Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 2589-2599
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5430

On the $(4\nu, 3)$-arcs in $PG(2, q)$ and the related linear codes

Hanan J. Al_Mayyahi[a], Mohammed A. Alabbood[b,*]

[a]University of Basrah, College of Science, Department of Mathematics
[b]University of Basrah, College of Science, Department of Mathematics

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, we use an irreducible plane-cubic curves in the projective plane $PG(2, q)$ to construct (k,3)-arcs of size 4ν where $\lceil \frac{q+1-2\sqrt{q}}{4} \rceil \leq \nu \leq \lfloor \frac{q+1+2\sqrt{q}}{4} \rfloor$. Each of these arcs gives rise to an error-correcting code that corrects the maximum possible number of errors for its length. Furthermore, we discuss the completeness of each arc. The isotropy subgroup of each arc are determined. All Griesmer codes that correspond to plane-cubic curves are given for $7 \leq q \leq 37, q$ is a prime.

Keywords: Cubic curves, Arcs, Projective plane, codes, stabilizer groups
2010 MSC: 14H52; 51E15, 51E22

1. Introduction

The subject of this paper depends on themes of Projective geometry over a finite field of prime order, Group theory, Field theory, Presentation theory. The strategy to construct the stabilizer groups and also to embedded the arcs is given as following:

A (k, d)-arc $\mathcal{K}$ in the projective plane over Galios field $\mathbb{F}_q$, $\mathbb{F}_q$ being the finite field with q elements, is a set of k elements such that no line in $PG(2, q)$ meets $\mathcal{K}$ in more than d points. The (k, d)-arc is called complete if it is not contained in a $(k + 1, d)$-arc. For the completeness of plane cubic curves over finite fields which are precisely $(k, 3)$-arcs, we can see [3]. For basic facts on arcs the reader is referred to [7] (see also the references therein), [8], [1], and [16].

*Corresponding author
Email addresses: hanan.alhusayn.sci_@gmail.com (Hanan J. Al_Mayyahi), mohna_l@yahoo.com (Mohammed A. Alabbood)

Received: March 2021 *Accepted:* June 2021

A natural example of a (k,d)-arc is the set $\mathcal{X}_d(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of a plane curve $\mathcal{X}_d$ without linear components and defined over $\mathbb{F}_q$, where k equal to the cardinality of $\mathcal{X}_d(\mathbb{F}_q)$ and d is the degree of $\mathcal{X}_d$. As a matter of terminology, we shall say that $\mathcal{X}_d$ has the arc property whenever $\mathcal{X}_d(\mathbb{F}_q)$ is a complete (k,d)-arc with k and d as above. As a matter of fact, the interplay between the theory of algebraic curves and finite geometries was initiated by Segre around 1955. In [13] (see also [7] he established an upper bound for the second largest size that a complete $(k,2)$-arc in $PG(2,q)$ can have.

Problems in combinatorics, especially in finite geometry, often require a count of the number of solutions of an equation in one or more unknowns defined over a finite field $\mathbb{F}_q$. When two unknowns, namely X, Y, occur, the equation is of type $f(X,Y) = 0$ with $f \in \mathbb{F}_q[X,Y]$, and the geometric approach for solving it depends on the theory of algebraic curves over finite fields.

In this paper we are concerning with the problem of determining plane-cubic curves having the arc property. This was asked around 1988 by Hirschfeld and Voloch [9]. Only few examples of such curves are known. Among plane curves we have the irreducibles conics in odd characteristic [7], certain cubics [9], and Hermitian curves [7].

For an (k,d)-arc $\mathcal{K}$ in $PG(2,q)$, we define the isotropy subgroup of $\mathcal{K}$ as follows:

$$G(\mathcal{K}) := \mathrm{PGL}_3(q)_{\mathcal{K}} = \Big\{\gamma \in \mathrm{PGL}_3(q) : \gamma(\mathcal{K}) = \mathcal{K}\Big\},$$

where $\mathrm{PGL}_3(q)$ is the projective general linear group over $\mathbb{F}_q$.

Assume that $\lceil \kappa \rceil$ denotes the smallest integer greater than or equal to κ and that $\lfloor \kappa \rfloor$ denotes the largest integer less than or equal to κ. Then the main results in our work are:

Theorem 1.1. *For a prime $q \geq 7$ and an integer ν where $\lceil \frac{q+1-2\sqrt{q}}{4} \rceil \leq \nu \leq \lfloor \frac{q+1+2\sqrt{q}}{4} \rfloor$; there is $(4\nu,3)$-arc in $PG(2,q)$. It follows that the maximum integer ν such that a plane cubic of size 4ν exists in $PG(2,q)$ is $\lfloor \frac{q+1+2\sqrt{q}}{4} \rfloor$. Moreover, incomplete $(8,3)$-arc exists for all prime $q \geq 7$.*

Corollary 1.2. *For a prime $q \geq 7$ and an integer ν where $\lfloor \frac{q+1-2\sqrt{q}}{4} \rfloor \leq \nu \leq \lfloor \frac{q+1+2\sqrt{q}}{4} \rfloor$; there exists a projective $[4\nu, 3, 4\nu-3]_q$-code of dimension 3. In particular, the codes $[12,3,9]_7$, $[16,3,13]_{11}$, $[20,3,17]_{13}$, $[24,3,21]_{17}$, $[24,3,21]_{19}$, $[28,3,25]_{19}$, $[28,3,25]_{23}$, $[32,3,29]_{23}$, $[36,3,33]_{29}$, $[40,3,37]_{29}$, $[36,3,33]_{31}$, $[40,3,37]_{31}$, $[44,3,41]_{37}$ and $[48,3,45]_{37}$ are Griesmer codes.*

