0}^{m} a_j L(x)$$

It is easily seen that

$$(\varphi_m u)^* = \varphi_m^* u^* \forall u \in H^k(\mathbb{R}_+), \tag{3.2}$$

Actually, since

$$u(x) = \sum_{i=0}^{\infty} a_i L_i(x)$$

whence (3.2). Now, from

$$|| u - \varphi_m u||_{H^k(\mathbb{R}_+)} \leq Cm^{k-\frac{m}{2}} || u||_{H^m_{w,m}(\mathbb{R}_+)} \forall u \in H^m_{w,m}(\mathbb{R}_+) \tag{3.3}$$

we get

$$|| u - \varphi_m u||_{H^k(\mathbb{R}_+)} = || \sum_{i=0}^{\infty} a_j L(x) - \sum_{i=0}^{m} a_j L(x) ||_{H^k(\mathbb{R}_+)}$$

$$= \|u^* - \varphi_m^* u^*\| \leq Cm^{k-\frac{m}{2}} || u^* ||_{H^m_{w,m}(\mathbb{R}_+)}$$

thus,

$$|| u - \varphi_m u||_{H^k(\mathbb{R}_+)} \leq Cm^{k-\frac{m}{2}} || u||_{H^m_{w,m}(\mathbb{R}_+)}$$

4. Illustrative Examples

In this section, we provide numerical experiments of the suggested method. In all examples, set the parameter $m = 5$ and denote the exact solution and approximate solution by $u(x)$ and $u_m(x)$ respectively.The error estimation is given to show the accuracy of the proposed method, and the following absolute error between the exact and approximate solutions is given by:

$$\text{Error} = |u(x) - u_m(x)| \qquad c \leq x \leq d,\ m = 1,2, \dots .$$

In Figures 1-3, we can observe that the approximate solution is highly consistent with the exact solution for different values of *x*. Furthermore, in Tables 1-3, the absolute errors and at $m = 5$ are reported, which shows the improved rate of convergence comparing with other mentioned methods [16, 17, 18].

Example 1 [16]**.** Consider the following linear mixed IDE,

$$u''(x) + Du'(x) + Bu(x)$$

$$= f(x) + \lambda_1 \int_{-1}^{x} k_1(x,t)u(t)dt + \lambda_2 \int_c^d k_2(x,t)u(t)dt,\ \ c \leq x,t \leq d,$$

$$u(0) = u'(0) = 1,$$

where $D = x,\ B = -x,\ f(x) = e^x - (x+1)sinx,\ k_1(x,t) = sinxe^{-t},\ k_2(x,t) = 0,$ $\lambda_1 = 1, \lambda_2 = 0.$

Solution. Using the combination of least squares with the Laguerre polynomial defined in the form, an approximate solution of $u(x)$ as (2.1) will be applied, when $m = 5$.

$$G = [-0.8143 \quad -0.5868 \quad 1.7778 \quad 6.1343 \quad 12.6433 \quad 21.7537]'$$

$$W = \begin{pmatrix} 0.0039e3 & 0.0060e3 & 0.0079e3 & 0.0118e3 & 0.0193e3 & 0.0323e3 \\ 0.0060e3 & 0.0100e3 & 0.0154e3 & 0.0254e3 & 0.0437e3 & 0.0741e3 \\ 0.0079e3 & 0.0154e3 & 0.0292e3 & 0.0549e3 & 0.0990e3 & 0.1693e3 \\ 0.0118e3 & 0.0254e3 & 0.0549e3 & 0.1098e3 & 0.2019e3 & 0.3464e3 \\ 0.0193e3 & 0.0437e3 & 0.0990e3 & 0.2019e3 & 0.3738e3 & 0.6424e3 \\ 0.0323e3 & 0.0741e3 & 0.1693e3 & 0.3464e3 & 0.6424e3 & 1.1052e3 \end{pmatrix}$$

the augmented matrices are respectively obtained from the given conditions, as follows.

