Numerical Solution of Singularly Perturbed Differential Difference Equation by Using Cubic B-Spline Method

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Doi 10.29072/basjs.20190209

Abstract

In this paper, a cubic B-spline method is applied to solve singularly perturbed differential difference equations with delay as well as advances whose solution exhibits boundary player behavior. Error analysis of the submitted method was discussed. We tested the method with three numerical examples found the presented method can be applicable and accurate.

Keywords: Singular perturbation problems, a cubic B-spline method, error analysis, exact solution.

1. Introduction

In this paper, we consider the following singularly perturbed differential-difference (SPDDE) [1–8]:

$$\varepsilon y''(x)+a_1(x)y'(x)+a_2(x)y(x-\delta)+a_3(x)y(x)+a_4(x)y(x+\eta)=f(x), \tag{1}$$

$\forall x\in(0,1)$ and subject to the interval and boundary conditions

$$y(x)=\phi(x), \text{ on } -\delta\le x\le 0 \tag{2}$$

$$y(x)=\gamma(x), \text{ on } 1\le x\le 1+\eta \tag{3}$$

Where $a_1(x),a_2(x),a_3(x),a_4(x), f(x),\phi(x)$ and $\gamma(x)$ are bounded and continuously differentiable functions on $(0,1)$, ε is the singular perturbation parameter $(0<\varepsilon<<1)$, δ and η are the delay and the advance parameters respectively $(0<\delta=o(\varepsilon);0<\eta=o(\varepsilon))$.

These equations are widespread in many branches of sciences and engineering and have been used for many years in control theory, description of the so-called human pupil-light reflex and evolutionary biology [9-10]. The arguments for small delay problems are found throughout the literature on epidemics and population where these small shifts play an important role in the modeling of various real life phenomena [11]. There is research dealing with the solution of these equations numerically, for example the mixed finite difference method[19], numerical integration method [10], a domain decomposition method[3], presented a fitted approach[1].

By using Taylor series expansion in the neighborhood of the point x, we have

$$y(x-\delta)\approx y(x)-\delta y'(x), \tag{4}$$

$$y(x+\eta)\approx y(x)+\eta y'(x), \tag{5}$$

by substituting Eqs. (4) and (5) into Eq. (1), we get an asymptotically equivalent singular perturbation problem of the form:

$$\varepsilon y''(x)+p(x)y'(x)+q(x)y(x)=f(x), \tag{6}$$

with boundary conditions

$$y(0)=\phi(0)=\phi_0, \tag{7}$$

$$y(1)=\gamma(1)=\gamma_1, \tag{8}$$

where

$$p(x)=a_1(x)+a_4(x)\eta-a_2(x)\delta, \tag{9}$$

$$q(x)=a_2(x)+a_4(x)+a_3(x). \tag{10}$$

Since $0<\delta<<1$ and $0<\eta<<1$, the transition from Eq. (1) to Eq. (6) is admitted ([1] and [4]) and the solution of Eq. (6) will provide a good approximation to the solution of Eq. (1).

2. Cubic B-spline method

In this section we use the cubic B-spline collocation method to compute the approximate solution of Eqs. (1)–(3). It can be written as a linear combination of cubic B-splines basis functions[2], [12-13].

Consider equally spaced knots of a partition $\Delta_n : a=x_0<x_1<x_2<...<x_n=b$ on $\left[a,b\right]$, with mesh size $h=\dfrac{b-a}{n}$. Let $S_3(\Delta_n)$ be the space of cubic spline functions over the partition Δ_n. The B-splines of degree zero are defined by

$$B_i^0(x)=\begin{cases}1 & if \quad x_i\le x<x_{i+1},\\ 0 & otherwise,\end{cases} \tag{11}$$

and those of degree $k\in Z^+$ are defined recursively in terms of B-splines of degree $k-1$ by

$$B_i^k(x)=\left(\frac{x-x_i}{x_{i+k}-x_i}\right)B_i^{k-1}(x)+\left(\frac{x_{i+k+1}-x}{x_{i+k+1}-x_{i+1}}\right)B_{i+1}^{k-1}(x). \tag{12}$$

