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ORIGINAL RESEARCH

Particular Solution of Linear Sequential Fractional Differential equation with Constant Coefficients by Inverse Fractional Differential Operators

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Abstract This paper adopts the inverse fractional differential operator method for obtaining the explicit particular solution to a linear sequential fractional differential equation, involving Jumarie's modification of Riemann–Liouville derivative, with constant coefficient s. This method depends on the classical inverse differential operator method and it is independent of the integral transforms. Several examples are then given to demonstrate the validity of our main results.

Keywords Fractional differential equations · Riemann–Liouville derivative · Jumarie's fractional derivation · Inverse differential operators · Inverse fractional differential operators

Introduction

Fractional differential equations are a generalization of the ordinary differential equation to arbitrary non-integer order . Fractional differential equations arise in many complex systems in nature and society with many dynamics, such as rheology, porous media, viscoelasticity, electrochemistry, electromagnetism, signal processing, dynamics of earthquakes, optics, geology, viscoelastic materials, biosciences, bioengineering, medicine, economics, probability and statistics, astrophysics, chemical engineering, physics, splines, tomography, fluid mechanics, electromagnetic waves, nonlinear control, control of power electronic, converters, chaotic dynamics, polymer science, proteins, polymer physics, electrochemistry, statistical physics, thermodynamics, neural networks and many more [10,13,14,30,31,37,39] . In recent years, there has been a significant development in the techniques of solving fractional differential equations, some recent contributions can be seen in [1–6,8–12,17,18,24–29,32–

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40], and the references therein. A wide description of the problem of the existence and uniqueness of solutions of Cauchy-type problems for fractional order differential equations together with its applications can be found in the literature [1,6,8,27,30,33–37].

Motivated and inspired by the on-going research in this field, we will consider the following non-homogeneous linear fractional differential equation with constant coefficients

$$\left(\mathcal{D}_x^{n\alpha} + a_1\mathcal{D}_x^{(n-1)\alpha} + a_2\mathcal{D}_x^{(n-2)\alpha} + \cdots + a_{n-1}\mathcal{D}_x^{\alpha} + a_n\right) y(x) = Q(x) \tag{1.1}$$

where $q = \frac{1}{\alpha}$ is integer number, a_k, $k = 1, 2, \ldots, n$ are real constant, $\mathcal{D}_x^{n\alpha} = \underbrace{D_x^\alpha D_x^\alpha \cdots D_x^\alpha}_{n-\text{times}}$ and D_x^α denotes Jumarie's fractional derivation [19–23], which is a modified Riemann–Liouville derivative defined as

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\zeta)^{-\alpha}(f(\zeta) - f(0))d\zeta, \quad 0 < \alpha < 1 \tag{1.2}$$

and

$$D_x^\alpha f(x) = \frac{d^n}{dx^n}(D^{(\alpha-n)}f(x)), \quad n < \alpha < n+1, n \geq 1 \tag{1.3}$$

Eq. (1.1) is called fractional linear differential equation with constant coefficients of order (n, q), or more briefly, a fractional differential equation of order (n, q) [34]. If $\alpha = 1$, then Eq. (1.1) become n^{th} order ordinary differential equations.

This paper is organized as follows: Sect. 2 presents Jumarie's Modification of Riemann–Liouville Derivative and their main properties. In Sect. 3, we study some properties of linear fractional differential operators with constant coefficients. In Sect. 4, we adopt the method of inverse fractional differential operators to find the particular solution to non-homogeneous LSFDE with constant coefficients while in Sect. 5, several examples are given to demonstrate the validity of our main results.

Jumarie's Modification of Riemann–Liouville Derivative

The fractional derivative has different definitions [27,33–36], and exploiting any of them depends on the boundary conditions and the specifics of the considered physical systems and processes. The first definition of fractional derivative which has been proposed in the literature is the so-called Riemann–Liouville definition which reads as follows

Definition 2.1 (*Riemann–Liouville derivative*) [34] Let $f(x) : R \to R$ be a continuous function then the fractional derivative of order α is defined by

$$D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x (x-\zeta)^{-\alpha-1}f(\zeta)d\zeta, \quad \alpha < 0 \tag{2.1}$$

and

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-\zeta)^{n-\alpha-1}f(\zeta)d\zeta, \quad n < \alpha < n+1 \tag{2.2}$$

where $\Gamma(.)$ is the Gamma function.

