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Abstract

This paper presents direct methods for obtaining the explicit general solution to a linear sequential fractional differential equation (LSFDE), involving Jumarie's modification of Riemann–Liouville derivative, with constant coefficients. The general solution to a homogenous LSFDE with constant coefficients is obtained by using the roots of the characteristic polynomial of the corresponding homogeneous equation. For the non-homogeneous case, two methods, undetermined coefficients and variation of parameter, are investigated to find the particular solution. The method of undetermined coefficients is independent of the integral transforms while the method of variation of parameter is not. Moreover, several examples are illustrative for demonstrating the advantage of our approach.
Keywords: Fractional differential equations, Riemann–Liouville derivative, Caputo derivative, undetermined coefficients, variation of parameter.

1. Introduction

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of any arbitrary real or complex order. The History of fractional derivatives were planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many great mathematicians (pure and applied) of their times, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A.K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl (Sabatier et al, 2007). But during this last decades fractional calculus have been applied in widespread fields of science and engineering (Machado et al, 2011).

Fractional differential equations arise in many complex systems in nature and society with many dynamics, such as charge transport in amorphous semiconductors, the spread of contaminants in underground water, relaxation in viscoelastic materials like polymers, the diffusion of pollution in the atmosphere, and many more (Podlubny, 1999; Kilbas et al, 2006). However, the problem of studying fractional differential equations has been dealt with by numerous authors throughout history, particularly in recent years(Mophou,2010; Rajeev and Kushwaha,2013, Khudair 2013, Khudair and Mahdi 2016. Eidelman and Kochubei, 2004; Xue et al, 2008; Guo et al,2012; Molliq et al, 2009). A wide description of the existence and uniqueness of solutions of initial value problem for fractional order differential equations together with its applications can be found in the literature (Samko, et al, 1993; Delbosco, 1996; Podlubny, 1999, Dielhelm,2002).

It is well known that the fractional derivative, in the sense of Riemann_Liouville definition of fractional derivative , of a constant is not zero. This encourage Caputo to introduce Caputo derivative such that the fractional derivative of a constant is zero (Podlubny, 1999; Kilbas et al, 2006). With Caputo definition, a fractional derivative would be defined for differentiable functions only. In order to deal with non-differentiable functions, Jumarie have recently proposed a modification of the Riemann_Liouville definition (Jumarie, 1993, 2006, 2007, 2009, 2010). This fractional derivative provides a Taylor's series of fractional order for non differentiable functions. He, et al, (He, et al, 2012) introduce the geometrical explanation of fractional complex transform and derivative chain rule for fractional calculus in the sense of Jumarie's modification of Riemann–Liouville derivative. Motivated and inspired by the on-going research in this field, we will consider the following non-homogeneous linear fractional differential equation with constant coefficient

$$(\mathcal{D}_x^{n\alpha} + a_1\mathcal{D}_x^{(n-1)\alpha} + a_2\mathcal{D}_x^{(n-2)\alpha} + \cdots + a_{n-1}\mathcal{D}_x^{\alpha} + a_n)y(x) = f(x) \qquad (1)$$

where $\alpha = \frac{1}{q}$ is constant rational number, $a_k,\ k = 1,2,\ldots,n$ are real constant ,

$\mathcal{D}_x^{n\alpha} = \underbrace{D_x^{\alpha}D_x^{\alpha}\cdots D_x^{\alpha}}_{n-\text{times}}$ and D_x^{α} denotes Jumarie's fractional derivation, which is a modified Riemann–Liouville derivative (Samko, et al, 1993; Podlubny, 1999, Kilbas et al, 2006) defined as

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\zeta)^{-\alpha}(f(\zeta) - f(0))d\zeta, \qquad 0 < \alpha < 1 \qquad (2)$$

and

$$D_x^{\alpha} f(x) = \frac{d^n}{dx^n}(D^{(\alpha-n)} f(x)), \qquad n < \alpha < n+1,\ n \geq 1 \qquad (3)$$

Eq.(1) is called fractional linear differential equation with constant coefficients of order (n,q), or more briefly, a fractional differential equation of order (n,q) (Podlubny, 1999). If $\alpha = 1$, then Eq.(1) become n^{th} order ordinary differential equations.

This paper is organised as follows. Sections 2 presents Jumarie's Modification of Riemann–Liouville Derivative and their main properties. In section 3, we develop a direct method for solving the homogeneous LSFDE with constant coefficients, using the roots of the characteristic polynomial and Mittag-Leffler functions. In section 4, the method of undetermined coefficients will be used to find the particular solution to non-homogeneous LSFDE with constant coefficients. In section 5, the method of variation of parameter will be used to find the particular solution to non-homogeneous LSFDE with constant coefficients, while in section 6, several examples are given to illustrative the advantage of our approach.

2. Jumarie's Modification of Riemann–Liouville Derivative

The first definition of fractional derivative which has been proposed in the literature is the so-called Riemann–Liouville definition which reads as follows

Definition 2.1 (Riemann_Liouville) Let $f(x) := R \to R$ be a continuous function then the

fractional
derivative of order α is defined by

$$D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\zeta)^{-\alpha-1} f(\zeta) d\zeta \quad , \quad \alpha < 0 \tag{4}$$

and

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\zeta)^{-\alpha} f(\zeta) d\zeta , \quad 0 < \alpha < 1 \tag{5}$$

and

$$D_x^\alpha f(x) = \frac{d^n}{dx^n} (D^{(\alpha-n)} f(x)) , \quad n < \alpha < n+1, \ n \geq 1 \tag{6}$$

Definition 2.2 (Jumarie's modification of Riemann–Liouville derivative): Let $f(x) := R \to R$ be a continuous function then the fractional derivative of order α is defined by

$$D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\zeta)^{-\alpha-1} (f(\zeta) - f(0)) d\zeta , \alpha < 0 \tag{7}$$

and

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\zeta)^{-\alpha} (f(\zeta) - f(0)) d\zeta , \quad 0 < \alpha < 1 \tag{8}$$

and

$$D_x^\alpha f(x) = \frac{d^n}{dx^n} (D^{(\alpha-n)} f(x)) , \quad n < \alpha < n+1, \ n \geq 1 \tag{9}$$

Remark the main difference between definition (2.1) and definition (2.2). The second one involves the constant $f(0)$ while the first one does not. Also, the fractional Riemann–Liouville derivative of a constant is not zero while the fractional Jumarie derivative of a constant is zero. In the rest of the paper , D_x^α will be used to refer to Jumarie's modification of Riemann–Liouville derivative.

Definition 2.3 (Principle of Derivative increasing orders) : The functional derivative of fractional $D_x^{\alpha+\beta}$ expressed in terms of D_x^{α} and D_x^{β} is definded by the equality

$$D_x^{\alpha+\beta} f(x) = D_x^{\max(\alpha,\beta)}(D_x^{\min(\alpha,\beta)} f(x)).$$

Proposition 2.4: Assume that the continuous function $f(x) := R \to R$ has a fractional derivative of order αk for any positive integer k and $0 < \alpha < 1$, then the following equality holds (Jumarie,2009),

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(\alpha k+1)} f^{(\alpha k)}(x), \quad 0 < \alpha \le 1 \tag{10}$$

where $f^{(\alpha k)}(x)$ is the fractional Jumarie derivative of order αk of $f(x)$. Formally, Eq.(10) can be written $f(x+h) = E_\alpha(h^\alpha D_x^\alpha) f(x), \quad 0 < \alpha \le 1$, where $E_\alpha(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(\alpha k+1)}$.