For $i=0,\pm1,\pm2,...$ [14-15]. The basis functions B_i^k which defined by (12) are called B-splines of degree k. Applies recurrence relation (12) and assuming the partition Δ_n, the non-uniform B-splines up to degree 3 are given by[16-18] :

$$B_i^3(x)=\begin{cases} \dfrac{(x-x_i)^3}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+1}-x_i)} & if \quad x_i \le x < x_{i+1} \\ \dfrac{(x-x_i)^2(x_{i+2}-x)}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+2}-x_{i+1})}+\dfrac{(x-x_i)(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & \\ +\dfrac{(x_{i+4}-x)(x-x_{i+1})^2}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & if \quad x_{i+1} \le x < x_{i+2} \\ \dfrac{(x-x_i)(x_{i+3}-x)^2}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+1})}+\dfrac{(x_{i+4}-x)(x-x_{i+1})(x_{i+3}-x)}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} & \\ +\dfrac{(x_{i+4}-x)^2(x-x_{i+2})}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+3}-x_{i+2})} & if \quad x_{i+2} \le x < x_{i+3} \\ \dfrac{(x_{i+4}-x)^3}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+4}-x_{i+3})} & if \quad x_{i+3} \le x < x_{i+4} \\ 0 & o.w \end{cases}$$

We apply this recursion to get the cubic B-spline, it is defined as follows:

$$B_i^3(x)=\frac{1}{6h^3}\begin{cases} (x-x_{i-2})^3 & if \quad x_{i-2} \le x < x_{i-1} \\ -3(x-x_{i-1})^3+3h(x-x_{i-1})^2+3h^2(x-x_{i-1})+h^3 & if \quad x_{i-1} \le x < x_i \\ -3(x_{i+1}-x)^3+3h(x_{i+1}-x)^2+3h^2(x_{i+1}-x)+h^3 & if \quad x_i \le x < x_{i+1} \\ (x_{i+2}-x)^3 & if \quad x_{i+1} \le x < x_{i+2} \\ 0 & if \quad otherwise \end{cases} \tag{13}$$

The numerical treatment for solving (1)–(3) using the collocation method with cubic B-spline is to find an approximate solution $Y(x)$ for the exact solution $y(x)$ in the form

$$Y(x)=\sum_{i=-1}^{n+1} c_i B_i(x) \tag{14}$$

where c_i is unknown real coefficient and $B_i(x)$ are cubic B-spline functions which defined in Eq.(13).

It is require that Eq. (14) satisfies our boundary value problem (BVP) (6-8) at $x=x_i$ where x_i is an interior point. That is

$$\varepsilon Y''(x_i)+p(x_i)Y'(x_i)+q(x_i)Y(x_i)=f(x_i), \tag{15}$$

and the boundary conditions are

$$Y(x_0)=\alpha \text{ for } x_0=a,$$
$$Y(x_n)=\beta \text{ for } x_n=b,$$

from Eq. (14), we have

$$\begin{aligned} Y(x_i) &= c_{i-1}B_{i-1}(x_i)+c_iB_i(x_i)+c_{i+1}B_{i+1}(x_i)+c_{i+2}B_{i+2}(x_i), \\ Y'(x_i) &= c_{i-1}B'_{i-1}(x_i)+c_iB'_i(x_i)+c_{i+1}B'_{i+1}(x_i)+c_{i+2}B'_{i+2}(x_i), \\ Y''(x_i) &= c_{i-1}B''_{i-1}(x_i)+c_iB''_i(x_i)+c_{i+1}B''_{i+1}(x_i)+c_{i+2}B''_{i+2}(x_i), \end{aligned} \tag{16}$$

and these yield

$$\begin{aligned} & c_{i-1}[\varepsilon B''_{i-1}(x_i)+p(x_i)B'_{i-1}(x_i)+q(x_i)B_{i-1}(x_i)] \\ & +c_i[\varepsilon B''_i(x_i)+p(x_i)B'_i(x_i)+q(x_i)B_i(x_i)] \\ & +c_{i+1}[\varepsilon B''_{i+1}(x_i)+p(x_i)B'_{i+1}(x_i)+q(x_i)B_{i+1}(x_i)] \\ & +c_{i+2}[\varepsilon B''_{i+2}(x_i)+p(x_i)B'_{i+2}(x_i)+q(x_i)B_{i+2}(x_i)] = f(x_i). \end{aligned}$$

The values of successive derivatives $B_i^{(r)}(x),\ i=-1,0,\ldots,n+1;\ r=0,1,2$ at nodes, are listed in Table 1.