$$U_0 = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,],\ U_1 = [0 \quad -1 \quad -2 \quad -3 \quad -4 \quad -5]$$

if we replace the last two rows of the matrices *W* and *G* by the values of U_0 and U_1then

$$\tilde{G} = [-0.8143 \quad -0.5868 \quad 1.7778 \quad 6.1343 \quad 1 \quad 1]'$$

$$\widetilde{W} = \begin{pmatrix} 0.0039e3 & 0.0060e3 & 0.0079e3 & 0.0118e3 & 0.0193e3 & 0.0323e3 \\ 0.0060e3 & 0.0100e3 & 0.0154e3 & 0.0254e3 & 0.0437e3 & 0.0741e3 \\ 0.0079e3 & 0.0154e3 & 0.0292e3 & 0.0549e3 & 0.0990e3 & 0.1693e3 \\ 0.0118e3 & 0.0254e3 & 0.0549e3 & 0.1098e3 & 0.2019e3 & 0.3464e3 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 & -5 \end{pmatrix}$$

$$A = \widetilde{W}^{-1}\tilde{G} = [\,6.1068 \quad -15.4749 \quad 20.8206 \quad -15.6838 \quad 6.2713 \quad -1.0401\,]'.$$

Therefore, the approximate solution of the problem taking $m = 5$ is the exact solution under the given conditions as follows:

$$u_5 = 0.999 + 1.0004x + 0.4980x^2 + 0.1666x^3 + 0.0446x^4 + 0.0087x^5.$$

Example 2 [17]**.** Consider the following linear mixed IDE second kind

$$u'(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)u(t)dt + \lambda_2 \int_c^d k_2(x,t)u(t)dt\,, c \le x, t \le d,$$

$$u(0) = 0$$

The exact solution is given as

$$u(x) = xe^x,$$

where $f(x) = xe^x + e^x - x, k_1(x,t) = 0, k_2(x,t) = x, \lambda_1 = 0, \lambda_2 = 1, c = 0, d = 1,$ $A = 0, B = 0.$

Solution. Using the combination of least squares with the Laguerre polynomial defined in the form, an approximate solution of $u(x)$ as (2.1) will be applied, when $m = 5$.

$$G = [-1.3849 \quad -2.9108 \quad -3.2824 \quad -2.9582 \quad -2.2696 \quad -1.4490]'$$

$$\mathrm{W} = \begin{pmatrix} 0.3333 & 0.6667 & 0.7222 & 0.6111 & 0.4139 & 0.1870 \\ 0.6667 & 1.5833 & 1.9444 & 1.9514 & 1.7528 & 1.4553 \\ 0.7222 & 1.9444 & 2.5648 & 2.7824 & 2.7468 & 2.5677 \\ 0.6111 & 1.9514 & 2.7824 & 3.2485 & 3.4616 & 3.5072 \\ 0.4139 & 1.7528 & 2.7468 & 3.4616 & 3.9534 & 4.2701 \\ 0.1870 & 1.4553 & 2.5677 & 3.5072 & 4.2701 & 4.8627 \end{pmatrix}$$

the augmented matrices are respectively obtained from the given conditions, as follows.

$$U_0 = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,],$$

if we replace the last rows of the matrices *W* and *G* by the values of U_0, then

$$\tilde{G} = [-1.3849 \quad -2.9108 \quad -3.2824 \quad -2.9582 \quad -2.2696 \quad 0]'$$

$$\widetilde{\mathrm{W}} = \begin{pmatrix} 0.3333 & 0.6667 & 0.7222 & 0.6111 & 0.4139 & 0.1870 \\ 0.6667 & 1.5833 & 1.9444 & 1.9514 & 1.7528 & 1.4553 \\ 0.7222 & 1.9444 & 2.5648 & 2.7824 & 2.7468 & 2.5677 \\ 0.6111 & 1.9514 & 2.7824 & 3.2485 & 3.4616 & 3.5072 \\ 0.4139 & 1.7528 & 2.7468 & 3.4616 & 3.9534 & 4.2701 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A = \widetilde{W}^{-1}\tilde{G} = [\,12.8131 \quad -41.9191 \quad 53.9630 \quad -32.4262 \quad 7.5692 \quad 0\,]'.$$