Corollary 2.5:The following equalities hold (Jumarie,2009), which are

$$D^\alpha x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha)x^{\gamma-\alpha}, \quad \gamma > 0 \tag{11}$$

or, what amounts to the same (we set $\alpha = n+\theta$)

$$D^{n+\theta} x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-n-\theta)x^{\gamma-n-\theta}, \quad 0 < \theta < 1 \tag{12}$$

$$D_x^\alpha(u(x)v(x)) = D_x^\alpha u(x)\, v(x) + u(x)\, D_x^\alpha v(x) \tag{13}$$

$$D_x^\alpha(f(u(x))) = \frac{df(u)}{du} D_x^\alpha u(x) \tag{14}$$

$$D_x^\alpha(f(u(x))) = D_u^\alpha f(u)\, D_x^\alpha\left(\frac{du(x)}{dx}\right) \tag{15}$$

Lemma 2.6: The following various formulae are hold (Jumarie,2009)

1. $\int \frac{d^\alpha x}{x} = \ln_\alpha\left(\frac{x}{c}\right), \quad x = E_\alpha(\ln_\alpha x), \quad xc > 0$ (16)

2. $\ln_\alpha(x^y) = y^\alpha \ln_\alpha x$ (17)

3. $E_\alpha(x^\alpha y^\alpha) = (E_\alpha(y^\alpha))^x$ (18)

4. $(\ln_\alpha(uv))^{\frac{1}{\alpha}} = (\ln_\alpha(u))^{\frac{1}{\alpha}} + (\ln_\alpha(v))^{\frac{1}{\alpha}}$ (19)

5. $E_\alpha(\lambda(x+y)^\alpha) = E_\alpha(\lambda x^\alpha) + E_\alpha(\lambda y^\alpha)$ (20)

6. $D_x^{\alpha} E_{\alpha}(\lambda x^{\alpha}) = \lambda E_{\alpha}(\lambda x^{\alpha})$ (21)

7. $E_{\alpha}(ix) = \cos_{\alpha} x + i \sin_{\alpha} x$ (22)

8. $E_{\alpha}(x) = \cosh_{\alpha} x + \sinh_{\alpha} x$ (23)

9. $D_x^{\alpha} \cos_{\alpha} x^{\alpha} = -\sin_{\alpha} x^{\alpha},\ D_x^{\alpha} \sin_{\alpha} x^{\alpha} = \cos_{\alpha} x^{\alpha}$ (24)

10. $D_x^{\alpha} \cosh_{\alpha} x^{\alpha} = -\sinh_{\alpha} x^{\alpha},\ D_x^{\alpha} \sinh_{\alpha} x^{\alpha} = \cosh_{\alpha} x^{\alpha}$ (25)

11. $D_x^{\alpha} E_{\alpha}(\lambda x) = \lambda \alpha^{-\alpha} x^{1-\alpha} E_{\alpha}(\lambda x)$ (26)

3. General Solution of homogeneous Linear Fractional Differential Equation with constant Coefficients :

Consider the following linear homogeneous linear fractional differential equation with constant coefficients of order (n, q)

$$(\mathcal{D}_x^{n\alpha} + a_1 \mathcal{D}_x^{(n-1)\alpha} + a_2 \mathcal{D}_x^{(n-2)\alpha} + \cdots + a_{n-1} \mathcal{D}_x^{\alpha} + a_n) y(x) = 0 \quad (27)$$

where $\alpha = \dfrac{1}{q}$ is constant rational number, $a_k,\ k = 1, 2, \ldots, n$ are real constant ,

$$\mathcal{D}_x^{n\alpha} = \underbrace{D_x^{\alpha} D_x^{\alpha} \cdots D_x^{\alpha}}_{n-\text{times}}.$$

Rewrite Eq.(28) in the form

$$P(\mathcal{D}_x^{\alpha}) y(x) = 0 \quad (28)$$

where $P(\mathcal{D}_x^{\alpha})$ is a linear fractional differential operator.

Lemma(3.1): $\mathcal{D}^{k\alpha} E_{\alpha}(\lambda x^{\alpha}) = \lambda^k E_{\alpha}(\lambda x^{\alpha}),\quad k = 0, 1, \ldots, n$ where $E_{\alpha}(u) = \sum_{k=0}^{\infty} \dfrac{u^k}{\Gamma(\alpha k + 1)}$ is the Mittag-Leffler function.

Proof:

$$\mathcal{D}^{n\alpha} E_{\alpha}(\lambda x^{\alpha}) = \underbrace{D^{\alpha} D^{\alpha} \cdots D^{\alpha}}_{n-\text{times}} E_{\alpha}(\lambda x^{\alpha})$$

$$= \underbrace{D^{\alpha} D^{\alpha} \cdots D^{\alpha}}_{(n-1)-\text{times}} \lambda E_{\alpha}(\lambda x^{\alpha})$$

$$=\underbrace{D^{\alpha}D^{\alpha}\cdots D^{\alpha}}_{(n-2)-\text{times}}\lambda^{2}E_{\alpha}(\lambda x^{\alpha})$$

$$\vdots$$

$$=\lambda^{n}E_{\alpha}(\lambda x^{\alpha}).$$

By using Lemma (3.1), one can have

$$P(\mathcal{D}_x^{\alpha})E_{\alpha}(\lambda x^{\alpha})=P(\lambda)E_{\alpha}(\lambda x^{\alpha}) \quad (29)$$

where $P(\lambda)=\lambda^{n}+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\cdots+a_n$

If λ is any root of the algebraic equation $P(\lambda)=0$, then Eq.(20) imply

$$P(\mathcal{D}_x^{\alpha})E_{\alpha}(\lambda x^{\alpha})=0$$

which means simply that $y(x)=E_{\alpha}(\lambda x^{\alpha})$ is a solution of Eq.(28). the equation

$$P(\lambda)=0 \quad (30)$$

is called the auxiliary equation associated with Eq.(27) or Eq.(28).

The auxiliary equation for Eq.(27) is of degree n.

Theorem 3.2:

1. if $P(\lambda)=0$ has r real distance roots say $m_1,m_2,\ldots,m_r$, for $1\le r\le n$ then its corresponding solution of Eq.(27) is

$$y(x)=c_1E_{\alpha}(m_1x^{\alpha})+c_2E_{\alpha}(m_2x^{\alpha})+\cdots+c_rE_{\alpha}(m_rx^{\alpha}).$$

2. if $P(\lambda)=0$ has r repeated roots say $m_1=m_2=\ldots=m_r$, for $1\le r\le n$ then its corresponding solution of Eq.(27) is

$$y(x)=c_1E_{\alpha}(m_1x^{\alpha})+c_2x^{\alpha}E_{\alpha}(m_1x^{\alpha})+c_3x^{2\alpha}E_{\alpha}(m_1x^{\alpha})+\cdots+c_rx^{(n-1)\alpha}E_{\alpha}(m_1x^{\alpha}).$$

3. if $P(\lambda)=0$ has complex roots say $m=\gamma\mp\beta=\rho e^{\mp i\theta}$, then its corresponding solution of Eq.(27) is

$$y(x)=c_1\sum_{k=0}^{\infty}\frac{\rho^{k}\cos(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k}+c_2\sum_{k=0}^{\infty}\frac{\rho^{k}\sin(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k}.$$

Proof (1): Let $P(\lambda)=0$ has r real distance roots say $m_1,m_2,\ldots,m_r$, then one can write $P(\lambda)=Q(\lambda)(\lambda-m_1)(\lambda-m_2)\cdots(\lambda-m_r)$ where $Q(\lambda)$ is a polynomial of degree $n-r$ satisfy $Q(m_k)\neq 0$ for $k=1,2,\ldots,r$.