2. Linear codes and plane-cubic arcs

A linear $[k,n,d]_q$-code $\mathcal{C}$ over $q = p^h$ is an n-dimensional subspace of the k-dimensional vector space $V = V(k,q)$ over $\mathbb{F}_q$. The minimum distance d of the code is the smallest number of positions in which two different elements of $\mathcal{C}$ differ. Equivalently, d is the smallest number of non-zero symbols in any non-zero vector of $\mathcal{C}$. From [2], the Singleton bound states that, if $\mathcal{C}$ is an $[k,n,d]_q$-code, then $d \leq k-n+1$. The Singleton defect for linear code $\mathcal{C}$ is defined as $\triangle(C) = k+1-n-d$. Linear codes meeting the singleton bound are Maximum Distance Separable (MDS). For more information about MDS codes, we can see [4]. In general, a linear code having singleton defect equal to 1 is Almost Maximum Distance Separable (AMDS). Furthermore, in the later reference, we have the following theorem:

Theorem 2.1. *There exists a projective $[k,n,d]_q$-code if and only if there exists an $(k,k-d)$-arc in $PG(n-1,q)$.*

A central problem in coding theory is that of optimizing one of the parameters n,k and d for given values of the other two and q-fixed. There are two versions introduced in [6], namely

1. Find $d_q(n,k)$, the largest value of d for which there exists an $[n,k,d]_q$-code.
2. Find $n_q(k,d)$, the smallest value of n for which there exists an $[n,k,d]_q$-code.

A code which achieves one of these two values is called d-optimal or n-optimal respectively. The well-known lower bound for $n_q(k,d)$ is the Griesmer bound [5], [14]

$$n_q(k,d) \geq g_q(k,d) = \sum_{j=0}^{k-1} \lceil \frac{d}{q^j} \rceil$$

Codes with parameters $[g_q(k,d),k,d]_q$, are called Griesmer codes.

Theorem 2.2. *[**Griesmer Bound** [6]] Let C be a linear $[n,k,d]$-code over $GF(q)$. Then we must have that $n_q(k,d) \geq \sum_{j=0}^{k-1} \lceil \frac{d}{q^j} \rceil$.*

In [6], we see that $n_q(k,d) = g_q(k,d)$ for all d when $k=1$ or 2. The problem of finding $n_q(k,d)$ for all d has been solved only in the next cases (See [11], [12]):

- $k \leq 8$ for codes over $GF(2)$,
- $k \leq 5$ for codes over $GF(3)$,
- $k \leq 4$ for codes over $GF(4)$,
- $k = 3$ for codes over $GF(q), 5 \leq q \leq 9$.

Thus, in the case of three-dimensional codes the problem remains open when $q \geq 11$.

Let $F(x_0,x_1,x_2)$ be a form, that is, a homogeneous polynomial in $\mathbb{F}_q[x_0,x_1,x_2]$. The vanishing set of this form,

$$\mathcal{V} = \{(x_0 : x_1 : x_2) \in PG(2,q) : F(x_0,x_1,x_2) = 0\},$$

is a curve in the projective plane $PG(2,q)$. The curve is irreducible if $F(x_0,x_1,x_2)$ does not factor in $\overline{\mathbb{F}}_q[x_0,x_1,x_2]$ where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$.

A point P lying on a curve is a singular point of the curve if there is more than one tangent line to the curve through P, [10]. If no such point exists in $PG(2,q)$, that is, if there is a unique tangent line at each point of the curve considered over $\mathbb{F}_q$, then the curve is a non-singular. This means that, working over the algebraic closure of $\mathbb{F}_q$, it is impossible to find a point P on $\mathcal{V}$ such that the three partial derivatives of F with respect to x_0,x_1,x_2 are all zero at P. If a curve $\mathcal{V}$ in $PG(2,q)$ has a singular point over the algebraic closure of $\mathbb{F}_q$, then the curve $\mathcal{V}$ is singular. Geometrically, the non-singularity of $\mathcal{V}$ means that it has no node or cusp or isolated double point; so there is a unique tangent line to the curve at every point P.

Let $\mathcal{X}$ be a projective, geometrically irreducible, non-singular, algebraic curve defined over $\mathbb{F}_q$, with $q = p^h$ and p a prime. The celebrated Hasse-Weil theorem states that the number $\#(\mathcal{X}_g(\mathbb{F}_q))$ of its rational points has an upper and lower bound:

$$|\#(\mathcal{X}_g(\mathbb{F}_q)) - (q+1)| \leq 2g\sqrt{q}; \tag{2.1}$$

where g is the genous of $\mathcal{X}_g$, see, for example, [15].

