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A Comparison of the Goodness of Fit for Three Theoretically-derived Infiltration Equations

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Experimental measurements of the cumulative infiltration over time were obtained using uniformly packed columns of porous materials of different particle size ranges and distribution. The relationship between cumulative infiltrated water and time was fitted by a non-linear regression program to three infiltration equations, namely, that of Green and Ampt, the three-term Philip equation and that of Knight, all of which are theoretically derived. The fit was generally good for all of the equations, with the Philip three-term equation being the best, the Green and Ampt equation second and the Knight equation third. The suction head at the wetting front was evaluated from the Green and Ampt equation and seems to be related to the size and distribution of the particles of the porous medium.

1. Introduction

Water infiltration into soil can be described theoretically by solving the govering partial differential equation presented by Richards[1] subject to appropriate boundary conditions. This equation has been solved numerically by Philip.[2] The so-called exact solution of Philip[3] is greatly complicated by experimental difficulties related to measurements of $K(\theta)$ and $D(\theta)$. On the other hand, a large amount of computing time is necessary to determine infiltration for certain conditions. These problems could be overcome by the concise, closed-form infiltration equations that have arisen from both theoretical and empirical arguments. The infiltration equations can be classified into empirical, such as those introduced by Kostiakov,[4] Horton,[5] and Holtan,[6] and theoretically-derived such as those presented by Green and Ampt[7] and Philip.[3] Empirical equations usually require the collection of large amounts of data to determine parameters that are used for different porous media and conditions. On the other hand, the exact solution requires difficult and time-consuming measurements of certain soil characteristics. Therefore, it would be useful to deal with approximate, but physically-based infiltration equations which are simple and do not require a large amount of data.

The most widely used physically-based model is that of Green and Ampt.[7] The Green and Ampt approach is based on the idea that the infiltration can be depicted by a very steep wetting front behind which the water content has a constant value θ_1. Water is infiltrated through the soil surface from ponded water of constant depth H and at any time the wetting front has descended to depth L in the soil. The suction head at the wetting front is supposed to be constant for a certain uniform porous medium. The Green and Ampt equation can be written in the following form:

$$t = (1/K_1)[I - a \ln (1 + I/a)], \tag{1}$$

where

$$I = (\theta_1 - \theta_0)L, \tag{2}$$

$$a = (\theta_1 - \theta_0)(H + P). \tag{3}$$

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Notation

θ	volumetric water content, m^3/m^3
$K(\theta)$	unsaturated hydraulic conductivity as a function of θ, m/min
$D(\theta)$	soil–water diffusivity as a function of θ, m^2/min
I	cumulative infiltration, m
t	time, min
K_1	a parameter of the Green and Ampt equation which is correlated to the hydraulic conductivity of the region behind the wetting front, m/min
a	a parameter of the Green and Ampt equation defined by Eqn (3)
θ_1	volumetric water content of the region behind the wetting front, m^3/m^3
θ_0	initial volumetric water content, m^3/m^3
L	depth of the region behind the wetting front, m
H	a constant depth of ponded water on the soil surface, m
P	suction head at the wetting front, m
S	sorptivity of the soil, $m/min^{1/2}$
A	a constant of the second term of the Philip equation, m/min
B	a constant of the third term of the Philip equation, $m/min^{3/2}$
K_s	the saturated hydraulic conductivity of the soil, m/min
A'	constant
B'	constant
C	a constant in the Knight equation, $min^{-1/2}$
w	a dummy variable in the error function term
K_d	the hydraulic conductivity measured from water dripping steadily from the bottom end of the column, m/min
f	total soil porosity calculated from $f = 1-(\rho_b/\rho_s)$, where ρ_b is the soil bulk density and ρ_s is the soil particle density

The parameter K_1 has been termed by Miller and Bresler[8] as the "satiated" hydraulic conductivity. They defined satiation as the near-saturated equilibrium water content resulting from free liquid water in contact with soil, but wherein complete saturation is prevented by the entrapment of air. The water content difference $(\theta_1-\theta_0)$ has been called by Onstad *et al.*,[9] and Brakensiek and Onstad[10] as the fillable porosity. Eqn (1) is applied to porous media initially in the drier region of water content range.