Note that,

$$(\mathcal{D}_x^\alpha-m_k)E_\alpha(m_k x^\alpha)=\mathcal{D}_x^\alpha E_\alpha(m_k x^\alpha)-m_k E_\alpha(m_k x^\alpha)=0 \quad \text{for} \quad k=1,2,\ldots,r$$

so, $P(\mathcal{D}_x^\alpha)E_\alpha(m_k x^\alpha)=0$ for $k=1,2,\ldots,r$.

$$P(\mathcal{D}_x^\alpha)y(x)=P(\mathcal{D}_x^\alpha)(c_1E_\alpha(m_1x^\alpha)+c_2E_\alpha(m_2x^\alpha)+\cdots+c_rE_\alpha(m_rx^\alpha))$$

$$=c_1P(\mathcal{D}_x^\alpha)E_\alpha(m_1x^\alpha)+c_2P(\mathcal{D}_x^\alpha)E_\alpha(m_2x^\alpha)+\cdots+c_rP(\mathcal{D}_x^\alpha)E_\alpha(m_rx^\alpha)$$

$$=0.$$

Proof (2): Let $P(\lambda)=0$ has r repeated roots say $m_1=m_2=\ldots=m_r$, for $1\le r\le n$ then one can write $P(\lambda)=Q(\lambda)(\lambda-m_1)^r$ where $Q(\lambda)$ is a polynomial of degree $n-r$ satisfy $Q(m_1)\neq 0$.

Note that,

$$(\mathcal{D}_x^\alpha-m_1)E_\alpha(m_1x^\alpha)=0$$

$$(\mathcal{D}_x^\alpha-m_1)^2x^\alpha E_\alpha(m_1x^\alpha)=(\mathcal{D}_x^\alpha-m_1)(\mathcal{D}_x^\alpha(x^\alpha E_\alpha(m_1x^\alpha))-m_1x^\alpha E_\alpha(m_1x^\alpha))$$

$$=(\mathcal{D}_x^\alpha-m_1)[x^\alpha m_1E_\alpha(m_1x^\alpha)+\Gamma(\alpha+1)E_\alpha(m_1x^\alpha)-m_1x^\alpha E_\alpha(m_1x^\alpha)]$$

$$=(\mathcal{D}_x^\alpha-m_1)[\Gamma(\alpha+1)E_\alpha(m_1x^\alpha)]$$

$$=\Gamma(\alpha+1)[\mathcal{D}_x^\alpha E_\alpha(m_1x^\alpha)-m_1E_\alpha(m_1x^\alpha)]$$

$$=\Gamma(\alpha+1)[m_1E_\alpha(m_1x^\alpha)-m_1E_\alpha(m_1x^\alpha)]$$

$$=0$$

Also,

$$(\mathcal{D}_x^\alpha-m_1)^3x^{2\alpha}E_\alpha(m_1x^\alpha)=(\mathcal{D}_x^\alpha-m_1)^2(\mathcal{D}_x^\alpha(x^{2\alpha}E_\alpha(m_1x^\alpha))-m_1x^{2\alpha}E_\alpha(m_1x^\alpha))$$

$$=(\mathcal{D}_x^{\alpha}-m_1)^2[x^{2\alpha}m_1E_\alpha(m_1x^\alpha)+\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}x^\alpha E_\alpha(m_1x^\alpha)-m_1x^{2\alpha}E_\alpha(m_1x^\alpha)]$$

$$=(\mathcal{D}_x^{\alpha}-m_1)^2[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}x^\alpha E_\alpha(m_1x^\alpha)]$$

$$=\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(\mathcal{D}_x^{\alpha}-m_1)x^\alpha E_\alpha(m_1x^\alpha)$$

$$=0$$

and so on, one can have,

$$(\mathcal{D}_x^{\alpha}-m_1)^k x^{(k-1)\alpha}E_\alpha(m_1x^\alpha)=(\mathcal{D}_x^{\alpha}-m_1)^{(k-1)}(\mathcal{D}_x^{\alpha}(x^{(k-1)\alpha}E_\alpha(m_1x^\alpha))-m_1x^{(k-1)\alpha}E_\alpha(m_1x^\alpha))$$

$$=(\mathcal{D}_x^{\alpha}-m_1)^{(k-1)}[x^{(k-1)\alpha}m_1E_\alpha(m_1x^\alpha)+\frac{\Gamma((k-1)\alpha+1)}{\Gamma((k-2)\alpha+1)}x^{(k-2)\alpha}E_\alpha(m_1x^\alpha)-m_1x^{(k-1)\alpha}E_\alpha(m_1x^\alpha)]$$

$$=(\mathcal{D}_x^{\alpha}-m_1)^{(k-1)}[\frac{\Gamma((k-1)\alpha+1)}{\Gamma((k-2)\alpha+1)}x^{(k-2)\alpha}E_\alpha(m_1x^\alpha)]$$

$$=\frac{\Gamma((k-1)\alpha+1)}{\Gamma((k-2)\alpha+1)}(\mathcal{D}_x^{\alpha}-m_1)x^{(k-2)\alpha}E_\alpha(m_1x^\alpha)$$

$$=0 \quad \text{for} \quad k=1,2,\ldots,r$$

so, $P(\mathcal{D}_x^{\alpha})x^{(k-1)\alpha}E_\alpha(m_kx^\alpha)=0$ for $k=1,2,\ldots,r$.

$$P(\mathcal{D}_x^{\alpha})y(x)=P(\mathcal{D}_x^{\alpha})(c_1E_\alpha(m_1x^\alpha)+c_2x^\alpha E_\alpha(m_1x^\alpha)+c_3x^{2\alpha}E_\alpha(m_1x^\alpha)+\cdots+c_rx^{(n-1)\alpha}E_\alpha(m_1x^\alpha))$$

$$=c_1P(\mathcal{D}_x^{\alpha})E_\alpha(m_1x^\alpha)+c_2P(\mathcal{D}_x^{\alpha})(x^\alpha E_\alpha(m_1x^\alpha))+c_3P(\mathcal{D}_x^{\alpha})(x^{2\alpha}E_\alpha(m_1x^\alpha))+$$
$$\cdots+c_rP(\mathcal{D}_x^{\alpha})(x^{(n-1)\alpha}E_\alpha(m_1x^\alpha))=0$$

Proof (3): Let $P(\lambda)=0$ has complex roots say $m=\gamma\mp\beta i=\rho e^{\mp i\theta}$, then one can write $P(\lambda)=Q(\lambda)((\lambda-\gamma)^2+\beta^2)=Q(\lambda)(\lambda^2-2\rho\cos(\theta)\lambda+\rho^2)$ where $Q(\lambda)$ is a polynomial of degree $n-2$ satisfy $Q(\gamma\mp\beta i)\neq 0$.