It follows, that if $\mathcal{X}$ is a plane curve, which may be singular, of degree n defined over $\mathbb{F}_q$, then the size of the latter set, namely $\mathcal{X}$ satisfies

$$(q+1) - (n-1)(n-2)\sqrt{q} \leq \#(\mathcal{X}(\mathbb{F}_q)) \leq (q+1) + (n-1)(n-2)\sqrt{q}. \tag{2.2}$$

3. Irreducible plane-cubic curves and $(k, 3)$-arcs

In this section, we construct $(k, 3)$-arcs in projective plane $PG(2, q)$ by using an irreducible plane-cubic curve. This method having the following steps: The vanishing set of general plane cubic is given by the following variety:

$$\mathscr{C} = \mathbb{V}(c_1x_0^3 + c_2x_1^3 + c_3x_2^3 + c_4x_0^2x_1 + c_5x_0^2x_2 + c_6x_1^2x_0 + c_7x_1^2x_2 + c_8x_2^2x_0 + c_9x_2^2x_1 + c_{10}x_0x_1x_2).$$

Consider the set of points

$$\mathcal{C}_{0,0} = \{(1:0:0), (0:1:0), (0:0:1)\}$$

and

$$\mathcal{C}_{1,1} = \{(1:1:0), (1:0:1), (0:1:1), (1:1:1)\}.$$

For q is a prime and $q \geq 7$, the points in $\mathcal{C}_{0,0} \cup \mathcal{C}_{1,1}$ forms a $(7, 3)$-arc in $PG(2, q)$ which is a quadrangle with vertices, namely the points

$$(1:0:0), (0:1:0), (0:0:1), (1:1:1),$$

and with the diagonal points, namely the points

$$(1:1:0), (1:0:1), (0:1:1).$$

By substituting these points in the homogeneous equation of $\mathscr{C}$, we get

$$c_1 = c_2 = c_3 = 0,$$

and $c_6 = -c_4$, $c_8 = -c_5$, $c_9 = -c_7$ and $\sum_{j=1}^{10} c_j = 0$. Consequently, we get $c_{10} = 0$.

It follows that the vanishing set of plane cubic passing through $\mathcal{C}_{0,0} \cup \mathcal{C}_{1,1}$ becomes

$$\mathscr{C} = \mathbb{V}(c_4x_0^2x_1 + c_5x_0^2x_2 - c_4x_1^2x_0 + c_7x_1^2x_2 - c_5x_2^2x_0 - c_7x_2^2x_1).$$

It is clear that if $\mathscr{C}$ is irreducible then $c_k \neq 0$ for $k = 4, 5, 7$. So assume that $c_k \neq 0$ for $k = 4, 5, 7$. Then

$$\mathscr{C}_{\lambda,\tau} := \mathscr{C} = \mathbb{V}(x_0^2x_1 + \lambda x_0^2x_2 - x_1^2x_0 + \tau x_1^2x_2 - \lambda x_2^2x_0 - \tau x_2^2x_1), \tag{3.1}$$

where $\lambda = c_5/c_4$ and $\tau = c_7/c_4$. Let us assume $x_2 = 1$, we have the following affine cubic curve of degree 3, namely

$$\widehat{\mathscr{C}}_{\lambda,\tau} := \mathbb{V}(x_0^2x_1 + \lambda x_0^2 - x_1^2x_0 + \tau x_1^2 - \lambda x_0 - \tau x_1).$$

$$\widehat{\mathscr{C}}_{\lambda,\tau} := \mathbb{V}(x_0^2x_1 + \lambda x_0^2 - x_1^2x_0 + \tau x_1^2 - \lambda x_0 - \tau x_1). \tag{3.2}$$

However, the cubic in Equation 3.2 is irreducible if $\lambda \neq 0, -1$, $\tau \neq 0, 1$ and $\tau \neq -\lambda \pmod q$.

Let us consider the plane-cubic, namely

$$\mathscr{C}_{\lambda,\tau} = \mathbb{V}(x_0^2x_1 + \lambda x_0^2x_2 - x_1^2x_0 + \tau x_1^2x_2 - \lambda x_2^2x_0 - \tau x_2^2x_1), \tag{3.3}$$

where $\lambda \in \mathbb{F}_7\backslash\{0, -1\}$,$\tau \in \mathbb{F}_7\backslash\{0, 1\}$ and $\tau \neq -\lambda \pmod 7$. If we assume

$$\mathcal{E}_{\lambda,\tau}(q) = \mathcal{C}_{0,0} \cup \mathcal{C}_{1,1} \cup \{(1 : -\lambda\tau^{-1} : \tau^{-1})\}$$

then we get the Table 1.

Table 1: $(k, 3)$-arcs as the set of points on the plane cubic $\mathscr{C}_{\lambda,\tau}(7)$ over $\mathbb{F}_7$