Philip[2] developed an equation of several terms to estimate the amount of infiltration of water into porous media based on the diffusion form of Richards's equation. He showed that the infiltration equation can be expressed by a simpler, rapidly converging power series in square root of time ($t^{1/2}$). Both Philip[3] and Watson[11] showed that the first two terms of Philip's solution were sufficient to give reasonable predictions of the infiltration rates. This equation is of the following form:

$$I = St^{1/2} + At. \tag{4}$$

The term S was proposed to be called the sorptivity of the soil (Philip[12]) and A is a constant reflecting an essentially steady rate at long time. One difficulty with using Eqn (4) is the uncertainty in the estimation of parameter A. Philip[13] noted that Eqn (4) is inappropriate for long time because in the limit, as time goes to infinity, $dI/dt = K_s$. However, A may not be equal to K_s, and there is no general analytical relationship between the two (Smiles and Knight,[14] Collis-George[15]). This problem has also been discussed by Youngs,[16] Philip[13] and Swartzendruber and Youngs.[17] Since Eqn (4) is inappropriate for long time, it was suggested that the first three terms of Philip's solution should be used as follows:

$$I = St^{1/2} + At + Bt^{3/2}, \quad (5)$$

where B is the third parameter of the Philip equation which is a function of θ_0 and θ_1. The parameter B is affected by the properties of the porous medium.

A so-called minimally non-linear approach was suggested by Philip[18] in which $dK/d\theta$ in the Richards equation is set equal to $A'\theta + B'$ where A' and B' are constants, but the diffusivity $D(\theta)$ is still constant. The govering partial differential equation then becomes the non-linear equation of Burgers.[19] Taking Burger's equation subject to the boundary conditions indicated in Philip,[3] Knight (reported by Philip[18]) obtained the complete mathematical solution which is of the following form:

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$$I = (K_1/C^2)[\ln(1 + \text{erf}(Ct^{1/2})] + K_1 t, \quad (6)$$

where

$C = K_1/\theta_1 D^{1/2}$,

$$\text{erf}(Ct^{1/2}) = \text{error function of } Ct^{1/2} = \frac{2}{\sqrt{\pi}} \int_0^{Ct^{1/2}} e^{-w} \, dw.$$

A detailed analysis for the derivation of this equation is given by Aoda.[20] Although the assumed constant $D(\theta)$ still remains as shortcoming, the form of $K(\theta)$ implied by $dK/d\theta = A'\theta + B'$ is of the correct shape. It would thus seem that Knight's equation is at present one of the better analytical descriptions for the complete time range. In seeking an approximate solution for infiltration that would hold over the complete time range, Philip[21] linearized the Richards equation by taking $D(\theta)$ and $dK/d\theta$ as constants. Despite these two rather drastic assumptions, the linearized solution remains deficient at large times.

There has been abundant work quantifying the infiltration process and comparing the different models of predicting it, Swartzendruber *et al.*,[22] compared the Green and Ampt and Philip two-term equations and concluded that the Green and Ampt approach and equation seem to possess some inherent advantage over the two-term Philip equation. Swartzendruber and Young[17] compared three physically-based infiltration equations and concluded that the Philip[3] two-term equation is preferred over Green Ampt's[7] and a linearized form of Philip's[21] equation. Aoda and Swartzenduber in unpublished work compared the Knight (reported by Philip[18]) equation with the Green and Ampt equation and reported that both equations predicted the cumulative infiltration of water very well, but the Green and Ampt equation gave the better fit and its fitted parameter K_1 was in better agreement with the hydraulic conductivity K_d measured from water dripping steadily from the bottom end of the sand column.

The purpose of this study is to compare some physically-based infiltration equations by fitting the data of cumulative water infiltration as a function of time from experiments using uniformly packed laboratory columns of several soil materials and various particle sizes.