Note that,

$$(\mathcal{D}_x^{2\alpha}-2\rho\cos(\theta)\mathcal{D}_x^{\alpha}+\rho^2)\sum_{k=0}^{\infty}\frac{\rho^k\cos(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k}=\sum_{k=0}^{\infty}\frac{\rho^k\cos(k\theta)}{\Gamma(\alpha k+1)}\mathcal{D}_x^{2\alpha}x^{\alpha k}-$$

$$2\rho\cos(\theta)\sum_{k=0}^{\infty}\frac{\rho^k\cos(k\theta)}{\Gamma(\alpha k+1)}\mathcal{D}_x^{\alpha}x^{\alpha k}+\rho^2\sum_{k=0}^{\infty}\frac{\rho^k\cos(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k}$$

$$=\sum_{k=2}^{\infty}\frac{\rho^{k}\cos(k\theta)}{\Gamma((k-2)\alpha+1)}x^{(k-2)\alpha}-2\rho\cos(\theta)\sum_{k=1}^{\infty}\frac{\rho^{k}\cos(k\theta)}{\Gamma((k-1)\alpha+1)}x^{(k-1)\alpha}+\rho^{2}\sum_{k=0}^{\infty}\frac{\rho^{k}\cos(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k}$$

$$=\sum_{j=0}^{\infty}\frac{\rho^{j+2}\cos((j+2)\theta)}{\Gamma(\alpha j+1)}x^{\alpha j}-2\rho\cos(\theta)\sum_{k=1}^{\infty}\frac{\rho^{j+1}\cos((j+1)\theta)}{\Gamma(\alpha j+1)}x^{\alpha j}+\rho^{2}\sum_{j=0}^{\infty}\frac{\rho^{j}\cos(j\theta)}{\Gamma(\alpha j+1)}x^{\alpha j}$$

$$=\rho^{2}\sum_{j=0}^{\infty}\frac{\rho^{j}[\cos((j+2)\theta)-2\cos(\theta)\cos((j+1)\theta)+\cos(j\theta)]}{\Gamma(\alpha j+1)}x^{\alpha j}$$

$$=\rho^{2}\sum_{j=0}^{\infty}\frac{\rho^{j}[\cos(2\theta)\cos(j\theta)-\sin(j\theta)\sin(2\theta)-2\cos^{2}(\theta)\cos(j\theta)+2\cos(\theta)\sin(j\theta)\sin(\theta)+\cos(j\theta)]}{\Gamma(\alpha j+1)}x^{\alpha j}$$

$$=\rho^{2}\sum_{j=0}^{\infty}\frac{\rho^{j}[\cos(2\theta)-2\cos^{2}(\theta)+1]\cos(j\theta)}{\Gamma(\alpha j+1)}x^{\alpha j}=0.$$

In similar manner, one can have

$$(\mathcal{D}_{x}^{2\alpha}-2\rho\cos(\theta)\mathcal{D}_{x}^{\alpha}+\rho^{2})\sum_{k=0}^{\infty}\frac{\rho^{k}\sin(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k}=0$$

So, $(\mathcal{D}_{x}^{2\alpha}-2\rho\cos(\theta)\mathcal{D}_{x}^{\alpha}+\rho^{2})(c_{1}\sum_{k=0}^{\infty}\frac{\rho^{k}\cos(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k}+c_{2}\sum_{k=0}^{\infty}\frac{\rho^{k}\sin(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k})=0$

and

$$P(\mathcal{D}_{x}^{\alpha})(c_{1}\sum_{k=0}^{\infty}\frac{\rho^{k}\cos(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k}+c_{2}\sum_{k=0}^{\infty}\frac{\rho^{k}\sin(k\theta)}{\Gamma(\alpha k+1)}x^{\alpha k})=0.$$

Theorem: Let $\{y_{1}(x),y_{2}(x),\ldots,y_{n}(x)\}$ be a set of solutions of the linear homogeneous fractional differential equation with constant coefficients of order (n,q) then $y_{1}(x),y_{2}(x),\ldots,y_{n}(x)$ are linearly independent if and only if $|W((y_{1}(x),y_{2}(x),\ldots,y_{n}(x))|\neq 0$, where

$$|W((y_{1}(x),y_{2}(x),\ldots,y_{n}(x))|=\begin{vmatrix} y_{1}(x) & y_{2}(x) & \cdots & y_{n}(x)\\ \mathcal{D}_{x}^{\alpha}y_{1}(x) & \mathcal{D}_{x}^{\alpha}y_{2}(x) & \cdots & \mathcal{D}_{x}^{\alpha}y_{n}(x)\\ \mathcal{D}_{x}^{2\alpha}y_{1}(x) & \mathcal{D}_{x}^{2\alpha}y_{2}(x) & \cdots & \mathcal{D}_{x}^{2\alpha}y_{n}(x)\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{D}_{x}^{(n-1)\alpha}y_{1}(x) & \mathcal{D}_{x}^{(n-1)\alpha}y_{2}(x) & \cdots & \mathcal{D}_{x}^{(n-1)\alpha}y_{n}(x)\end{vmatrix}$$

The above determinant is called α-wronskian determinant.

Proof:

Let $c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0$

By successive α differentiation, we have

$$c_1 \mathcal{D}_x^{\alpha} y_1(x) + c_2 \mathcal{D}_x^{\alpha} y_2(x) + \ldots + c_n \mathcal{D}_x^{\alpha} y_n(x) = 0$$

$$c_1 \mathcal{D}_x^{2\alpha} y_1(x) + c_2 \mathcal{D}_x^{2\alpha} y_2(x) + \ldots + c_n \mathcal{D}_x^{2\alpha} y_n(x) = 0$$

$$c_1 \mathcal{D}_x^{3\alpha} y_1(x) + c_2 \mathcal{D}_x^{3\alpha} y_2(x) + \ldots + c_n \mathcal{D}_x^{3\alpha} y_n(x) = 0$$

$$\vdots$$

$$c_1 \mathcal{D}_x^{(n-1)\alpha} y_1(x) + c_2 \mathcal{D}_x^{(n-1)\alpha} y_2(x) + \ldots + c_n \mathcal{D}_x^{(n-1)\alpha} y_n(x) = 0$$

In order to find the constants, $c_1, c_2, \ldots, c_n$, one can solve the following linear system

$$\begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ \mathcal{D}_x^{\alpha} y_1(x) & \mathcal{D}_x^{\alpha} y_2(x) & \cdots & \mathcal{D}_x^{\alpha} y_n(x) \\ \mathcal{D}_x^{2\alpha} y_1(x) & \mathcal{D}_x^{2\alpha} y_2(x) & \cdots & \mathcal{D}_x^{2\alpha} y_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_x^{(n-1)\alpha} y_1(x) & \mathcal{D}_x^{(n-1)\alpha} y_2(x) & \cdots & \mathcal{D}_x^{(n-1)\alpha} y_n(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (31)$$

The above system has zero solution, $c_1 = c_2 = \ldots = c_n = 0$,if and only if $|W((y_1(x), y_2(x), \ldots, y_n(x))| \neq 0$. That is, $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent if and only if $|W((y_1(x), y_2(x), \ldots, y_n(x))| \neq 0$.