The plane cubic $\mathscr{C}_{\lambda,\tau}(7)$	Points of $\mathscr{C}_{\lambda,\tau}(7)$ as $(k, 3)$-arc	$\#(\mathscr{C}_{\lambda,\tau}(7))$
$\mathscr{C}_{1,2}(7)$	$\mathcal{E}_{1,2}(7)$	8
$\mathscr{C}_{2,2}(7)$	$\mathcal{E}_{2,2}(7) \cup \{(1:4:6),(1:5:6),(1:4:3),(1:5:3)\}$	12
$\mathscr{C}_{3,2}(7)$	$\mathcal{E}_{3,2}(7) \cup \{(1:4:2),(1:3:5),(1:3:2),(1:4:5)\}\}$	12
$\mathscr{C}_{4,2}(7)$	$\mathcal{E}_{4,2}(7)$	8
$\mathscr{C}_{1,3}(7)$	$\mathcal{E}_{1,3}(7) \cup \{(1:6:2),(1:6:3),(1:5:4),(1:5:6)\}\}$	12
$\mathscr{C}_{2,3}(7)$	$\mathcal{E}_{2,3}(7)$	8
$\mathscr{C}_{3,3}(7)$	$\mathcal{E}_{3,3}(7)$	8
$\mathscr{C}_{5,3}(7)$	$\mathcal{E}_{5,3}(7)$	8
$\mathscr{C}_{1,4}(7)$	$\mathcal{E}_{1,4}(7) \cup \{(1:3:6),(1:2:6),(1:4:5),(1:6:5)\}\}$	12
$\mathscr{C}_{2,4}(7)$	$\mathcal{E}_{2,4}(7)$	8
$\mathscr{C}_{4,4}(7)$	$\mathcal{E}_{4,4}(7)$	8
$\mathscr{C}_{5,4}(7)$	$\mathcal{E}_{5,4}(7) \cup \{(1:5:4),(1:2:4),(1:2:3),(1:5:3)\}\}$	12
$\mathscr{C}_{1,5}(7)$	$\mathcal{E}_{1,5}(7)$	8
$\mathscr{C}_{3,5}(7)$	$\mathcal{E}_{3,5}(7)$	8
$\mathscr{C}_{4,5}(7)$	$\mathcal{E}_{4,5}(7)$	8
$\mathscr{C}_{5,5}(7)$	$\mathcal{E}_{5,5}(7) \cup \{(1:2:6),(1:3:2),(1:2:5),(1:3:4)\}\}$	12
$\mathscr{C}_{2,6}(7)$	$\mathcal{E}_{2,6}(7)$	8
$\mathscr{C}_{3,6}(7)$	$\mathcal{E}_{3,6}(7) \cup \{(1:6:2),(1:2:3),(1:5:2),(1:4:3)\}\}$	12
$\mathscr{C}_{4,6}(7)$	$\mathcal{E}_{4,6}(7) \cup \{(1:3:5),(1:6:4),(1:6:5),(1:3:4)\}\}$	12
$\mathscr{C}_{5,6}(7)$	$\mathcal{E}_{5,6}(7)$	8

For a prime $q \geq 7$ and $5 \leq q \leq 37$, our program give us:

$$\begin{aligned}
\mathcal{E}^{\star}_{2,2}(7) &= \{(1:4:6),(1:5:6),(1:4:3),(1:5:3)\},\\
\mathcal{E}^{\star}_{3,2}(11) &= \{(1:5:7),(1:3:4),(1:3:2),(1:5:3)\},\\
\mathcal{E}^{\star}_{2,2}(13) &= \{(1:6:11),(1:2:3),(1:3:11),(1:12:5)\},\\
\mathcal{E}^{\star}_{6,2}(17) &= \{(1:16:15),(1:3:4),(1:16:4),(1:3:15)\},\\
\mathcal{E}^{\star}_{11,3}(19) &= \{(1:8:12),(1:13:9),(1:13:12),(1:8:9)\},\\
\mathcal{E}^{\star}_{2,2}(11) &= \{(1:9:4),(1:8:4),(1:9:2),(1:6:3),(1:4:7),(1:4:3),\\
&\qquad (1:8:2),(1:6:7)\},\\
\mathcal{E}^{\star}_{1,2}(13) &= \{(1:11:6),(1:7:12),(1:12:4),(1:10:5),(1:7:5),(1:11:4),\\
&\qquad (1:12:6),(1:10:12)\},\\
\mathcal{E}^{\star}_{1,2}(17) &= \{(1:13:11),(1:5:3),(1:5:14),(1:12:3),(1:15:7),(1:16:7),\\
&\qquad (1:12:14),(1:9:11)\},\\
\mathcal{E}^{\star}_{2,2}(19) &= \{(1:14:6),(1:7:3),(1:4:8),(1:4:3),(1:7:8),(1:8:6),\\
&\qquad (1:8:16),(1:14:16)\},\\
\mathcal{E}^{\star}_{3,2}(23) &= \{(1:8:6),(1:7:2),(1:8:13),(1:3:8),(1:11:6),(1:11:13),\\
&\qquad (1:7:8),(1:3:2)\},\\
\mathcal{E}^{\star}_{3,2}(13) &= \{(1:10:3),(1:6:10),(1:8:2),(1:12:10),(1:6:8),(1:8:9),\\
&\qquad (1:12:8),(1:3:2),(1:9:3),(1:2:11),(1:7:11),(1:3:9)\},\\
\mathcal{E}^{\star}_{3,2}(17) &= \{(1:16:11),(1:3:6),(1:10:7),(1:8:10),(1:6:11),(1:4:7),\\
&\qquad (1:8:13),(1:3:2),(1:15:13),(1:15:10),(1:5:2),(1:5:6)\},\\
\mathcal{E}^{\star}_{5,2}(19) &= \{(1:4:11),(1:16:13),(1:12:11),(1:5:12),(1:5:17),(1:12:5),\\
&\qquad (1:9:4),(1:18:13),(1:16:2),(1:9:14),(1:18:2),(1:4:5)\},\\
\mathcal{E}^{\star}_{4,2}(23) &= \{(1:17:10),(1:5:19),(1:19:17),(1:8:10),(1:12:19),(1:19:20),\\
&\qquad (1:17:15),(1:8:15),(1:18:20),(1:12:21),(1:5:21),(1:18:17)\},\\
\mathcal{E}^{\star}_{10,4}(29) &= \{(1:3:4),(1:3:6),(1:16:19),(1:7:4),(1:16:13),(1:10:20),\\
&\qquad (1:8:24),(1:8:21),(1:10:23),(1:4:23),(1:4:20),(1:7:6)\},\\
\mathcal{E}^{\star}_{2,2}(17) &= \{(1:2:8),(1:12:4),(1:3:16),(1:3:12),(1:11:8),(1:13:3),\\
&\qquad (1:4:3),(1:7:15),(1:12:15),(1:8:12),(1:7:4),(1:8:16),\\
&\qquad (1:4:14),(1:2:5),(1:11:5),(1:13:14)\},\\
\mathcal{E}^{\star}_{1,2}(19) &= \{(1:17:5),(1:6:17),(1:18:5),(1:6:12),(1:5:2),(1:4:12),\\
&\qquad (1:11:4),(1:7:14),(1:7:4),(1:11:14),(1:10:13),(1:10:2),\\
&\qquad (1:18:11),(1:4:17),(1:5:13),(1:17:11)\},\\
\mathcal{E}^{\star}_{2,2}(23) &= \{(1:4:5),(1:3:17),(1:19:15),(1:6:10),(1:19:10),(1:6:15),\\
&\qquad (1:15:19),(1:17:4),(1:11:20),(1:4:3),(1:3:20),(1:2:21),\\
&\qquad (1:2:19),(1:15:21),(1:17:7),(1:11:17)\},\\
\mathcal{E}^{\star}_{1,2}(29) &= \{(1:21:7),(1:6:28),(1:25:27),(1:11:7),(1:20:18),(1:6:20),\\
&\qquad (1:12:14),(1:11:25),(1:21:25),(1:10:8),(1:13:28),(1:25:18),\\
&\qquad (1:13:20),(1:12:8),(1:10:14),(1:20:27)\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}^{\star}_{1,2}(31) = \{&(1:23:30),(1:6:15),(1:23:11),(1:2:24),(1:6:24),(1:5:12),\\
&(1:5:18),(1:25:18),(1:3:25),(1:25:12),(1:18:30),(1:13:25),\\
&(1:13:22),(1:3:22),(1:18:11),(1:2:15)\},\\
\mathcal{E}^{\star}_{4,2}(19) = \{&(1:14:12),(1:8:4),(1:10:4),(1:3:7),(1:2:3),(1:3:18),\\
&(1:2:8),(1:9:3),(1:18:6),(1:12:9),(1:15:12),(1:8:14),\\
&(1:14:17),(1:12:15),(1:9:8),(1:18:16),(1:4:6),(1:10:14),\\
&(1:4:16),(1:15:17)\},\\
\mathcal{E}^{\star}_{5,2}(23) = \{&(1:13:19),(1:11:2),(1:8:17),(1:22:8),(1:3:14),(1:11:8),\\
&(1:14:6),(1:20:9),(1:6:20),(1:4:19),(1:4:21),(1:3:9),\\
&(1:5:6),(1:13:21),(1:14:13),(1:20:14),(1:5:13),(1:6:17),\\
&(1:8:20),(1:22:2)\},\\
\mathcal{E}^{\star}_{8,2}(29) = \{&(1:28:17),(1:22:17),(1:28:4),(1:21:2),(1:15:16),(1:11:9),\\
&(1:21:10),(1:20:10),(1:16:21),(1:22:4),(1:13:6),(1:13:11),\\
&(1:26:21),(1:15:22),(1:4:11),(1:23:16),(1:7:9),(1:4:6),\\
&(1:23:22),(1:20:2)\},\\
\mathcal{E}^{\star}_{2,2}(31) = \{&(1:17:21),(1:18:3),(1:9:5),(1:4:3),(1:4:19),(1:6:2),\\
&(1:12:26),(1:6:21),(1:24:5),(1:18:19),(1:12:9),(1:24:28),\\
&(1:23:26),(1:23:9),(1:20:8),(1:9:28),(1:5:8),(1:17:2),\\
&(1:20:17),(1:5:17)\},\\
\mathcal{E}^{\star}_{6,2}(37) = \{&(1:31:34),(1:32:29),(1:35:22),(1:28:29),(1:31:27),(1:8:3),\\
&(1:20:16),(1:30:34),(1:12:16),(1:9:22),(1:11:24),(1:32:31),\\
&(1:30:27),(1:11:6),(1:19:6),(1:25:3),(1:19:24),(1:25:30),\\
&(1:28:31),(1:8:30)\},\\
\mathcal{E}^{\star}_{1,2}(23) = \{&(1:9:4),(1:3:18),(1:17:3),(1:15:13),(1:7:15),(1:20:2),\\
&(1:9:7),(1:13:8),(1:20:8),(1:15:6),(1:2:4),(1:18:10),\\
&(1:4:6),(1:3:11),(1:2:7),(1:18:15),(1:14:3),(1:4:13),\\
&(1:14:5),(1:17:5),(1:19:22),(1:13:2),(1:7:10),(1:19:16)\},\\
\mathcal{E}^{\star}_{2,2}(29) = \{&(1:21:23),(1:11:3),(1:9:19),(1:3:6),(1:14:6),(1:16:13),\\
&(1:19:4),(1:2:17),(1:5:19),(1:9:24),(1:7:23),(1:14:11),\\
&(1:3:11),(1:4:3),(1:4:12),(1:5:24),(1:23:13),(1:11:12),\\
&(1:19:17),(1:7:5),(1:21:5),(1:2:4),(1:16:26),(1:23:26)\},\\
\mathcal{E}^{\star}_{3,2}(31) = \{&(1:8:28),(1:5:25),(1:20:21),(1:15:8),(1:13:27),(1:11:22),\\
&(1:20:2),(1:21:18),(1:21:12),(1:10:17),(1:15:17),(1:5:22),\\
&(1:25:5),(1:25:28),(1:8:5),(1:3:21),(1:11:25),(1:9:12),\\
&(1:3:2),(1:9:18),(1:13:4),(1:18:4),(1:10:8),(1:18:27)\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}^{\star}_{1,2}(37) = \{&(1:9:6),(1:20:35),(1:2:34),(1:4:7),(1:22:27),(1:34:28),\\
&(1:12:9),(1:2:27),(1:14:20),(1:9:24),(1:23:35),(1:33:25),\\
&(1:20:8),(1:31:25),(1:31:2),(1:21:24),(1:12:7),(1:33:2),\\
&(1:22:34),(1:34:20),(1:14:28),(1:23:8),(1:21:6),(1:4:9)\},\\
\mathcal{E}^{\star}_{3,2}(29) = \{&(1:27:9),(1:18:26),(1:12:19),(1:3:10),(1:9:10),(1:18:13),\\
&(1:24:16),(1:26:8),(1:14:16),(1:2:24),(1:9:2),(1:20:9),\\
&(1:12:24),(1:15:20),(1:4:28),(1:15:28),(1:21:26),(1:25:14),\\
&(1:3:2),(1:24:22),(1:21:13),(1:5:21),(1:14:22),(1:8:21),\\
&(1:4:20),(1:26:14),(1:25:8),(1:2:19)\},\\
\mathcal{E}^{\star}_{5,2}(31) = \{&(1:28:24),(1:18:17),(1:30:2),(1:27:6),(1:30:21),(1:28:15),\\
&(1:7:17),(1:4:30),(1:18:8),(1:22:29),(1:23:28),(1:22:13),\\
&(1:20:29),(1:2:6),(1:6:11),(1:6:30),(1:24:2),(1:7:8),\\
&(1:20:13),(1:10:5),(1:11:24),(1:11:15),(1:10:28),(1:27:23),\\
&(1:23:5),(1:2:23),(1:24:21),(1:4:11)\},\\
\mathcal{E}^{\star}_{2,2}(37) = \{&(1:11:26),(1:29:3),(1:13:18),(1:30:26),(1:34:16),(1:15:18),\\
&(1:4:30),(1:14:29),(1:30:15),(1:16:21),(1:4,3),(1:15:10),\\
&(1:9:29),(1:17:36),(1:10:21),(1:14:31),(1:29:30),(1:32:13),\\
&(1:35:16),(1:10:5),(1:32:36),(1:19:22),(1:9:31),(1:17:13),\\
&(1:16:5),(1:11:15),(1:13:10),(1:25:22)\},\\
\mathcal{E}^{\star}_{9,2}(29) = \{&(1:17:3),(1:17:12),(1:18:7),(1:26:22),(1:27:3),(1:15:24),\\
&(1:19:13),(1:24:5),(1:16:2),(1:14:7),(1:9:27),(1:14:25),\\
&(1:16:10),(1:18:25),(1:25:10),(1:20:13),(1:24:23),(1:15:19),\\
&(1:26:16),(1:12:16),(1:4:5),(1:28:19),(1:7:27),(1:4:23),\\
&(1:9:18),(1:20:26),(1:12:22),(1:19:26),(1:28:24),(1:27:12),\\
&(1:7:18),(1:25:2)\},\\
\mathcal{E}^{\star}_{6,2}(31) = \{&(1:21:20),(1:4:25),(1:6:17),(1:20:27),(1:21:14),(1:7:12),\\
&(1:4:22),(1:15:21),(1:30:28),(1:19:17),(1:3:28),(1:12:22),\\
&(1:13:14),(1:8:21),(1:19:8),(1:26:10),(1:22:10),(1:26:7),\\
&(1:3:5),(1:8:2),(1:13:20),(1:7:18),(1:6:8),(1:30:5),\\
&(1:15:2),(1:23:18),(1:23:12),(1:12:25),(1:22:7),(1:11:27),\\
&(1:20:4),(1:11:4)\},\\
\mathcal{E}^{\star}_{4,2}(37) = \{&(1:26:20),(1:13:26),(1:13:15),(1:32:18),(1:34:21),(1:27:11),\\
&(1:22:28),(1:25:4),(1:33:10),(1:31:14),(1:29:5),(1:25:11),\\
&(1:30:13),(1:15:32),(1:31:32),(1:10:2),(1:28:26),(1:19:13),\\
&(1:29:21),(1:22:20),(1:34:5),(1:28:15),(1:17:2),(1:30:36),\\
&(1:33:18),(1:15:14),(1:10:25),(1:32:10),(1:17:25),(1:26:28),\\
&(1:19:36),(1:27:4)\},
\end{aligned}$$