Therefore, the approximate solution of the problem taking $m = 5$ is the exact solution under the given conditions as follows:

$$u_5 = -1.7764e - 15 + 0.9949x + 1.0498x^2 + 0.3582x^3 + 0.3154x^4.$$

Example 3 [18]**.** Considered the following linear mixed IDE second kind.

$$u'''(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)u(t)dt + \lambda_2 \int_c^d k_2(x,t)u(t)dt\,, c \le x, t \le d,$$

$$u(0) = 0, u'(0) = 1, u''(0) = -1,$$

where $f(x) = sinx - x, k_1(x,t) = 0, k_2(x,t) = xt, \lambda_1 = 0, \lambda_2 = 1, c = 0, d = \pi/2$.

Solution. Using the combination of least squares with the Laguerre polynomial defined in the form, an approximate solution of $u(x)$ as (2.1) will be applied, when $m = 5$.

$$G = [\,0 \quad 0.3602 \quad 0.3432 \quad 0.4049 \quad 0.6162 \quad 0.8836]'$$

$$\mathrm{W} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.9663 & 1.8735 & 2.4565 & 4.3484 & 7.1853 \\ 0 & 1.8735 & 1.7851 & 2.3406 & 4.1431 & 6.8461 \\ 0 & 2.4565 & 2.3406 & 3.4616 & 7.0032 & 12.8228 \\ 0 & 4.3484 & 4.1431 & 7.0032 & 15.8992 & 31.2745 \\ 0 & 7.1853 & 6.8461 & 12.8228 & 31.2745 & 63.9409 \end{pmatrix},$$

the augmented matrices are respectively obtained from the given conditions, as follows.

$$U_0 = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,], \quad U_1 = [\,0 \quad -1 \quad -2 \quad -3 \quad -4 \quad -5\,],$$

$$U_2 = [0 \quad 0 \quad 1 \quad 3 \quad 6 \quad 10].$$

By replacing the last three rows of the matrices *W* and *G* by the values of U_0, U_1 and U_2 then

$$\tilde{G} = [\,0 \quad 0.3602 \quad 0.3432 \quad 1 \quad 1 \quad -1]'$$

$$\widetilde{W} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.9663 & 1.8735 & 2.4565 & 4.3484 & 7.1853 \\ 0 & 1.8735 & 1.7851 & 2.3406 & 4.1431 & 6.8461 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & 0 & 1 & 3 & 6 & 10 \end{pmatrix},$$

$$A = \widetilde{W}^{-1}\tilde{G} = [\; 0.6212 \quad 0.1364 \quad 0.8636 \quad -0.6212 \quad 0 \quad 0\;]'.$$

Therefore, the approximate solution of the problem taking $m = 5$ is the exact solution under the given conditions as follows:

$$u_5 = 1 + 2.204e - 16x - 0.5x^2 + 0.1035x^3.$$

Table 1. Comparison of absolute errors of Example 1 for $m = 5$ at different values of x.

x	Exact Solution	Approximated Solution	$\lvert u - \tilde{u} \rvert$ $m = 5$	Method [16] $m = 8$
-0.1	0.36787944117144	0.3668	1.1e-03	2.0697374 e-2
-0.8	0.44932896411722	0.448418144	9.1082e-4	1.1470384 e-2
-0.6	0.54881163609403	0.548058048	7.5359e-4	4.420981e-3
-0.4	0.67032004603564	0.669810272	5.0977e-4	9.73181e-4
-0.2	0.81873075307798	0.818475776	2.5498e-4	5.9298e-5
0	1.0000	0.9999	1.0000e-4	1.0e-8
0.2	1.2214027581602	1.221306944	9.5814e-5	8.849e-5
0.4	1.4918246976413	1.491633248	1.9145e-4	1.21776e-3
0.6	1.8221188003905	1.821862272	2.5653e-4	5.26806e-3
0.8	2.2255409284925	2.225358176	1.8275e-4	1.334832e-2
1	2.718281828459	2.7182	8.1828e-5	2.336246e-2