4. General Solution of nonhomogeneous Linear Fractional Differential Equation with constant Coefficients by using undetermined coefficients

The general solution of Eq.(1) is $y(x) = y_c(x) + y_p(x)$, where $y_c(x)$ is the general solution of the homogenous equation Eq.(27) and $y_p(x)$ is any particular solution of the Eq.(1). In this section, the method of undetermined coefficients will be used to find a particular solution of the Eq.(1).

We will summarized the method in the following steps:

1. Write Eq.(1) in form of linear fractional differential operator $P(\mathcal{D}_x^{\alpha}) y(x) = f(x)$.
2. Suppose that the right member $f(x)$ of Eq.(1) is itself a particular solution of some homogeneous linear fractional differential equation with constant coefficients,

$$Q(\mathcal{D}_x^{\alpha})f(x) \tag{32}$$

whose auxiliary equation has the roots $m_1', m_2', \ldots, m_s'$ where s is the degree of the polynomial $Q(\lambda)$.

3. Find the general solution of the following homogeneous linear fractional differential equation with constant coefficients of degree $(n+s, q)$,

$$Q(\mathcal{D}_x^{\alpha})P(\mathcal{D}_x^{\alpha})y(x) = 0, \tag{33}$$

Hence the general solution of Eq.(33) contains the $y_c(x)$ of Eq.(1) and so is of the form $y(x) = y_c(x) + y_q(x)$, But also any particular solution of Eq.(1) must satisfy Eq.(33). Now, if $P(\mathcal{D}_x^{\alpha})(y_c(x) + y_q(x)) = f(x)$, then $P(\mathcal{D}_x^{\alpha})(y_q(x)) = f(x)$ because $P(\mathcal{D}_x^{\alpha})(y_c(x)) = 0$. Then deleting the $y_c(x)$ from the general solution of Eq.(33) leaves a function $y_q(x)$ that for some numerical value of its coefficients must satisfy Eq.(1). The determination of those numerical coefficient may be accomplished as in the following examples.

It must be kept in mind that the undetermined coefficients method is applicable when, and only when, the right member of the equation is $e^{ax}, \cos(ax)$, $\cosh(ax)$, $\sin(ax), \sinh(ax), x^a, E_\alpha(ax), E_b(ax^b)$ or any combination of these functions.

5. General Solution of nonhomogeneous Linear Fractional Differential Equation with constant variation of parameters

The general solution of Eq.(1) is $y(x) = y_c(x) + y_p(x)$, where $y_c(x)$ is the general solution of the homogenous equation Eq.(27) and $y_p(x)$ is any particular solution of the Eq.(1). In this section, the method of variation of parameters will be used to find a particular solution of the Eq.(1).We will summarized the method in the following steps:

1. Find the general solution of the homogenous equation Eq.(27),

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \tag{34}$$

2. Replace each $c_k, k = 1, 2, \ldots, n$ by unknown functions $v_k(x), k = 1, 2, \ldots, n$, so that the particular solution is $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) + \cdots + v_n(x)y_n(x)$.

3. Compute $v_k(x)$, $k = 1, 2, \ldots, n$ from

$$\mathcal{D}_x^{\alpha} v_k(x) = \frac{W_k(x)}{W(x)}, \qquad k = 1, 2, \ldots, n,$$

Or,

$$v_k(x) = \mathcal{D}_x^{-\alpha} \frac{W_k(x)}{W(x)}, \qquad k = 1, 2, \ldots, n$$

$$v_k(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\zeta)^{\alpha-1} \left(\frac{W_k(\zeta)}{W(\zeta)} - \frac{W_k(0)}{W(0)}\right), \qquad k = 1, 2, \ldots, n$$

where $W(x)$ is the α-wronskian determinant and $W_i(x)$ is the α-wronskian determinant with the k^{th} column replaced by $(0, 0, \ldots, f(x))$. So that, the particular solution to the non-homogeneous equation Eq.(1) can be written as

$$y_p(x) = \sum_{k=1}^{n} \left[\frac{1}{\Gamma(\alpha)} \int_0^x (x-\zeta)^{\alpha-1} \left(\frac{W_k(\zeta)}{W(\zeta)} - \frac{W_k(0)}{W(0)}\right)\right] y_k(x),$$

6. Illustrated Examples:

Example 1 : we consider the homogeneous fractional differential equation

$$(\mathcal{D} + \mathcal{D}^{\frac{1}{2}} - 2)y(x) = 0$$

Clearly, the auxiliary equation is $p(m) = m^2 + m - 2 = 0$ and its roots are $m = 1, -2$. then the general solution is seen to be $y(x) = c_1 E_{\frac{1}{2}}(x^{\frac{1}{2}}) + c_2 E_{\frac{1}{2}}(-2x^{\frac{1}{2}})$

Example 2 : we consider the homogeneous fractional differential equation

$(\mathcal{D}^{\frac{4}{3}} - 7\mathcal{D} + 18\mathcal{D}^{\frac{2}{3}} - 20\mathcal{D}^{\frac{1}{3}} + 8)y(x) = 0$, Clearly, the auxiliary equation is $p(m) = m^4 - 7m^3 + 18m^2 - 20m + 8 = 0$ and its roots are $m_1 = 1, 2, 2, 2$. then the general solution is seen to be

$$y(x) = c_1 E_{\frac{1}{3}}(x^{\frac{1}{3}}) + c_2 E_{\frac{1}{3}}(2x^{\frac{1}{3}}) + c_3 x^{\frac{1}{3}} E_{\frac{1}{3}}(2x^{\frac{1}{3}}) + c_4 x^{\frac{2}{3}} E_{\frac{1}{3}}(2x^{\frac{1}{3}}).$$

Example 3 : we consider the homogeneous fractional differential equation

$$(\mathcal{D}^{\frac{3}{2}} - 3\mathcal{D} + 9\mathcal{D}^{\frac{1}{2}} + 13)y(x) = 0$$

Clearly, the auxiliary equation is $p(m)=m^3-3m^2+9m+13=0$ and its roots are

$m=-1,2\mp 3i$. then the general solution is seen to be

$$y(x)=c_1E_{\frac{1}{2}}(-x^{\frac{1}{2}})+c_2\sum_{k=0}^{\infty}\frac{13^{\left(\frac{k}{2}\right)}\cos\left(k\tan^{-1}\left(\frac{3}{2}\right)\right)}{\Gamma(\alpha k+1)}x^{\alpha k}+c_3\sum_{k=0}^{\infty}\frac{13^{\left(\frac{k}{2}\right)}\sin\left(k\tan^{-1}\left(\frac{3}{2}\right)\right)}{\Gamma(\alpha k+1)}x^{\alpha k}$$

Example 4 : we consider the nonhomogeneous fractional differential equation

$$(\mathcal{D}^{\frac{1}{2}}-2)y(x)=e^x$$

Clearly, the auxiliary equation is $p(m)=m-2=0$ and its root is $m=2$. then

$$y_c(x)=c_1E_{\frac{1}{2}}(2x^{\frac{1}{2}})=c_1\left(e^{4t}-\frac{1}{2}\sqrt{4}e^{4t}\left(-1+\operatorname{erfc}\left(2\sqrt{t}\right)\right)\right)$$