2. Materials and methods

Glass columns 75 cm long and 4 cm inside diameter were packed with sand of particle of size ranges and average bulk densities ρ_b for each experiment as shown in Table 1. In experiments 6 and 7, sandy loam (from Zubair) and sandy clay loam (from Mussyab) were used, respectively. The sand material was acid-washed with dilute hydrochloric acid before fractionation by sieving. All porous materials were air dried before packing. The initial average volumetric water content θ_0 for each column was determined gravimetrically and found to be negligibly small. The uniformity of bulk density for each column was achieved by packing known masses of the material into known incremental volumes for each column. The application of water to soil was done by using a Mariotte-type water reservoir which

Table 1
Some physical characteristics of the materials used

Experiment number	*Particle size range of sand, m × 10⁻³, and soil texture*	*Average bulk density, Mg/m³*
1	0·300–0·250	1·438
2	0·250–0·180	1·404
3	0·180–0·125	1·365
4	0·125–0·045	1·194
5	0·300–0·045	1·426
6	Sandy loam	1·590
7	Sandy clay loam	1·342

was adjusted to produce 0·005 m of water as an instantaneous head (H) of free water at the soil surface.

The infiltration experiment was started by opening the stopcock to produce the desired instantaneous head on the top end of the soil column. Water reservoir readings were recorded with time. The visual wet front penetration was also recorded with time. These readings were continued until the water front reached the bottom end of the sand column. From these readings with time, the flux was determined and the hydraulic conductivity K_d was calculated using Darcy's law.

To determine the parameters of the infiltration equations used [Eqns (1), (4)–(6)], a non-linear regression program based on the least squares technique was used for fitting each equation to the experimental data. The computer work was done at the University of Rostock Computing Centre, in the German Democratic Republic (GDR).

3. Results and discussion

The Green and Ampt [Eqn (1)], Philip [Eqn (5)], and Knight [Eqn (6)] equations were fitted to experimental data of cumulative infiltration I *vs* time t for all experiments, from $t = 0$ until the visual wet front reached the bottom end of the soil column. Results of these fittings are given in Table 2. For all experiments in the table, the Philip three-term equation (5) produced the smallest residual mean square of I (RMSI), while the Knight equation (6) produced the largest RMSI. In terms of the average for all experiments, the RMSI for the Green and Ampt equation (1) is 4·25 times larger than that for the Philip equation (5), while the RMSI for the Knight equation (6) is about 5·86 times larger than that for the Philip equation (5). On the other hand, the average RMSI for the Knight equation (6) is 1·38 times larger than that for the Green and Ampt equation (1). Hence, with regard to the goodness of fit, the Philip three-term equation (5) is superior to both Green and Ampt and Knight equations and the Green and Ampt equation is superior to the Knight equation. Graphical representation of Eqn (6) for experiment 2 is shown in *Fig. 1*. The theoretical curve of this equation was drawn using values of the fitted parameters $K_1 = 1{\cdot}081 \times 10^{-2}$ m/min and $C = 0{\cdot}372\ \text{min}^{-1/2}$. The experimental points are close to the fitted curve even though the value of RMSI is the largest of all for all experiments and equations (Table 2). Since the fitted curve of Eqn (6) has the largest RMSI with experiment 2 (Table 2), and hence its poorest fit, graphical plots for the other experiments and equations would be even better than that in *Fig. 1*.

In an attempt to compare the Philip two-term [Eqn (4)] and three-term [Eqn (5)] equations, Eqn (4) was fitted to experimental (I, t) pairs for all experiments. Values of the fitted parameters along with the RMSI are listed in Table 3. In terms of the average for all

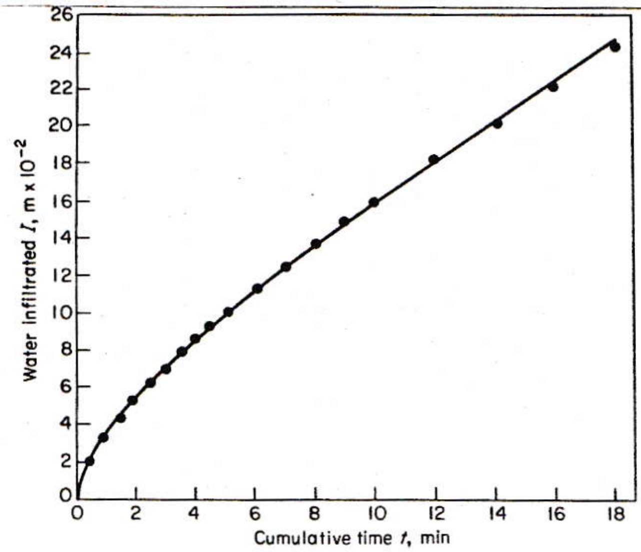

Fig. 1. Fitting of the Knight equation (6) to infiltration data (●) of experiment 2. The curve represents the largest RMSI in Table 2