Table 1: Coefficients of cubic B-spline and its derivative at nodes x_i .

	x_{i-1}	x_i	x_{i+1}	else
$B_i(x)$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0
$B_i^{(1)}(x)$	$-\frac{1}{2h}$	0	$\frac{1}{2h}$	0
$B_i^{(2)}(x)$	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

If we combine the values of Table 1 and Eq. (16), we obtain

$$\begin{aligned} & c_{i-1}[6\varepsilon-3p(x_i)h+q(x_i)h^2]+c_i[-12\varepsilon+4q(x_i)h^2] \\ & \qquad +c_{i+1}[6\varepsilon+3p(x_i)h+q(x_i)h^2]=6h^2 f(x_i). \end{aligned} \tag{17}$$

Now we apply the boundary conditions:

$$\begin{aligned} Y(x_0) &= c_{-1}B_{-1}(x_0)+c_0B_0(x_0)+c_1B_1(x_0)+c_2B_2(x_0)=\alpha, \\ Y(x_n) &= c_{n-1}B_{n-1}(x_n)+c_nB_n(x_n)+c_{n+1}B_{n+1}(x_n)+c_{n+2}B_{n+2}(x_n)=\beta, \end{aligned} \tag{18}$$

where the value of $B_i(x)$ at $x=x_0$ and $x=x_n$ are given

$$B_{-1}(x_0)=\frac{1}{6}=B_{n-1}(x_n),$$
$$B_0(x_0)=\frac{4}{6}=B_n(x_n),$$
$$B_1(x_0)=\frac{1}{6}=B_{n+1}(x_n),$$
$$B_2(x_0)=0=B_{n+2}(x_n),$$
(19)

therefore,

$$c_{-1}+4c_0+c_1=6\alpha, \quad (20)$$

$$c_{n-1}+4c_n+c_{n+1}=6\beta, \quad (21)$$

coupling Eqs. (17)- (21) lead to a system of $(n+1)$ linear equations $AY=B$ in the $(n+1)$ unknowns, where

$$Y=[c_0,c_1,...c_{n-1},c_n]^T,$$
$$B=6[w_0,h^2 f(x_1),h^2 f(x_2),...,h^2 f(x_{n-1}),w_n]^T,$$

and the coefficient matrix A given by

$$A=\begin{pmatrix} g_1 & g_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_1 & b_1 & r_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & r_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & b_{n-2} & r_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{n-1} & b_{n-1} & r_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & g_3 & g_4 \end{pmatrix}$$

where a_i, b_i, and g_i are define below

$$a_i=6\varepsilon-3p(x_i)h+q(x_i)h^2,$$
$$b_i=-12\varepsilon+4r(x_i)h^2,$$
$$r_i=6\varepsilon+3p(x_i)h+q(x_i)h^2$$

$$g_1 = b_0 - 4a_0,$$
$$g_2 = r_0 - a_0,$$
$$g_3 = a_n - r_n,$$
$$g_4 = b_n - 4r_n,$$
$$w_0 = h^2 f(x_0) - \alpha a_0,$$
$$w_n = h^2 f(x_n) - \beta r_n,$$

since A is a non-singular matrix, so can solve the system $AY = B$ for $c_0, c_1, \ldots c_{n-1}, c_n$ substituting these values in Eq. (14), to get the required approximate solution.

3. Error Analysis

By substituting the blending function of Table 1 into Eq. (16), we have

$$Y(x_i) = \frac{1}{6}c_{i-1} + \frac{2}{3}c_i + \frac{1}{6}c_{i+1} \approx y(x_i), \tag{22a}$$

$$Y'(x_i) = -\frac{1}{2h}c_{i-1} + \frac{1}{2h}c_{i+1} \approx y'(x_i), \tag{22b}$$

$$Y''(x_i) = \frac{1}{h^2}c_{i-1} - \frac{2}{h^2}c_i + \frac{1}{h^2}c_{i+1} \approx y''(x_i), \tag{22c}$$

then, the following relationships can be obtain:

$$\frac{h}{6}\left[Y'(x_{i-1}) + 4Y'(x_i) + Y'(x_{i+1})\right] = \frac{1}{2}\left[Y(x_{i+1}) - Y(x_{i-1})\right], \tag{23a}$$

$$h^2 Y''(x_i) = 6\left[Y(x_{i+1}) - Y(x_i)\right] - 2h\left[2Y'(x_i) + Y'(x_{i+1})\right], \tag{23b}$$

now, define $E(Y(x_i)) = Y(x_{i+1})$, Eq. (23a) can be written as[6]

$$\frac{h}{6}\left[E^{-1} + 4 + E\right]Y'(x_i) = \frac{1}{2}\left[E - E^{-1}\right]y(x_i). \tag{24}$$