It is well known that the fractional derivative, in the sense of Riemann–Liouville definition of fractional derivative, of a constant is not zero. This encourages Caputo to introduce Caputo derivative such that the fractional derivative of a constant is zero [7].

Definition 2.2 (*Caputo derivative*) [34] Let f(x) : R → R be a continuous function then the fractional derivative of order α is defined by

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\zeta)^{n-\alpha-1} \frac{d^n f(\zeta)}{d\zeta^n} d\zeta, \quad n < \alpha < n+1 \tag{2.3}$$

Definition 2.3 (*Grunwald–Letnikov*) [34] The Grunwald–Letnikov definition is given by

$${}_{t_0}^{GL} D_t^{\alpha} x(t) = \lim_{h \to 0} \sum_{j=0}^{\left[\frac{t-t_0}{h}\right]} (-1)^j \binom{\alpha}{j} x(t-jh),$$

Where [.] means the integer part.

With Caputo definition, a fractional derivative would be defined for differentiable functions only. In order to deal with non-differentiable functions, Jumarie have recently proposed a modification of the Riemann–Liouville definition [19–23]. This fractional derivative provides a Taylor's series of fractional order for non differentiable functions.

Definition 2.4 (*Jumarie's modification of Riemann–Liouville derivative*) [19–23] Let f(x) := R → R be a continuous function then the fractional derivative of order α is defined by

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\zeta)^{-\alpha-1} (f(\zeta) - f(0)) d\zeta, \quad \alpha < 0 \tag{2.4}$$

and

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\zeta)^{n-\alpha-1} (f(\zeta) - f(0)) d\zeta, \quad n < \alpha < n+1 \tag{2.5}$$

He et al. [15] introduce the geometrical explanation of fractional complex transform and derivative chain rule for fractional calculus in the sense of Jumarie's modification of Riemann–Liouville derivative. Remark the main difference between Definitions (2.2) and (2.3). The second one involves the constant f(0) while the first one does not. Also, the fractional Riemann–Liouville derivative of a constant is not zero while the fractional Jumarie derivative of a constant is zero. In the rest of the paper, D_x^{α} will be used to refer to Jumarie's modification of Riemann–Liouville derivative

Definition 2.5 (*Principle of Derivative increasing orders*) [22] The functional derivative of fractional $D_x^{\alpha+\beta}$ expressed in terms of D_x^{α} and D_x^{β} is defined by the equality $D_x^{\alpha+\beta} f(x) = D_x^{\max(\alpha,\beta)}(D_x^{\min(\alpha,\beta)} f(x))$.

The function $E_{\alpha}(t)$ was defined by Mittag–Leffler in the year 1903. These functions will play the main role in investigating the stability of fractional order system.

Definition 2.6 (*Mittag–Leffler function*) [27] The one-parameter and two-parameter Mittag–Leffler functions are defined as, respectively

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+1)}, \quad (\alpha > 0),$$

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+\beta)}, \quad (\alpha > 0, \beta > 0),$$

For $\beta = 1$, we have $E_{\alpha}(t) = E_{\alpha,1}(t)$. Also, $E_{1,1}(t) = e^t$.

Proposition 2.7 *Assume that the continuous function $f(x): R \to R$ has a fractional derivative of order αk for any positive integer k and $0 < \alpha < 1$, then the following equality holds* [21],

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(\alpha k + 1)} f^{(\alpha k)}(x), \quad 0 < \alpha \leq 1 \tag{2.6}$$

where $f^{(\alpha k)}(x)$ is the fractional Jumarie derivative of order αk of $f(x)$. Formally, Eq. (2.6) *can be written $f(x+h) = E_{\alpha}(h^{\alpha} D_x^{\alpha}) f(x), \quad 0 < \alpha \leq 1$.*

Corollary 2.8 *The following equalities hold* [22], *which are*

$$D^{\alpha} x^{\gamma} = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha) x^{\gamma-\alpha}, \quad \gamma > 0 \tag{2.7}$$

or, what amounts to the same (we set $\alpha = n + \theta$)

$$D^{n+\theta} x^{\gamma} = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-n-\theta) x^{\gamma-n-\theta}, \quad 0 < \theta < 1 \tag{2.8}$$

$$D_x^{\alpha}(u(x)v(x)) = D_x^{\alpha} u(x) v(x) + u(x) D_x^{\alpha} v(x) \tag{2.9}$$

Lemma 2.9 *The following various formulae are hold* [22]