$$\begin{aligned}\mathcal{E}^{\star}_{3,2}(37) = \{&(1:35:21),(1:19:34),(1:33:6),(1:5:27),(1:30:20),(1:26:10),\\&(1:2:10),(1:28:21),(1:22:13),(1:33:24),(1:27:13),(1:15:22),\\&(1:2:18),(1:11:16),(1:5:34),(1:31:26),(1:29:22),(1:28:5),\\&(1:22:36),(1:19:27),(1:30:28),(1:35:5),(1:10:15),(1:26:18),\\&(1:18:28),(1:18:20),(1:27:36),(1:24:2),(1:24:25),(1:31:15),\\&(1:34:24),(1:3:2),(1:21:16),(1:34:6),(1:3:25),(1:10:26)\},\\ \mathcal{E}^{\star}_{5,2}(37) = \{&(1:36:2),(1:19:15),(1:29:12),(1:22:15),(1:27:20),(1:35:33),\\&(1:10:29),(1:31:5),(1:6:12),(1:27:28),(1:36:25),(1:35:17),\\&(1:6:23),(1:13:31),(1:28:2),(1:15:33),(1:18:35),(1:22:26),\\&(1:29:23),(1:9:30),(1:5:9),(1:24:3),(1:32:5),(1:5:7),\\&(1:15:17),(1:25:8),(1:11:9),(1:21:28),(1:25:35),(1:11:7),\\&(1:24:30),(1:28:25),(1:10:31),(1:18:8),(1:9:3),(1:32:21),\\&(1:19:26),(1:21:20),(1:13:29),(1:31:21)\}.\end{aligned}$$