Morever, we have

$$EY(x) = Y(x+h) = \sum_{i=0}^{\infty}\frac{h^i Y^{(i)}(x)}{i!} = \left[\sum_{i=0}^{\infty}\frac{(hD)^i}{i!}\right]Y(x) = e^{hD}Y(x), \text{ where } D = \frac{d}{dx},$$

It implies that $E = e^{hD}$. Similarly, we have

$$E^{-1} = e^{-hD}, E^m = e^{mhD}, E^{-m} = e^{-mhD},$$

can be write in the expansion form of powers hD. Therefore, the above Eq. (24) can be expresses as [14].

$$\left[1 + \frac{1}{3}\left(\frac{(hD)^2}{2!} + \frac{(hD)^4}{4!} + \frac{(hD)^6}{6!} + \ldots\right)\right]Y'(x_i) = \left(D + \frac{h^2D^3}{3!} + \frac{h^4D^5}{5!} + \frac{h^6D^7}{7!} + \ldots\right)y(x_i),$$

and, it can be simplify

$$Y'(x_i)=\frac{\left(D+\frac{h^2D^3}{3!}+\frac{h^4D^5}{5!}+\frac{h^6D^7}{7!}+\ldots\right)}{\left[1+\left(\frac{(hD)^2}{6}+\frac{(hD)^4}{72}+\frac{(hD)^6}{2160}+\ldots\right)\right]}y(x_i),$$

$$Y'(x_i)=\left(D+\frac{h^2D^3}{3!}+\frac{h^4D^5}{5!}+\frac{h^6D^7}{7!}+\ldots\right)$$
$$\left[1-\left(\frac{(hD)^2}{6}+\frac{(hD)^4}{72}+\frac{(hD)^6}{2160}+\ldots\right)+\left(\frac{(hD)^2}{6}+\frac{(hD)^4}{72}+\frac{(hD)^6}{2160}+\ldots\right)^2+\ldots\right]y(x_i)$$
$$=\left(D+\frac{h^2D^3}{3!}+\frac{h^4D^5}{5!}+\frac{h^6D^7}{7!}+\ldots\right)\left(1-\frac{(hD)^2}{6}+\frac{(hD)^4}{72}-\frac{(hD)^6}{2160}-\ldots\right)y(x_i)$$
$$=\left(D-\frac{h^4D^5}{180}+\frac{h^6D^7}{1512}-\ldots\right)y(x_i),$$

hence,

$$Y'(x_i)=y'(x_i)-\left(\frac{1}{180}\right)h^4y^{(5)}(x_i)+O(h^6).$$

By using the same approach for Eq. (23b), we can derive

$$Y''(x_i)=y''(x_i)-\left(\frac{1}{12}\right)h^2y^{(4)}(x_i)+\left(\frac{1}{360}\right)h^4y^{(6)}(x_i)+O(h^6).$$

4. Numerical Examples

The exact solution of singularly perturbed differential-difference equation:[10]

$$\varepsilon y''(x)+a_1(x)y'(x)+a_2(x)y(x-\delta)+a_3(x)y(x)+a_4(x)y(x+\eta)=f(x),$$
$$0<x<1$$

under the boundary conditions

$$y(x)=\phi(x),\quad \text{on}\quad -\gamma\le x\le 0,$$
$$y(x)=\gamma(x),\quad \text{on}\quad 1\le x\le 1+\eta,$$

with constant coefficients is given by

$$y(x)=c_1e^{m_1x}+c_2e^{m_2x}+\frac{f}{c}$$

$$c_1=\frac{[-f+\gamma c_3+e^{m_2}(f-\phi c_3)]}{[(e^{m_1}-e^{m_2})c_3]},$$

$$c_2=\frac{[-f+\gamma c_3+e^{m_1}(f-\phi c_3)]}{[(e^{m_1}-e^{m_2})c_3]},$$

$$m_1 = \frac{[-(a_1 - a_2\delta + a_4\eta) + \sqrt{(a_1 - a_2\delta + a_4\eta)^2 - 4\varepsilon c_3}}{2\varepsilon},$$

$$m_1 = \frac{[-(a_1 - a_2\delta + a_4\eta) - \sqrt{(a_1 - a_2\delta + a_4\eta)^2 - 4\varepsilon c_3}}{2\varepsilon},$$

$c = (a_2 + a_3 + a_4)$,

we now consider three numerical examples to illustrate the comparative performance of our method. All calculations are implemented by Maple. In Examples 1, 2 and 4, we applied the scheme to solve these problems for different values of and compared with exact solution in Figures 1, 2 and 3 respectively.