1.
$$E_{\alpha}(x^{\alpha} y^{\alpha}) = (E_{\alpha}(y^{\alpha}))^{x} \tag{2.10}$$

2.
$$D_x^{\alpha} E_{\alpha}(\lambda x^{\alpha}) = \lambda E_{\alpha}(\lambda x^{\alpha}) \tag{2.11}$$

3.
$$E_{\alpha}(ix) = \cos_{\alpha} x + i \sin_{\alpha} x \tag{2.12}$$

4.
$$E_{\alpha}(x) = \cosh_{\alpha} x + \sinh_{\alpha} x \tag{2.13}$$

5.
$$D_x^{\alpha} \cos_{\alpha} x^{\alpha} = -\sin_{\alpha} x^{\alpha}, \quad D_x^{\alpha} \sin_{\alpha} x^{\alpha} = \cos_{\alpha} x^{\alpha} \tag{2.14}$$

6.
$$D_x^{\alpha} \cosh_{\alpha} x^{\alpha} = -\sinh_{\alpha} x^{\alpha}, \quad D_x^{\alpha} \sinh_{\alpha} x^{\alpha} = \cosh_{\alpha} x^{\alpha} \tag{2.15}$$

Some Properties of Linear Fractional Differential Operators with constant Coefficients:

Consider the following linear non homogeneous LSFDE with constant coefficients of order (n, q)

$$(\mathcal{D}_x^{n\alpha} + a_1 \mathcal{D}_x^{(n-1)\alpha} + a_2 \mathcal{D}_x^{(n-2)\alpha} + \cdots + a_{n-1} \mathcal{D}_x^{\alpha} + a_n) y(x) = Q(x) \tag{3.1}$$

where $\alpha = \frac{1}{q}$ is constant rational number, $a_k, k = 1, 2, \ldots, n$ are real constant, $\mathcal{D}_x^{n\alpha} = \underbrace{D_x^{\alpha} D_x^{\alpha} \cdots D_x^{\alpha}}_{n-\text{times}}$.

Rewrite Eq. (3.1) in the form

$$\mathrm{P}(\mathcal{D}_{\mathrm{x}}^{\alpha})\mathrm{y}(\mathrm{x}) = \mathrm{Q}(\mathrm{x}) \tag{3.2}$$

where $\mathrm{P}(\mathcal{D}_{\mathrm{x}}^{\alpha})$ is a linear fractional differential operator.

Lemma 3.1 (The Exponential Mittag–Leffler shift) $E_{\alpha}(\lambda\, x^{\alpha})P(\mathcal{D}_{\mathrm{x}}^{\alpha})y(x) = P(\mathcal{D}_{\mathrm{x}}^{\alpha} - \lambda)E_{\alpha}(\lambda\, x^{\alpha})y(x)$ *where* $E_{\alpha}(u) = \sum_{k=0}^{\infty} \frac{u^{\mathrm{k}}}{\Gamma(\alpha\, k+1)}$ *is the Exponential Mittag–Leffler function.*

Proof Consider the effect of the operator $\mathcal{D}_{\mathrm{x}}^{\alpha} - \lambda$ on the product of $\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})$ and a function y(x), one can have

$$\begin{aligned}
(\mathcal{D}_{\mathrm{x}}^{\alpha} - \lambda)\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) &= \mathcal{D}_{\mathrm{x}}^{\alpha}\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) - \lambda\, \mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) \\
&= (\mathcal{D}_{\mathrm{x}}^{\alpha}\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha}))\mathrm{y}(\mathrm{x}) + \mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathcal{D}_{\mathrm{x}}^{\alpha}\mathrm{y}(\mathrm{x}) - \lambda\, \mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) \\
&= \lambda\, \mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) + \mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathcal{D}_{\mathrm{x}}^{\alpha}\mathrm{y}(\mathrm{x}) - \lambda\, \mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) \\
&= E_{\alpha}(\lambda x^{\alpha})\mathcal{D}_{x}^{\alpha}\, y(x).
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{D}_{\mathrm{x}}^{\alpha} - \lambda)^2\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) &= (\mathcal{D}_{\mathrm{x}}^{\alpha} - \lambda)(\mathcal{D}_{\mathrm{x}}^{\alpha} - \lambda)\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) \\
&= (\mathcal{D}_{\mathrm{x}}^{\alpha} - \lambda)(\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathcal{D}_{\mathrm{x}}^{\alpha}\mathrm{y}(\mathrm{x})) \\
&= \mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathcal{D}_{\mathrm{x}}^{\alpha}\mathcal{D}_{\mathrm{x}}^{\alpha}\mathrm{y}(\mathrm{x}) \\
&= \mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathcal{D}_{\mathrm{x}}^{2\,\alpha}\mathrm{y}(\mathrm{x}).
\end{aligned}$$