experiments, the RMSI for Eqn (4) is 2·83 times larger than that for Eqn (5). This suggests that the Philip three-term equation is superior to the two-term equation.

We also attempted to find if there was any correlation between the parameter A in both equations of Philip [Eqn (4) and (5)] and the measured hydraulic conductivity K_d from the dripping condition. The fitted values of A from Eqn (4) are meaningful since the ratios of A/K_d, for all experiments, fall between 0·5 and 1·0 which would be even smaller if the ratio of A/K_s were taken. Talsma[23] and Whisler and Bouwer[24] suggested the ratios of A/K_s to be 0·36 and 0·67, respectively. Ratios of fitted parameter A from Eqn (5) are not meaningful for experiments 1, 2 and 4. Some negative values of the fitted parameters A and B of Eqn (5) were obtained (Table 2), a case that is not true physically. Negative values of parameter A of Eqn (4) have been reported by Skaggs *et al.*,[25] Taylor and Ashcroft,[26] Cook *et al.*,[27] Fahad *et al.*[28] and Gosh.[29]

Another important aspect of the fitting is the meaning that can be attached to the fitted parameters. The parameter K_1 should be equal to the hydraulic conductivity K_d for the case where steady dripping of water occurs from the bottom end of the infiltration column. To remove the temperature effects from the comparison, the values of K_1 and K_d in Table 2 have been corrected to a standard temperature of 25°C. From Table 2, the parameter K_1 from the Green and Ampt equation (1) is closer to K_d than K_1 from the Knight equation (6). The Green and Ampt equation (1) has overestimated the parameter K_1 (in comparison with K_d) for some experiments and underestimated it for others. The highest overestimate was 21·0% for experiment 2 and the lowest underestimate was 33·4% for experiment 6. The Knight equation (6) has overestimated the parameter K_1 for all experiments except for experiment 4. The highest overestimate was 107% for experiment 7 and the lowest underprediction was 2·1% for experiment 4. Therefore, the Green and Ampt equation (1) produced a fitted parameter K_1 that is closer to K_d than did the Knight equation (6) which is another advantage of the Green and Ampt equation over the Knight equation, beside its smaller RMSI.

The water content difference $(\theta_1 - \theta_0)$ in Table 3 was calculated from the experimental (I, L) pairs, where L is the length of the visual wetting front penetration. Eqn (2) was fitted

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Table 2

Fitted parameters of the Green and Ampt equation (1), Philip equation (5) and Knight equation (6) for water infiltration into different porous materials

				Green and Ampt [Eqn (1)]			Philip [Eqn (5)]				Knight [Eqn (6)]		
Expt number	Materials	Particle size $m \times 10^{-3}$, and soil texture	K_d at 25°C, m/min $\times 10^{-2}$	K_1 at 25°C, m/min $\times 10^{-2}$	a, $m \times 10^{-2}$	RMSI*, $m^2 \times 10^{-6}$	S, $m/min^{1/2} \times 10^{-2}$	A, m/min $\times 10^{-2}$	B, $m/min^{3/2} \times 10^{-3}$	RMSI*, $m^2 \times 10^{-6}$	K_1 at 25°C, m/min $\times 10^{-2}$	C, $min^{-1/2}$	RMSI*, $m^2 \times 10^{-6}$
1	Sand	0·300–0·250	1·455	1·630	5·746	2·982	3·232	1·796	−1·208	0·302	2·069	0·478	5·795
2	Sand	0·250–0·180	0·757	0·925	5·671	5·501	2·332	1·085	−0·666	1·032	1·183	0·372	7·301
3	Sand	0·180–0·125	0·299	0·242	14·546	0·353	2·598	0·125	0·039	0·270	0·368	0·149	1·850
4	Sand	0·125–0·045	0·061	0·047	15·644	1·728	1·302	−0·009	0·024	0·290	0·060	0·050	1·182
5	Sand	0·300–0·045	0·148	0·131	5·476	1·881	1·330	0·025	0·048	0·072	0·165	0·136	0·472
6	Soil	Sandy loam	0·038	0·025	9·468	0·296	0·608	0·021	−0·002	0·184	0·044	0·068	1·243
7	Soil	Sandy clay loam	0·027	0·027	9·427	2·704	0·634	0·010	−0·004	0·686	0·039	0·055	2·781