Moreover, we computed solutions at grid point, the observed maximum absolute errors $L_\infty = \max|y_i - y(x_i)|$) where y_i is numerical solution and $y(x_i)$ is exact solution are tabulated in Tables 2,3 and 4 (for $\varepsilon = 10^{-3}, 10^{-4}$) compared our result with the results given in domain decomposition method [3] and mixed finite difference method [19]**.** This shows that our results are more accurate.

Example 1: Consider the singularly perturbed differential difference equation with left end boundary layer: [3] and [19]

$$\varepsilon y''(x) + y'(x) + 2y(x-\delta) - 3y(x) = 0$$

$$y_i(x) = 1, -\delta \le x \le 0,\ y(x) = 1, 1 \le x \le 1+\eta.$$

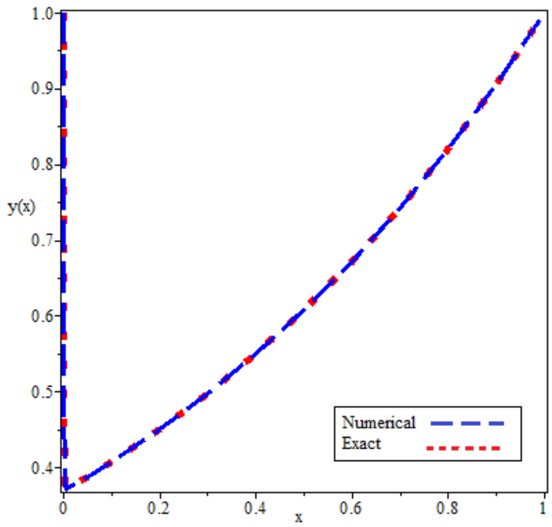

Figure 1: Comparison the exact and numerical solution ($\varepsilon = 10^{-3}$ *and* $\delta = 0.1\varepsilon = \eta$) for Example 1.

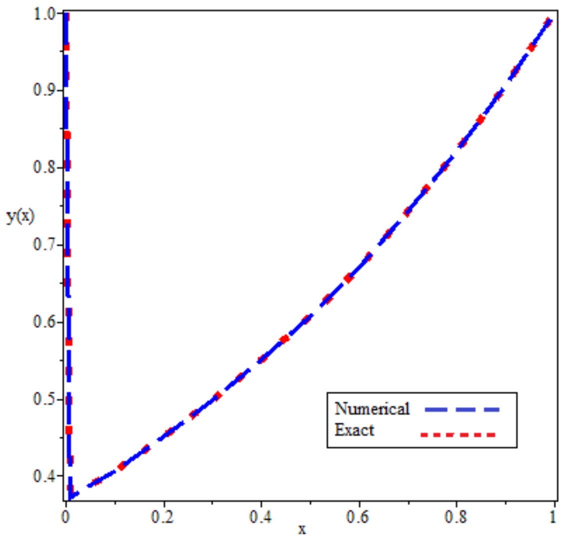

Figure 2: Comparison the exact and numerical solution ($\varepsilon = 10^{-4}$ *and* $\delta = 0.1\varepsilon = \eta$) for Example 1.

Table 2: Comparison of the maximum absolute errors of cubic B- spline method of Example 1 with the maximum absolute errors of [3] and [19].

x	$\varepsilon = 10^{-3},\ \delta = 0.1\varepsilon$			$\varepsilon = 10^{-4},\ \delta = 0.1\varepsilon$		
	Cubic B-Spline	[3]	[19]	Cubic B-Spline	[3]	[19]
0.1	2.30E-8	3.694E-4	3.694E-4	5.2 E-9	5.89661 E-5	1.E-6
0.2	2.28E-8	4.127 E-4	4.127E-4	4.6 E-9	5.7782 E-5	3.E-6
0.4	2.70E-8	5.152 E-4	5.152E-4	4.4 E-9	5.29203E-5	1.E-6
0.6	7.8E -9	6.428 E-4	6.429E-4	1.25 E-8	4.30033 E-5	3.E-6
0.8	3.67E-8	8.017 E-4	8.017E-4	1.215 E-7	2.62473 E-5	6.E-6
0.9	2.27E-8	8.956 E-4	8.956E-3	8.09 E-8	1.44433 E-5	4.E-6