Repeating the operation, one have

$$(\mathcal{D}_{\mathrm{x}}^{\alpha} - \lambda)^{\mathrm{k}}\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) = \mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathcal{D}_{\mathrm{x}}^{\mathrm{k}\,\alpha}\mathrm{y}(\mathrm{x}), \quad \mathrm{k} = 1, 2, \ldots \tag{3.3}$$

Using the linearity of fractional differential operators, we conclude that when $\mathrm{P}(\mathcal{D}_{\mathrm{x}}^{\alpha})$ is a polynomial in $\mathcal{D}_{\mathrm{x}}^{\alpha}$ with constant coefficients, then

$$\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{P}(\mathcal{D}_{\mathrm{x}}^{\alpha})\mathrm{y}(\mathrm{x}) = \mathrm{P}(\mathcal{D}_{\mathrm{x}}^{\alpha} - \lambda)\mathrm{E}_{\alpha}(\lambda\, \mathrm{x}^{\alpha})\mathrm{y}(\mathrm{x}) \tag{3.4}$$

□

As a direct computation, one has the following :

Lemma 3.2 *The following various formulae are hold*

1.

$$P(\mathcal{D}_{\mathrm{x}}^{\alpha})E_{\alpha}(\lambda x^{\alpha}) = P(\lambda)E_{\alpha}(\lambda x^{\alpha}) \tag{3.5}$$

2.

$$P(\mathcal{D}_{\mathrm{x}}^{2\,\alpha}) \cos_{\alpha} bx^{\alpha} = P(-b^2) \cos_{\alpha} bx^{\alpha}, \tag{3.6}$$

3.

$$P(\mathcal{D}_{\mathrm{x}}^{2\,\alpha}) \sin_{\alpha} bx^{\alpha} = P(-b^2) \sin_{\alpha} bx^{\alpha}, \tag{3.7}$$

4.

$$P(\mathcal{D}_{\mathrm{x}}^{2\,\alpha}) \cosh_{\alpha} bx^{\alpha} = P(-b^2) \cosh_{\alpha} bx^{\alpha}, \tag{3.8}$$

5.

$$P(\mathcal{D}_{\mathrm{x}}^{2\,\alpha}) \sinh_{\alpha} bx^{\alpha} = P(-b^2) \sinh_{\alpha} bx^{\alpha}, \tag{3.9}$$

Particular Solution of Nonhomogeneous LSFDE with Constant Coefficients by Using Inverse Fractional Differential Operators

In seeking a particular solution of Eq. (3.2), it is natural to write

$$y(x) = \frac{1}{P(\mathcal{D}_x^\alpha)} Q(x) \tag{4.1}$$

and to try to define an operator $\frac{1}{P(\mathcal{D}_x^\alpha)}$ so that the function $y(x)$ of Eq.(4.1) will have meaning and will satisfy Eq. (3.2).

Instead of building a theory of such inverse fractional differential operators, we shall adopt the following method of attack. Purely formal (unjustified) manipulations of the symbols will be performed, thus leading to a tentative evaluation of $\frac{1}{P(\mathcal{D}_x^\alpha)} Q(x)$. After all, the only thing that we require of evaluation is that

$$P(\mathcal{D}_x^\alpha) \frac{1}{P(\mathcal{D}_x^\alpha)} Q(x) = Q(x) \tag{4.2}$$

Hence the proof will be placed on a direct verification of the Eq. (4.2) in each instance.