The particular solution by using undetermined coefficients, first we find $Q(\mathcal{D}_x^{\frac{1}{2}})$ such that $Q(\mathcal{D}_x^{\frac{1}{2}})e^x=0$,

Clearly, $(\mathcal{D}_x^{\frac{2}{2}}-1)e^x=0$ and the auxiliary equation is $Q(m)=m^2-1=0$

and its root is $m=1,-1$. then $y_q(x)=c_1E_{\frac{1}{2}}(x^{\frac{1}{2}})+c_2E_{\frac{1}{2}}(-x^{\frac{1}{2}})$. One can see that each solution

in $y_q(x)$ not exist in $y_c(x)$ so that the particular solution has the form

$y_p(x)=c_1E_{\frac{1}{2}}(x^{\frac{1}{2}})+c_2E_{\frac{1}{2}}(-x^{\frac{1}{2}})$. Now, substitute $y_p(x)$ in given equation to find the

numerical value for c_1,c_2 , as follows

$$(\mathcal{D}^{\frac{1}{2}}-2)\left(c_1E_{\frac{1}{2}}(x^{\frac{1}{2}})+c_2E_{\frac{1}{2}}(-x^{\frac{1}{2}})\right)=e^x$$

$$c_1\mathcal{D}^{\frac{1}{2}}E_{\frac{1}{2}}(x^{\frac{1}{2}})+c_2\mathcal{D}^{\frac{1}{2}}E_{\frac{1}{2}}(-x^{\frac{1}{2}})-2c_1E_{\frac{1}{2}}(x^{\frac{1}{2}})-2c_2E_{\frac{1}{2}}(-x^{\frac{1}{2}})=e^x$$

$$c_1E_{\frac{1}{2}}(x^{\frac{1}{2}})-c_2E_{\frac{1}{2}}(-x^{\frac{1}{2}})-2c_1E_{\frac{1}{2}}(x^{\frac{1}{2}})-2c_2E_{\frac{1}{2}}(-x^{\frac{1}{2}})=e^x$$

$$c_1E_{\frac{1}{2}}(x^{\frac{1}{2}})+3c_2E_{\frac{1}{2}}(-x^{\frac{1}{2}})=-e^x$$

$$c_1\sum_{k=0}^{\infty}\frac{x^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}+3c_2\sum_{k=0}^{\infty}\frac{(-1)^k x^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}=-\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+1)}$$

$$\sum_{k=0}^{\infty}\frac{[c_1+(-1)^k 3c_2]x^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}=-\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+1)}$$

For k is even in the left member of above equation, we have

$$c_1+3c_2=-1$$

For k is odd in the left member of above equation, we have

$$c_1-3c_2=0$$

So, one can have $c_1=\frac{-1}{2},c_2=\frac{-1}{6}$ and the particular solution is

$y_p(x)=\frac{-1}{2}E_{\frac{1}{2}}(x^{\frac{1}{2}})+\frac{-1}{6}E_{\frac{1}{2}}(-x^{\frac{1}{2}})$. and the general solution is

$$y(x)=c_1E_{\frac{1}{2}}(2x^{\frac{1}{2}})-\frac{1}{2}E_{\frac{1}{2}}(x^{\frac{1}{2}})+\frac{-1}{6}E_{\frac{1}{2}}(-x^{\frac{1}{2}})$$

$$y(x)=c_1e^{4t}\left(1+\operatorname{erf}\left(2\sqrt{t}\right)\right)-\frac{1}{2}e^t\left(1+\operatorname{erf}\left(\sqrt{t}\right)\right)+\frac{1}{6}e^t\left(-1+\operatorname{erf}\left(\sqrt{t}\right)\right)$$

$$y(x)=c_1e^{4t}\left(1+\operatorname{erf}\left(2\sqrt{t}\right)\right)-\frac{2}{3}e^t-\frac{1}{3}e^t\operatorname{erf}\left(\sqrt{t}\right)$$

The particular solution by using variation of parameters,

Replace c_1 by unknown functions $v_1(x)$, so that the particular solution is $y_p(x)=v_1(x)$.

$$\mathcal{D}_x^{\frac{1}{2}}v_1(x)=\frac{W_1(x)}{W(x)}=\frac{e^x}{E_{\frac{1}{2}}(2x^{\frac{1}{2}})}=\frac{1}{1+\operatorname{erf}\left(\sqrt{x}\right)},$$

$$v_1(x)=\frac{-1}{\Gamma(\frac{1}{2})}\int_0^x(x-\zeta)^{\alpha-1}\frac{\operatorname{erf}\left(\sqrt{x}\right)}{1+\operatorname{erf}\left(\sqrt{x}\right)}d\zeta,$$

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$$v_1(x)=\int_0^x -\frac{\operatorname{erf}\left(\sqrt{\zeta}\right)}{\sqrt{\pi(x-\zeta)}\left(1+\operatorname{erf}\left(\sqrt{\zeta}\right)\right)}d\zeta$$

So, the particular solution is

$$y_p(x)=\left(e^{4t}-\frac{1}{2}\sqrt{4}e^{4t}\left(-1+\operatorname{erfc}\left(2\sqrt{t}\right)\right)\right)\int_0^x -\frac{\operatorname{erf}\left(\sqrt{\zeta}\right)}{\sqrt{\pi(x-\zeta)}\left(1+\operatorname{erf}\left(\sqrt{\zeta}\right)\right)}d\zeta,$$

and the general solution is

$$y(x)=\left(e^{4t}-\frac{1}{2}\sqrt{4}e^{4t}\left(-1+\operatorname{erfc}\left(2\sqrt{t}\right)\right)\right)\left(c_1+\int_0^x -\frac{\operatorname{erf}\left(\sqrt{\zeta}\right)}{\sqrt{\pi(x-\zeta)}\left(1+\operatorname{erf}\left(\sqrt{\zeta}\right)\right)}d\zeta\right)$$

Example 5 : we consider the homogeneous fractional differential equation

$$(\mathcal{D}+\mathcal{D}^{\frac{1}{2}}-2)y(x)=\cos(x)$$

Clearly, the auxiliary equation is $p(m)=m^2+m-2=0$ and its roots are $m=1,-2$. then

$$y_c(x)=c_1E_{\frac{1}{2}}(x^{\frac{1}{2}})+c_2E_{\frac{1}{2}}(-2x^{\frac{1}{2}}) \quad .$$

The particular solution by using undetermined coefficients, first we find $Q(\mathcal{D}_x^{\frac{1}{2}})$ such that $Q(\mathcal{D}_x^{\frac{1}{2}})\cos(x)=0$,