Table 2. Comparison of absolute errors of Example 2 for $m = 5$ at different values of x.

x	Exact Solution	Approximated Solution	$\lvert u - \tilde{u} \rvert$ $m = 5$	Method [17] $m = 10$
0.1	0.11051709180756	0.11037774	1.3935e-04	1.6806e-3
0.2	0.24428055163203	0.24434224	6.1688e-05	1.2580e-3
0.3	0.4049576422728	0.40517814	2.2050e-04	3.2146e-3
0.4	0.59672987905651	0.59692704	1.9716e-04	3.0893e-3
0.5	0.82436063535006	0.8243875	2.6865e-05	5.3670e-3
0.6	1.0932712802343	1.09311504	1.5624e-04	5.5896e-3
0.7	1.4096268952293	1.40942214	2.0476e-04	8.2456e-3
0.8	1.780432742794	1.78037824	5.4503e-05	8.8806e-3
0.9	2.2136428000413	2.21380974	1.6694e-04	1.1987e-2
1	2.718281828459	2.7183	1.8172e-05	1.3116e-2

Table 3. Comparison of absolute errors of Example 3 for $m = 5$ at different values of x.

x	Exact Solution	Approximated Solution	$\lvert u - \tilde{u} \rvert$ $m = 5$	Method [18] $m = 10$
0	1	1	0	0
0.1	0.99500416527803	0.9951035	9.9335e-5	4.99847830e-3
0.2	0.98006657784124	0.980828	7.6142e-4	2.00095694e-2
0.3	0.95533648912561	0.9577945	2.5e-3	4.51238431e-2
0.4	0.92106099400289	0.926624	5.6e-3	8.05195987e-2
0.5	0.87758256189037	0.8879375	1.04e-2	1.26461140e-1
0.6	0.82533561490968	0.842356	1.70e-2	1.83296917e-1
0.7	0.76484218728449	0.7905005	2.57e-2	2.51457501e-1
0.8	0.69670670934717	0.732992	3.63e-2	3.31453382e-1
0.9	0.62160996827066	0.6704515	4.88e-2	4.23872552e-1
1	0.54030230586814	0.6035	6.32e-2	5.29377859e-1

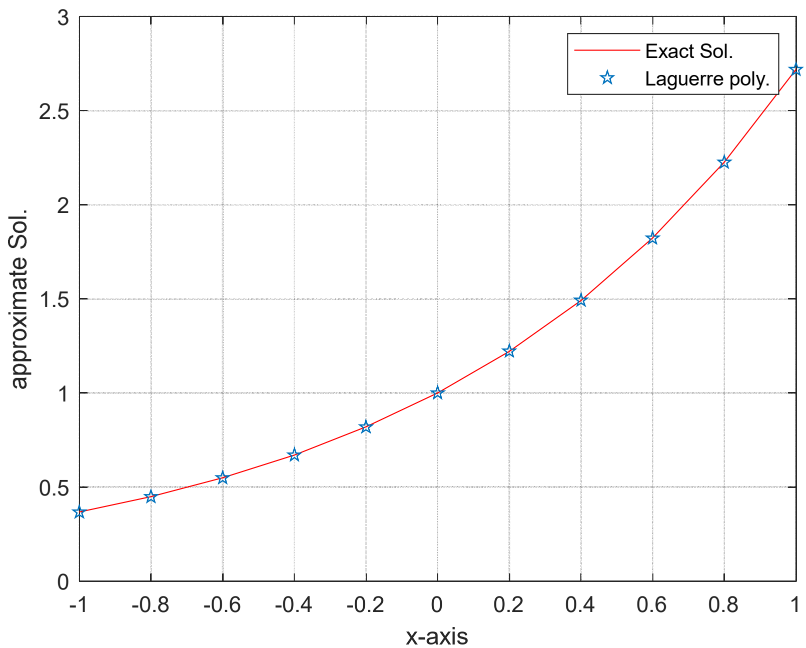

Figure 1. Exact and approximate solutions of Example 1 for $m = 5$.