By using Lemma 3.2, one has the following:

Lemma 4.1 *The following various formulae are hold*

1.

$$\frac{1}{P(\mathcal{D}_x^\alpha)} E_\alpha(\lambda x^\alpha) = \frac{1}{P(\lambda)} E_\alpha(\lambda x^\alpha), \quad \textit{such that } P(\lambda) \neq \tag{4.3}$$

2.

$$\frac{1}{P(\mathcal{D}_x^{2\alpha})} \sin_\alpha bx^\alpha = \frac{1}{P(-b^2)} \sin_\alpha bx^\alpha, \quad \textit{such that } P(-b^2) \neq 0 \tag{4.4}$$

3.

$$\frac{1}{P(\mathcal{D}_x^{2\alpha})} \cos_\alpha bx^\alpha = \frac{1}{P(-b^2)} \cos_\alpha bx^\alpha, \quad \textit{such that } P(-b^2) \neq 0 \tag{4.5}$$

4.

$$\frac{1}{P(\mathcal{D}_x^{2\alpha})} \sinh_\alpha bx^\alpha = \frac{1}{P(-b^2)} \sinh_\alpha bx^\alpha, \quad \textit{such that } P(-b^2) \neq 0 \tag{4.6}$$

5.

$$\frac{1}{P(\mathcal{D}_x^{2\alpha})} \cosh_\alpha bx^\alpha = \frac{1}{P(-b^2)} \cosh_\alpha bx^\alpha, \quad \textit{such that } P(-b^2) \neq 0 \tag{4.7}$$

By using Lemma 3.1, one has the following :

Lemma 4.2 *The following various formulae are hold for any $f(x)$*

1.

$$\frac{1}{P(\mathcal{D}_x^\alpha)} \frac{f(x)}{E_\alpha(\lambda x^\alpha)} = \frac{1}{E_\alpha(\lambda x^\alpha)} \frac{1}{P(\mathcal{D}_x^\alpha - \lambda)} f(x), \tag{4.8}$$

2.

$$\frac{1}{P(\mathcal{D}_x^{\alpha})}E_{\alpha}(\lambda\, x^{\alpha})f(x) = E_{\alpha}(\lambda\, x^{\alpha})\frac{1}{P(\mathcal{D}_x^{\alpha}+\lambda)}f(x), \tag{4.9}$$

Proof To prove (1), let $y(x) = \frac{1}{P(\mathcal{D}_x^{\alpha})}\frac{f(x)}{E_{\alpha}(\lambda\, x^{\alpha})}$. So, one can have

$$P(\mathcal{D}_x^{\alpha})y(x) = \frac{f(x)}{E_{\alpha}(\lambda\, x^{\alpha})}$$

So, $E_{\alpha}(\lambda\, x^{\alpha})P(\mathcal{D}_x^{\alpha})y(x) = f(x)$. By using Lemma 3.1, we have

$$P(\mathcal{D}_x^{\alpha}-\lambda)E_{\alpha}(\lambda\, x^{\alpha})y(x) = f(x)$$

So, $E_{\alpha}(\lambda\, x^{\alpha})y(x) = \frac{1}{P(\mathcal{D}_x^{\alpha}-\lambda)}f(x)$. Then $y(x) = \frac{1}{E_{\alpha}(\lambda\, x^{\alpha})}\frac{1}{P(\mathcal{D}_x^{\alpha}-\lambda)}f(x)$
Hence (1) is satisfied . □

To prove (2), let $y(x) = \frac{1}{P(\mathcal{D}_x^{\alpha})}E_{\alpha}(\lambda\, x^{\alpha})f(x)$
So, $y(x) = \frac{E_{\alpha}(\lambda\, x^{\alpha})}{E_{\alpha}(\lambda\, x^{\alpha})}\frac{1}{P(\mathcal{D}_x^{\alpha})}E_{\alpha}(\lambda\, x^{\alpha})f(x)$. According to (1), we have

$$y(x) = E_{\alpha}(\lambda\, x^{\alpha})\frac{1}{P(\mathcal{D}_x^{\alpha}+1)}\frac{E_{\alpha}(\lambda\, x^{\alpha})}{E_{\alpha}(\lambda\, x^{\alpha})}f(x)$$
$$y(x) = E_{\alpha}(\lambda\, x^{\alpha})\frac{1}{P(\mathcal{D}_x^{\alpha}+1)}f(x)$$

Hence (2) is satisfied.

Lemma 4.3 *Let $P(m)$ be a polynomial of degree n and its roots are β_k, $k = 1, 2, \dots, n$, i.e. $P(m) = \prod_{k=1}^{n}(m-\beta_k)$ then*

$$\frac{1}{P(\mathcal{D}_x^{\alpha})}f(x) = \frac{\prod_{k=1}^{n}\left(\mathcal{D}_x^{(q-1)\,\alpha}+\beta_k\mathcal{D}_x^{(q-2)\,\alpha}+\beta_k^2\mathcal{D}_x^{(q-3)\,\alpha}+\cdots+\beta_k^{q-1}\right)}{\prod_{k=1}^{n}\left(D-\beta_k^{q}\right)}f(x) \tag{4.10}$$

where $D = \frac{d}{dx}$ and $q = \frac{1}{\alpha}$ is integer number.