Clearly, $(\mathcal{D}_x^{\frac{4}{2}}+1)\cos x=0$ and the auxiliary equation is $Q(m)=m^4+1=0$ and its root is $m=\frac{1}{\sqrt{2}}\mp\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\mp\frac{1}{\sqrt{2}}\equiv e^{\frac{\mp\pi i}{4}},e^{\frac{\mp 3\pi i}{4}}$. then

$$y_q(x)=c_1\sum_{k=0}^{\infty}\frac{\cos(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+c_2\sum_{k=0}^{\infty}\frac{\sin(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+c_3\sum_{k=0}^{\infty}\frac{\cos(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\alpha k}+c_4\sum_{k=0}^{\infty}\frac{\sin(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}$$ One can see that each solution in $y_q(x)$ not exist in $y_c(x)$ so that the particular solution has the form

$$y_p(x)=c_1\sum_{k=0}^{\infty}\frac{\cos(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+c_2\sum_{k=0}^{\infty}\frac{\sin(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+c_3\sum_{k=0}^{\infty}\frac{\cos(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\alpha k}+c_4\sum_{k=0}^{\infty}\frac{\sin(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}$$ Now, substitute $y_p(x)$ in given equation to find the numerical value for c_1,c_2 , as follows

$$(\mathcal{D}+\mathcal{D}^{\frac{1}{2}}-2)\left(c_1\sum_{k=0}^{\infty}\frac{\cos(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+c_2\sum_{k=0}^{\infty}\frac{\sin(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+c_3\sum_{k=0}^{\infty}\frac{\cos(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\alpha k}+c_4\sum_{k=0}^{\infty}\frac{\sin(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}\right)=\cos(x)$$

$$\sum_{k=0}^{\infty}\frac{[c_1\cos(\frac{k\pi}{4})+c_2\sin(\frac{k\pi}{4})+c_3\cos(\frac{3k\pi}{4})+c_4\sin(\frac{3k\pi}{4})]}{\Gamma(\frac{k}{2}+1)}(\mathcal{D}+\mathcal{D}^{\frac{1}{2}}-2)x^{\frac{k}{2}}=\cos(x)$$

$$\sum_{k=0}^{\infty}\frac{[c_1\cos(\frac{k\pi}{4})+c_2\sin(\frac{k\pi}{4})+c_3\cos(\frac{3k\pi}{4})+c_4\sin(\frac{3k\pi}{4})]}{\Gamma(\frac{k}{2}+1)}(\mathcal{D}+\mathcal{D}^{\frac{1}{2}}-2)x^{\frac{k}{2}}=\cos(x)$$

$$\sum_{k=2}^{\infty}\frac{[c_1\cos(\frac{k\pi}{4})+c_2\sin(\frac{k\pi}{4})+c_3\cos(\frac{3k\pi}{4})+c_4\sin(\frac{3k\pi}{4})]}{\Gamma(\frac{k-2}{2}+1)}x^{\frac{k}{2}-1}+$$

$$\sum_{k=1}^{\infty}\frac{[c_1\cos(\frac{k\pi}{4})+c_2\sin(\frac{k\pi}{4})+c_3\cos(\frac{3k\pi}{4})+c_4\sin(\frac{3k\pi}{4})]}{\Gamma(\frac{k-1}{2}+1)}x^{\frac{k-1}{2}}-$$

$$2\sum_{k=0}^{\infty}\frac{[c_1\cos(\frac{k\pi}{4})+c_2\sin(\frac{k\pi}{4})+c_3\cos(\frac{3k\pi}{4})+c_4\sin(\frac{3k\pi}{4})]}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}=\cos(x)$$

$$\sum_{j=0}^{\infty}\frac{[c_1\cos(\frac{(j+2)\pi}{4})+c_2\sin(\frac{(j+2)\pi}{4})+c_3\cos(\frac{3(j+2)\pi}{4})+c_4\sin(\frac{3(j+2)\pi}{4})]}{\Gamma(\frac{j}{2}+1)}x^{\frac{j}{2}}+$$

$$\sum_{j=0}^{\infty}\frac{[c_1\cos(\frac{(j+1)\pi}{4})+c_2\sin(\frac{(j+1)\pi}{4})+c_3\cos(\frac{3(j+1)\pi}{4})+c_4\sin(\frac{3(j+1)\pi}{4})]}{\Gamma(\frac{j}{2}+1)}x^{\frac{j}{2}}-$$

$$2\sum_{j=0}^{\infty}\frac{[c_1\cos(\frac{j\pi}{4})+c_2\sin(\frac{j\pi}{4})+c_3\cos(\frac{3j\pi}{4})+c_4\sin(\frac{3j\pi}{4})]}{\Gamma(\frac{j}{2}+1)}x^{\frac{j}{2}}=\cos(x)$$

$$\sum_{j=0}^{\infty}\frac{[c_1\{\cos(\frac{(j+2)\pi}{4})+\cos(\frac{(j+1)\pi}{4})-2\cos(\frac{j\pi}{4})\}+c_2\{\sin(\frac{(j+2)\pi}{4})+\sin(\frac{(j+1)\pi}{4})-2\sin(\frac{j\pi}{4})\}]}{\Gamma(\frac{j}{2}+1)}x^{\frac{j}{2}}+$$

$$\sum_{j=0}^{\infty}\frac{[c_3\{\cos(\frac{3(j+2)\pi}{4})++\cos(\frac{3(j+1)\pi}{4})-2\cos(\frac{3j\pi}{4})\}\}+c_4\{\sin(\frac{3(j+2)\pi}{4})+\sin(\frac{3(j+1)\pi}{4})-2\sin(\frac{3j\pi}{4})\}]}{\Gamma(\frac{j}{2}+1)}x^{\frac{j}{2}}$$

$$=\sum_{j=0}^{\infty}(-1)^j\frac{x^{2j}}{\Gamma(2j+1)}$$

So, we have the following linear system

$$c_1\left(1-\frac{1}{2}\sqrt{2}\right)+c_2\left(-1-\frac{1}{2}\sqrt{2}\right)+c_3\left(1+\frac{1}{2}\sqrt{2}\right)+c_4\left(1-\frac{1}{2}\sqrt{2}\right)=-1$$

$$c_1\sqrt{2}-c_2-c_3\sqrt{2}+c_4=0$$

$$c_1\left(1+\frac{1}{2}\sqrt{2}\right)+c_2\left(1-\frac{1}{2}\sqrt{2}\right)+c_3\left(1-\frac{1}{2}\sqrt{2}\right)+c_4\left(-1-\frac{1}{2}\sqrt{2}\right)=0$$

$$c_1+c_2\sqrt{2}+c_3+c_4\sqrt{2}=0$$

The solution of the above system is

$$c_1=-\frac{1}{6}+\frac{1}{12}\sqrt{2},c_2=\frac{1}{6}+\frac{1}{12}\sqrt{2},c_3=-\frac{1}{12}\sqrt{2}-\frac{1}{6},c_4=-\frac{1}{6}+\frac{1}{12}\sqrt{2}$$

Therefore , the particular solution is

$$y_p(x)=\left(-\frac{1}{6}+\frac{1}{12}\sqrt{2}\right)\sum_{k=0}^{\infty}\frac{\cos(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+\left(\frac{1}{6}+\frac{1}{12}\sqrt{2}\right)\sum_{k=0}^{\infty}\frac{\sin(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+$$

$$\left(-\frac{1}{12}\sqrt{2}-\frac{1}{6}\right)\sum_{k=0}^{\infty}\frac{\cos(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+\left(-\frac{1}{6}+\frac{1}{12}\sqrt{2}\right)\sum_{k=0}^{\infty}\frac{\sin(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}$$

and the general solution is

$$y(x)=c_1\left(e^x+e^x\left(1-\operatorname{erfc}\left(\sqrt{x}\right)\right)\right)+c_2\left(e^{4x}+\frac{1}{2}\sqrt{4}e^{4x}\left(-1+\operatorname{erfc}\left(2\sqrt{x}\right)\right)\right)$$

$$+\left(-\frac{1}{6}+\frac{1}{12}\sqrt{2}\right)\sum_{k=0}^{\infty}\frac{\cos(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+\left(\frac{1}{6}+\frac{1}{12}\sqrt{2}\right)\sum_{k=0}^{\infty}\frac{\sin(\frac{k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+\left(-\frac{1}{12}\sqrt{2}-\frac{1}{6}\right)\sum_{k=0}^{\infty}\frac{\cos(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}+$$

$$\left(-\frac{1}{6}+\frac{1}{12}\sqrt{2}\right)\sum_{k=0}^{\infty}\frac{\sin(\frac{3k\pi}{4})}{\Gamma(\frac{k}{2}+1)}x^{\frac{k}{2}}$$