From the above results, we construct the Table 2 that illustrates the possible size of the plane cubic $\mathscr{C}_{\lambda,\tau}(q)$ over $\mathbb{F}_q$. Furthermore, in Table 2, we determined which of these plane cubics are complete as $(k,3)$-arcs. Also, all the correspond isotropy subgroups of $(k,3)$-arcs are given in Table 2.

3.1. Proof Theorem 1.1

For a prime $q \geq 7$, the Hasse-Weil theorem states that the number $\#(\mathcal{X}_1(\mathbb{F}_q))$ of its rational points has an upper and lower bound:

$$|\#(\mathcal{X}_1(\mathbb{F}_q)) - (q+1)| \leq 2\sqrt{q};$$

where $g = 1$ is the genous of $\mathcal{X}_1$.

Our computer programs, states that the size of the plane cubic is 4ν (See Table 2). It follows that the integer ν satisfies $\lceil \frac{q+1-2\sqrt{q}}{4} \rceil \leq \nu \leq \lfloor \frac{q+1+2\sqrt{q}}{4} \rfloor$. Consequently, there is $(4\nu,3)$-arc in $PG(2,q)$ for all values of ν in the above range. It follows that the maximal value of ν such that the plane cubic of size 4ν exists is $\nu = \lfloor \frac{q+1+2\sqrt{q}}{4} \rfloor$.

Recall that $\mathcal{E}_{\lambda,\tau}(q)$ is the set of all points in $\mathcal{C}_{0,0} \cup \mathcal{C}_{1,1} \cup \{(1:-\lambda\tau^{-1}:\tau^{-1})\}$ where

$$\mathcal{C}_{0,0} = \{(1:0:0),(0:1:0),(0:0:1)\}$$

and

$$\mathcal{C}_{1,1} = \{(1:1:0),(1:0:1),(0:1:1),(1:1:1)\}.$$

Note that these points form the projective plane $PG(2,2)$ which is in turn a $(7,3)$-arc. Furthermore, if we assume $\lambda \in \mathbb{F}_q \backslash \{0,-1\}$,$\tau \in \mathbb{F}_q \backslash \{0,1\}$ and $\tau \neq -\lambda \,(\mathrm{mod}\, q)$. Then for odd prime $q \geq 7$, the set $\mathcal{E}_{\lambda,\tau}(q)$ forms incomplete $(8,3)$-arc on the plane cubic $\mathscr{C}_{\lambda,\tau}(q)$.

Table 2: Size of points on the plane cubic $\mathscr{C}_{\lambda,\tau}(q)$ over $\mathbb{F}_q$, where q is a prime and $7 \leq q \leq 37$