Proof Note that

$$\begin{aligned}
&\prod_{k=1}^{n}(D-\beta_k^{q})\\
&=\prod_{k=1}^{n}(\mathcal{D}_x^{q\,\alpha}-\beta_k^{q})\\
&=\prod_{k=1}^{n}(\mathcal{D}_x^{\alpha}-\beta_k)\left(\mathcal{D}_x^{(q-1)\,\alpha}+\beta_k\mathcal{D}_x^{(q-2)\,\alpha}+\beta_{k^2}\mathcal{D}_x^{(q-3)\,\alpha}+\cdots+\beta q_{k-1}\right)\\
&=\left(\prod_{k=1}^{n}(\mathcal{D}_x^{\alpha}-\beta_k)\right)\left(\prod_{k=1}^{n}\left(\mathcal{D}_x^{(q-1)\,\alpha}+\beta_k\mathcal{D}_x^{(q-2)\,\alpha}+\beta_{k^2}\mathcal{D}_x^{(q-3)\,\alpha}+\cdots+\beta_k^{q-1}\right)\right)\\
&=P(\mathcal{D}_x^{\alpha})\left(\prod_{k=1}^{n}\left(\mathcal{D}_x^{(q-1)\,\alpha}+\beta_k\mathcal{D}_x^{(q-2)\,\alpha}+\beta_k^2\mathcal{D}_x^{(q-3)\,\alpha}+\cdots+\beta_k^{q-1}\right)\right)
\end{aligned}$$

So, one can have

$$\prod_{k=1}^{n}(D-\beta_k^q)f(x)=P(\mathcal{D}_x^{\alpha})\left(\prod_{k=1}^{n}(\mathcal{D}_x^{(q-1)\,\alpha}+\beta_k\mathcal{D}_x^{(q-2)\,\alpha}+\beta_{k^2}\mathcal{D}_x^{(q-3)\,\alpha}+\cdots+\beta_k^{q-1})\right)f(x)$$

This imply Eq. (4.10). □

Remark that Lemma 4.3 is very important specially, when the right member of Eq. (3.2) is e^{ax}, cos(ax), cosh(ax), sin(ax), sinh(ax), x^m or any combination of these functions. In fact, Lemma 4.3 will be used the classical inverse differential operator in order to compute the inverse fractional differential operator. In the next section, we adopt several examples to illustrate the advantage of Method.

Illustrated Examples

Example 1 we consider the nonhomogeneous fractional differential equation

$$\left(\mathcal{D}^{\frac{1}{2}}-2\right)y(x)=e^x \tag{5.1}$$

Clearly, the auxiliary equation is $p(m)=m-2=0$ and its root is $m=2$. Then by using Lemma 4.2, one have

$$\begin{aligned}
y_p(x)&=\frac{\mathcal{D}^{\frac{1}{2}}+2}{D-4}e^x\\
y_p(x)&=\left(\mathcal{D}^{\frac{1}{2}}+2\right)\frac{1}{-3}e^x \quad \text{since} \quad \frac{1}{D-4}e^x=\frac{1}{-3}e^x\\
y_p(x)&=\frac{-1}{3}\mathcal{D}^{\frac{1}{2}}e^x-\frac{2}{3}e^x\\
y_p(x)&=\frac{-1}{3}\mathcal{D}^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+1)}-\frac{2}{3}e^x\\
y_p(x)&=\frac{-1}{3}\sum_{k=1}^{\infty}\frac{x^{k-\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)}-\frac{2}{3}e^x\\
y_p(x)&=\frac{-1}{3}\sum_{j=0}^{\infty}\frac{x^{j+\frac{1}{2}}}{\Gamma\left(j+\frac{3}{2}\right)}-\frac{2}{3}\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+1)}\\
y_p(x)&=\frac{-1}{2}E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)-\frac{1}{6}E_{\frac{1}{2}}\left(-x^{\frac{1}{2}}\right)
\end{aligned} \tag{5.2}$$

It is easily verify that $y_p(x)$ in Eq. (5.2) is particular solution to Eq. (5.1)