The particular solution by using variation of parameters, Replace c_1 and c_2 by unknown functions $v_1(x)$ and $v_2(x)$ respectively , so that the particular solution is

$$y_p(x)=v_1(x)E_{\frac{1}{2}}(x^{\frac{1}{2}})+v_2(x)E_{\frac{1}{2}}(-2x^{\frac{1}{2}}) \ .$$

$$\mathcal{D}_x^{\frac{1}{2}}v_1(x)=\frac{\begin{vmatrix}0 & E_{\frac{1}{2}}(-2x^{\frac{1}{2}})\\ \cos(x) & -2E_{\frac{1}{2}}(-2x^{\frac{1}{2}})\end{vmatrix}}{\begin{vmatrix}E_{\frac{1}{2}}(x^{\frac{1}{2}}) & E_{\frac{1}{2}}(-2x^{\frac{1}{2}})\\ E_{\frac{1}{2}}(x^{\frac{1}{2}}) & -2E_{\frac{1}{2}}(-2x^{\frac{1}{2}})\end{vmatrix}}=\frac{\cos(x)}{3E_{\frac{1}{2}}(x^{\frac{1}{2}})}=\frac{\cos(x)}{3\left(e^x+e^x\left(1-\operatorname{erfc}\left(\sqrt{x}\right)\right)\right)},$$

$$\mathcal{D}_x^{\frac{1}{2}}v_2(x)==\frac{\begin{vmatrix}E_{\frac{1}{2}}(x^{\frac{1}{2}}) & 0\\ E_{\frac{1}{2}}(x^{\frac{1}{2}}) & \cos(x)\end{vmatrix}}{\begin{vmatrix}E_{\frac{1}{2}}(x^{\frac{1}{2}}) & E_{\frac{1}{2}}(-2x^{\frac{1}{2}})\\ E_{\frac{1}{2}}(x^{\frac{1}{2}}) & -2E_{\frac{1}{2}}(-2x^{\frac{1}{2}})\end{vmatrix}}=\frac{\cos(x)}{-3E_{\frac{1}{2}}(-2x^{\frac{1}{2}})}=\frac{-\cos(x)}{3\left(e^{4x}+\frac{1}{2}\sqrt{4}e^{4x}\left(-1+\operatorname{erfc}\left(2\sqrt{x}\right)\right)\right)}$$

$$v_1(x)=\int_0^x\left(-\frac{1}{3}\frac{\cos(\zeta)-2e^{\zeta}+e^{\zeta}\operatorname{erfc}\left(\sqrt{\zeta}\right)}{\sqrt{x-\zeta}\sqrt{\pi}e^{\zeta}\left(-2+\operatorname{erfc}\left(\sqrt{\zeta}\right)\right)}\right)d\zeta$$

$$v_2(x)=\int_0^x\frac{1}{3}\frac{-2\cos(\zeta)+2e^{4\zeta}-e^{4\zeta}\sqrt{4}+e^{4\zeta}\sqrt{4}\operatorname{erfc}\left(2\sqrt{\zeta}\right)}{\sqrt{x-\zeta}\sqrt{\pi}e^{4\zeta}\left(2-\sqrt{4}+\sqrt{4}\operatorname{erfc}\left(2\sqrt{\zeta}\right)\right)}d\zeta$$

So, the particular solution is

$$y_p(x)=\left(e^x+e^x\left(1-\operatorname{erfc}\left(\sqrt{x}\right)\right)\right)\int_0^x\left(-\frac{1}{3}\frac{\cos(\zeta)-2e^{\zeta}+e^{\zeta}\operatorname{erfc}\left(\sqrt{\zeta}\right)}{\sqrt{x-\zeta}\sqrt{\pi}e^{\zeta}\left(-2+\operatorname{erfc}\left(\sqrt{\zeta}\right)\right)}\right)d\zeta+$$

$$\left(e^{4x}+\frac{1}{2}\sqrt{4}e^{4x}\left(-1+\operatorname{erfc}\left(2\sqrt{x}\right)\right)\right)\int_0^x\frac{1}{3}\frac{-2\cos(\zeta)+2e^{4\zeta}-e^{4\zeta}\sqrt{4}+e^{4\zeta}\sqrt{4}\operatorname{erfc}\left(2\sqrt{\zeta}\right)}{\sqrt{x-\zeta}\sqrt{\pi}e^{4\zeta}\left(2-\sqrt{4}+\sqrt{4}\operatorname{erfc}\left(2\sqrt{\zeta}\right)\right)}d\zeta$$

and the general solution is

$$y(x)=\left(e^x+e^x\left(1-\operatorname{erfc}\left(\sqrt{x}\right)\right)\right)\left(c_1+\int_0^x\left(-\frac{1}{3}\frac{\cos(\zeta)-2e^{\zeta}+e^{\zeta}\operatorname{erfc}\left(\sqrt{\zeta}\right)}{\sqrt{x-\zeta}\sqrt{\pi}e^{\zeta}\left(-2+\operatorname{erfc}\left(\sqrt{\zeta}\right)\right)}\right)d\zeta\right)+$$

$$+\left(e^{4x}+\frac{1}{2}\sqrt{4}e^{4x}\left(-1+\operatorname{erfc}\left(2\sqrt{x}\right)\right)\right)\left(c_2+\int_0^x\frac{1}{3}\frac{-2\cos(\zeta)+2e^{4\zeta}-e^{4\zeta}\sqrt{4}+e^{4\zeta}\sqrt{4}\operatorname{erfc}\left(2\sqrt{\zeta}\right)}{\sqrt{x-\zeta}\sqrt{\pi}e^{4\zeta}\left(2-\sqrt{4}+\sqrt{4}\operatorname{erfc}\left(2\sqrt{\zeta}\right)\right)}d\zeta\right)$$

6. Conclusion:

Depending on the roots of the characteristic polynomial of the corresponding homogeneous equation, The general solution to a homogenous LSFDE with constant coefficients is obtained in theorem (3.2). For the non-homogeneous case, two methods, undetermined coefficients and variation of parameter, are investigated to find the particular solution. The method of undetermined coefficients is independent of the integral transforms but it is applicable when, and only when, the right member of the Eq.(1) is e^{ax}, $\cos(ax)$, $\cosh(ax)$, $\sin(ax)$, $\sinh(ax)$, x^a, $E_\alpha(ax)$, $E_b(ax^b)$ or any combination of these functions. while the method of variation of parameter depend on the integral transforms and it is applicable when the right member of the Eq.(1) is any function.

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