Example 2 : Consider the singularly perturbed differential difference equation with left end boundary layer: :[3]and[19]

$$\varepsilon y''(x) + y'(x) - 3y(x) + 2y(x+\eta) = 0$$

$$y(x) = 1, -\delta \le x \le 0,\ y(x) = 1, 1 \le x \le 1+\eta.$$

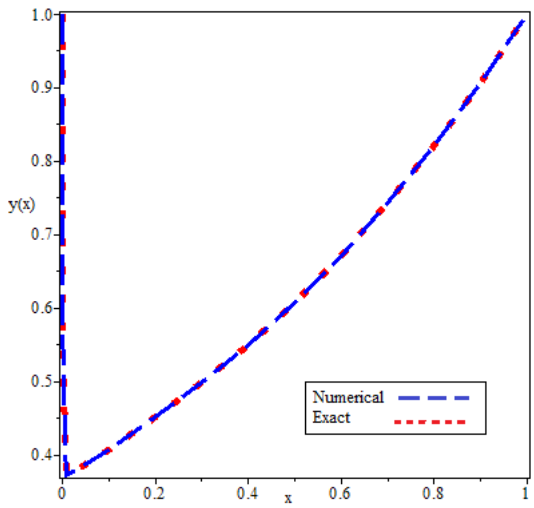

Figure 3: Comparison the exact and numerical solution ($\varepsilon = 10^{-3}$ *and* $\delta = 0.1\varepsilon = \eta$) for Example 2.

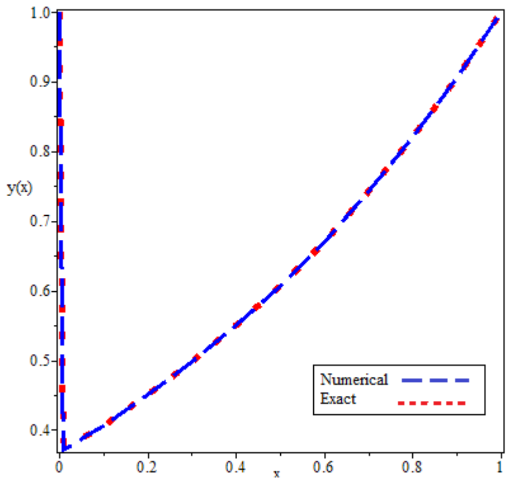

Figure 4: Comparison the exact and numerical solution ($\varepsilon = 10^{-4}$ *and* $\delta = 0.1\varepsilon = \eta$) for Example 2.

Table 3: Comparison the maximum absolute errors of Example 2 with the maximum absolute errors of [3]and[19].

X	$\varepsilon = 10^{-3}$, $\delta = 0.1\varepsilon$			$\varepsilon = 10^{-4}$, $\delta = 0.1\varepsilon$		
	Cubic B-Spline	[3]	[19]	Cubic B-Spline	[3]	[19]
0.1	4.88E-8	4.962 E-4	4.9E-5	9.7E-9	4.962 E-4	0.6E-5
0.2	4.82E-8	4.988 E-4	5.3 E-5	1.04E-8	4.988 E-4	1.9 E-5
0.4	2.76E-8	4.464 E-4	6.2 E-5	7.4E-9	4.464 E-4	1.0 E-5
0.6	4.67E-8	3.614 E-4	0.73E-4	2.20E-8	3.14 E-4	1.7 E-5
0.8	1.263E-7	2.172 E-4	0.85E-4	1.008E-7	2.172 E-4	1.9 E-5
0.9	1.182E-7	1.207 E-4	9.2 E-5	6.57E-8	1.207 E-4	1.5 E-5