* Residual mean square of $I = \sum_{i=1}^{n} (I_i - \hat{I}_i)^2/(n-n_e)$, where I_i is the ith experimental values of I, $\hat{I}_i$ is the fitted value of I_i, n is the number of (I, t) data points and n_e is the number of parameters in the fitted equation

Table 3

Fitted parameters of Philip's two-term equation (4) and some calculated values of parameters in the Green and Ampt equation (1)

				Philip two-term [Eqn (4)]			Green and Ampt [Eqn (1)]			
Expt number	Materials	Particle size, $m \times 10^{-3}$, and soil texture	K_d, m/min $\times 10^{-2}$	S, $m/min^{1/2} \times 10^{-2}$	A, m/min $\times 10^{-2}$	RMSI, $m^2 \times 10^{-6}$	$\theta_1 - \theta_0$, m^3/m^3	P_*, $m \times 10^{-2}$	f	% fractional saturation
1	Sand	0·300–0·250	1·330	3·852	1·226	1·696	0·360	15·5	0·457	78·7
2	Sand	0·250–0·180	0·692	2·848	0·695	2·942	0·358	15·4	0·470	76·1
3	Sand	0·180–0·125	0·273	2·522	0·162	0·416	0·360	39·9	0·485	74·4
4	Sand	0·125–0·045	0·056	1·189	0·026	1·325	0·377	41·0	0·549	68·7
5	Sand	0·300–0·045	0·135	1·146	0·089	2·155	0·312	17·1	0·462	67·5
6	Soil	Sandy loam	0·032	0·627	0·016	0·226	0·270	34·5	0·400	67·6
7	Soil	Sandy clay loam	0·022	0·578	0·020	1·208	0·353	26·2	0·494	71·5

to the (I, L) data points by linear regression, through the origin, yielding $(\theta_1-\theta_0)$ as the slope of the regression line. Goodness of fit was generally excellent. Since θ_0 is negligibly small for all porous materials used, θ_1 is nearly equal to $(\theta_1-\theta_0)$ which is a mean water content for the region behind the wetting front. From this, the fractional saturation (θ_1/f) was calculated and is listed in Table 3, with values lower than unity (saturation). Hence, for all experiments, satiation is clearly lower than saturation.

The Green and Ampt parameter a in Table 3 was used along with $(\theta_1-\theta_0)$ and the head H ($H=0{\cdot}005$ m) in Eqn (3) to calculate the suction head P at the wetting front. As shown in Table 3, values of P increase as the size of the particles becomes smaller. Mixing particles of different size ranges also increases the value of P. This can probably be explained by the fact that when particles of different sizes are packed together, the fine particles get in between the coarse ones producing small pore sizes which are of high suction head P. This is illustrated by the values of P obtained in experiments 6 and 7.

4. Conclusions

Choice of an appropriate infiltration equation that can describe the infiltration phenomenon is an important task in soil and water management. From the work reported here one can conclude that the three physically-based infiltration equations predicted the cumulative infiltration with time very well, with the Philip three-term equation being superior to the Green and Ampt equation. Also the Green and Ampt equation appeared to be superior to the Knight equation.

Based on the residual mean square of cumulative water infiltrated, it can be concluded that the Philip three-term equation fitted the infiltration data better than the two-term one, even though the fitted parameters of the two-term equation are capable of physical interpretation.

The size of particles and their distribution in the porous media seem to influence the soil-water flow characteristics and hence the water infiltration process.

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