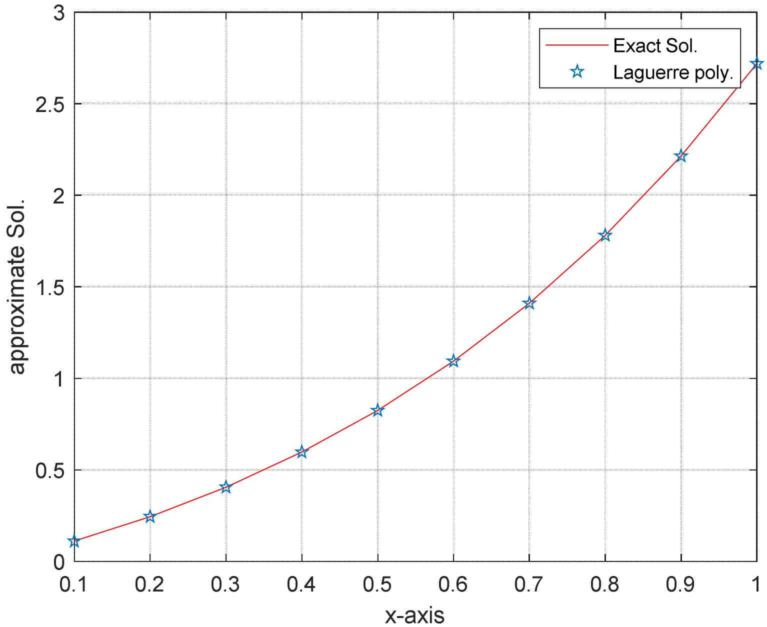

Figure 2. Exact and Approximate solutions of Example 2 for $m = 5$.

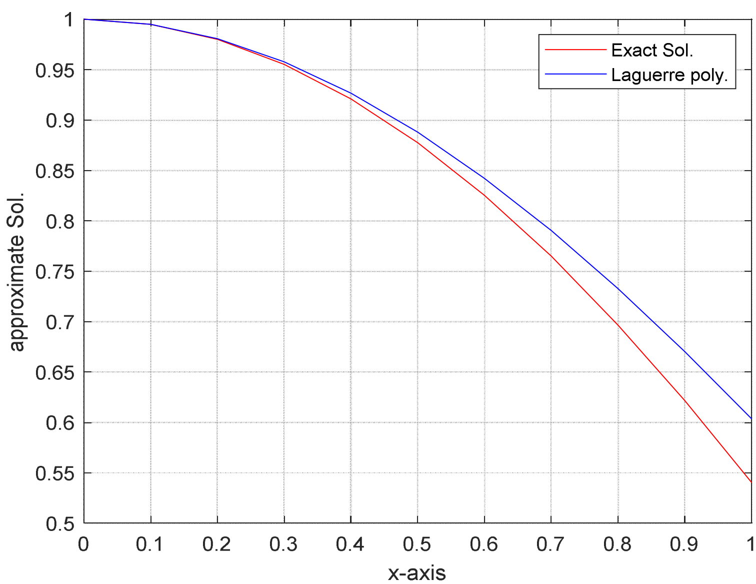

Figure 3. Exact and approximate solutions of Example 3 for $m = 5$.

5. Conclusions

In this paper, we developed a method based on the least-squares method (LSM) and Laguerre polynomials to solve mixed integro-differential equations. To illustrate the effectiveness of the proposed method, three examples were solved based on the present technique. In addition, the convergence of this method was analyzed. The results show that the proposed method is practically reliable and consistent in comparison with other mentioned methods [16, 17, 18].

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