Example 3 :Consider the singularly perturbed differential difference equation with left end boundary layer: [3] and [19]

$$\varepsilon y''(x)+y'(x)-2y(x-\delta)-5y(x)+y(x+\eta)=0$$

$$y(x)=1,-\delta\le x\le 0,\ y(x)=1,1\le x\le 1+\eta.$$

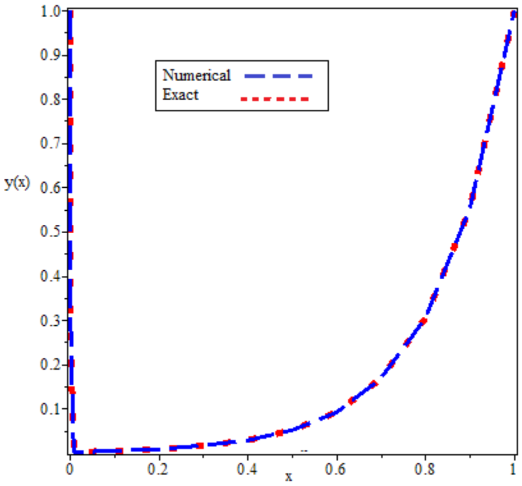

Figure 5: Comparison the exact and numerical solution ($\varepsilon=10^{-3}$ *and* $\delta=0.1\varepsilon=\eta$) for Example 3.

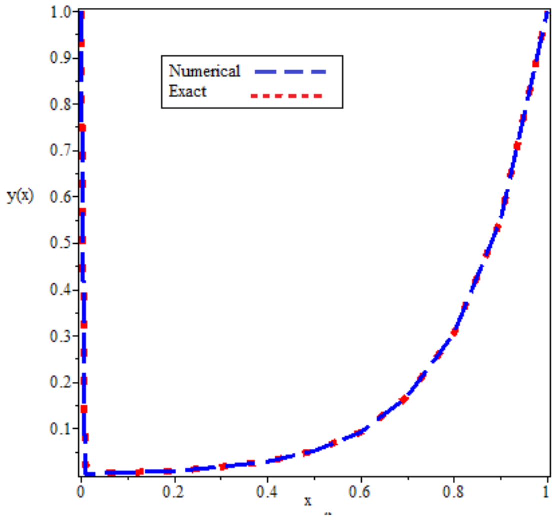

Figure 6: Comparison the exact and numerical solution ($\varepsilon=10^{-4}$ *and* $\delta=0.1\varepsilon=\eta$) for Example 3.

Table 4 :Comparison the maximum absolute errors of Example 3 with the maximum absolute errors of [3] and [19].

x	$\varepsilon=10^{-3},\ \delta=0.1\varepsilon$			$\varepsilon=10^{-4},\ \delta=0.1\varepsilon$		
	Cubic B-Spline	[3]	[19]	Cubic B-Spline	[3]	[19]
0.1	7.4E-11	2.13 E-5	6.8 E-6	1.417E-9	4.3E-6	1.346461 E-6
0.2	1.46E-10	4.27 E-4	1.43 E-5	2.259E-9	6.9 E-6	1.9593 E-6
0.4	4.0E-10	1.351 E-4	2.74E-4	6.68E-9	1.78 E-5	5.71252 E-6
0.6	2.3E-10	4.804 E-4	7.82 E-5	7.10E-9	4.10 E-5	1.63443 E-5
0.8	1.04E-8	1.6990E-2	2.185E-3	2.7E-9	7.69 E-5	4.5098E-4
0.9	2.97E-8	3.0291 E-3	2.006E-3	6.15E-8	2.E-7	7.371 E-5

5. Conclusion

The cubic B-spline method is developed for the approximate solution of singularly perturbed delay differential equations of second order with left and right boundary in this paper. The approximation errors are discussed. Three examples are considered for numerical illustration of the method. Numerical result are presented in Figure (1, 2, 3, 4, 5 and 6) with $\varepsilon=10^{-3},10^{-4}$ and compared with the exact solutions, as for the Tables (2, 3 and 4) we compared the numerical solution with other methods. The numerical results obtained indicate that the proposed method has high accuracy, which makes it very encouraging to deal with solving this type of problems.

6. References

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الحل العددي لمعادلات الفرق التفاضلي المضطرب باستخدام طريقة B-Spline التكعيبية

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المستخلص

في هذا البحث تم تطبيق طريقة B-spline التكعيبية لحل معادلات الفرق التفاضلي المضطرب الفردي من الدرجة الثانية مع الحدود اليسرى واليمنى. تمت مناقشة تحليل الخطأ للطريقة المقدمة. واختبرت الطريقة بثلاث امثلة وضحت بان الطريقة قابلة للتطبيق ودقيقة.

الكلمات المفتاحية: مسائل الاضطراب المفرد, طريقة B-spline المكعبة, تحليل الخطأ, الحل الحقيقي.