Example 2 we consider the homogeneous fractional differential equation

$$\left(\mathcal{D}+\mathcal{D}^{\frac{1}{2}}-2\right)y(x)=\cos(x) \tag{5.3}$$

Clearly, the auxiliary equation is $p(m)=m^2+m-2=0$ and its roots are $m=1,-2$. Then by using Lemma 4.2, one have

$$y_p(x) = \frac{\left(\mathcal{D}^{\frac{1}{2}}+1\right)\left(\mathcal{D}^{\frac{1}{2}}-2\right)}{(D^2-4)(D^2-1)}\cos(x)$$
$$y_p(x) = \frac{\left(\mathcal{D}^{\frac{1}{2}}+1\right)\left(\mathcal{D}^{\frac{1}{2}}-2\right)}{10}\cos(x) \quad \text{since} \quad \frac{1}{(D^2-4)(D^2-1)}\cos(x) = \frac{\cos(x)}{10}$$
$$y_p(x) = \frac{\left(D-\mathcal{D}^{\frac{1}{2}}-2\right)}{10}\cos(x)$$
$$y_p(x) = \frac{-\sin(x)-\mathcal{D}^{\frac{1}{2}}\cos(x)-2\cos(x)}{10}$$
$$y_p(x) = \frac{-\sin(x)-2\cos(x)-\mathcal{D}^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{\Gamma(2k+1)}}{10}$$
$$y_p(x) = \frac{-\sin(x)-2\cos(x)-\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k-\frac{1}{2}}}{\Gamma\left(2k+\frac{1}{2}\right)}}{10} \tag{5.4}$$

It is easily verify that $y_p(x)$ in Eq. (5.4) is particular solution to Eq. (5.3).

Example 3 we consider the homogeneous fractional differential equation

$$\left(\mathcal{D}^{\frac{3}{2}}+2\mathcal{D}^{\frac{1}{2}}-2\right)y(x) = E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right) \tag{5.5}$$

By using Lemma 4.1, one can have

$$y_p(x) = \frac{1}{\left(\mathcal{D}^{\frac{3}{2}}+2\mathcal{D}^{\frac{1}{2}}-2\right)}E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)$$
$$y_p(x) = \frac{1}{(1+2-2)}E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)$$
$$y_p(x) = E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right) \tag{5.6}$$

It is easily verify that $y_p(x)$ in Eq. (5.6) is particular solution to Eq. (5.5).

Example 4 we consider the homogeneous fractional differential equation

$$\left(\mathcal{D}+2\mathcal{D}^{\frac{1}{2}}-3\right)y(x) = E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right) \tag{5.7}$$

By using Lemma 4.1, one can have

$$y_p(x) = \frac{1}{\left(\mathcal{D}+2\mathcal{D}^{\frac{1}{2}}-3\right)}E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)$$
$$y_p(x) = \frac{1}{\left(\mathcal{D}^{\frac{1}{2}}+3\right)\left(\mathcal{D}^{\frac{1}{2}}-1\right)}E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)$$
$$y_p(x) = \frac{1}{4\left(\mathcal{D}^{\frac{1}{2}}-1\right)}E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)$$
$$y_p(x) = \frac{E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)}{4\mathcal{D}^{\frac{1}{2}}} = \frac{E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)\mathcal{D}^{\frac{1}{2}}}{4}\frac{1}{D}$$

$$y_p(x) = \frac{E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)\mathcal{D}^{\frac{1}{2}}x}{4} \quad \text{since} \quad \frac{1}{D} = x$$

$$y_p(x) = \frac{E_{\frac{1}{2}}\left(x^{\frac{1}{2}}\right)\sqrt{x}}{2\sqrt{\pi}} \tag{5.8}$$

It is easily verify that $y_p(x)$ in Eq. (5.8) is particular solution to Eq. (5.7).

Conclusion

Depending on the roots of the characteristic polynomial of the corresponding homogeneous equation, the inverse fractional differential operators method is established to obtain an explicit particular solution to a linear sequential fractional differential equation (LSFDE), involving Jumarie's modification of Riemann–Liouville derivative, with constant coefficients. This method is independent of the integral transforms but it is applicable when, and only when, the right member of the Eq. (1) is e^{ax}, $\cos(ax)$, $\cosh(ax)$, $\sin(ax)$, $\sinh(ax)$, x^a, $E_\alpha(ax)$, $E_b(ax^b)$ or any combination of these functions.

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