$\mathscr{C}_{\lambda,\tau}(q)$	$\#(\mathscr{C}_{\lambda,\tau}(q))$	complete/incomplete	isotropy subgroup
$\mathscr{C}_{1,2}(5) = \mathcal{E}_{1,2}(5)$	8	incomplete	$\mathbb{Z}/4\mathbb{Z}$
$\mathscr{C}_{1,2}(7) = \mathcal{E}_{1,2}(7)$	8	incomplete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{2,2}(7) = \mathcal{E}_{2,2}(7) \cup \mathcal{E}^{\star}_{2,2}(7)$	12	complete	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{1,2}(11) = \mathcal{E}_{1,2}(11)$	8	incomplete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{2,2}(11) = \mathcal{E}_{2,2}(11) \cup \mathcal{E}^{\star}_{2,2}(11)$	16	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{3,2}(11) = \mathcal{E}_{3,2}(11) \cup \mathcal{E}^{\star}_{3,2}(11)$	12	incomplete	$\mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{1,2}(13) = \mathcal{E}_{1,2}(13) \cup \mathcal{E}^{\star}_{1,2}(13)$	16	incomplete	$\mathbb{Z}/6\mathbb{Z}$
$\mathscr{C}_{2,2}(13) = \mathcal{E}_{2,2}(13) \cup \mathcal{E}^{\star}_{2,2}(13)$	12	incomplete	$\mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{3,2}(13) = \mathcal{E}_{3,2}(13) \cup \mathcal{E}^{\star}_{3,2}(13)$	20	complete	$\mathbb{Z}/4\mathbb{Z}$
$\mathscr{C}_{2,3}(13) = \mathcal{E}_{2,3}(13)$	8	incomplete	$\mathbb{Z}/4\mathbb{Z}$
$\mathscr{C}_{1,2}(17) = \mathcal{E}_{1,2}(17) \cup \mathcal{E}^{\star}_{1,2}(17)$	16	incomplete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{2,2}(17) = \mathcal{E}_{2,2}(17) \cup \mathcal{E}^{\star}_{2,2}(17)$	24	complete	$\mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{3,2}(17) = \mathcal{E}_{3,2}(17) \cup \mathcal{E}^{\star}_{3,2}(17)$	20	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{6,2}(17) = \mathcal{E}_{6,2}(17) \cup \mathcal{E}^{\star}_{6,2}(17)$	12	incomplete	$\mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{1,2}(19) = \mathcal{E}_{1,2}(19) \cup \mathcal{E}^{\star}_{1,2}(19)$	24	complete	$\mathfrak{S}_3$
$\mathscr{C}_{2,2}(19) = \mathcal{E}_{2,2}(19) \cup \mathcal{E}^{\star}_{2,2}(19)$	16	incomplete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{4,2}(19) = \mathcal{E}_{4,2}(19) \cup \mathcal{E}^{\star}_{4,2}(19)$	28	complete	$\mathbb{Z}/6\mathbb{Z}$
$\mathscr{C}_{5,2}(19) = \mathcal{E}_{5,2}(19) \cup \mathcal{E}^{\star}_{5,2}(19)$	20	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{11,3}(19) = \mathcal{E}_{11,3}(19) \cup \mathcal{E}^{\star}_{11,3}(19)$	12	incomplete	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{1,2}(23) = \mathcal{E}_{1,2}(23) \cup \mathcal{E}^{\star}_{1,2}(23)$	32	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{2,2}(23) = \mathcal{E}_{2,2}(23) \cup \mathcal{E}^{\star}_{2,2}(23)$	24	complete	$\mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{3,2}(23) = \mathcal{E}_{3,2}(23) \cup \mathcal{E}^{\star}_{3,2}(23)$	16	incomplete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{4,2}(23) = \mathcal{E}_{4,2}(23) \cup \mathcal{E}^{\star}_{4,2}(23)$	20	incomplete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{5,2}(23) = \mathcal{E}_{5,2}(23) \cup \mathcal{E}^{\star}_{5,2}(23)$	28	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{1,2}(29) = \mathcal{E}_{1,2}(29) \cup \mathcal{E}^{\star}_{1,2}(29)$	24	complete	$\mathfrak{S}_3$
$\mathscr{C}_{2,2}(29) = \mathcal{E}_{2,2}(29) \cup \mathcal{E}^{\star}_{2,2}(29)$	32	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{3,2}(29) = \mathcal{E}_{3,2}(29) \cup \mathcal{E}^{\star}_{3,2}(29)$	36	complete	$\mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{8,2}(29) = \mathcal{E}_{8,2}(29) \cup \mathcal{E}^{\star}_{8,2}(29)$	28	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{9,2}(29) = \mathcal{E}_{9,2}(29) \cup \mathcal{E}^{\star}_{9,2}(29)$	40	complete	$\mathbb{Z}/4\mathbb{Z}$
$\mathscr{C}_{10,4}(29) = \mathcal{E}_{10,4}(29) \cup \mathcal{E}^{\star}_{10,4}(29)$	20	incomplete	$\mathbb{Z}/4\mathbb{Z}$
$\mathscr{C}_{1,2}(31) = \mathcal{E}_{1,2}(31) \cup \mathcal{E}^{\star}_{1,2}(31)$	24	complete	$\mathfrak{S}_3$
$\mathscr{C}_{2,2}(31) = \mathcal{E}_{2,2}(31) \cup \mathcal{E}^{\star}_{2,2}(31)$	28	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{3,2}(31) = \mathcal{E}_{3,2}(31) \cup \mathcal{E}^{\star}_{3,2}(31)$	32	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{5,2}(31) = \mathcal{E}_{5,2}(31) \cup \mathcal{E}^{\star}_{5,2}(31)$	36	complete	$\mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{6,2}(31) = \mathcal{E}_{6,2}(31) \cup \mathcal{E}^{\star}_{6,2}(31)$	40	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{1,2}(37) = \mathcal{E}_{1,2}(37) \cup \mathcal{E}^{\star}_{1,2}(37)$	32	incomplete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{2,2}(37) = \mathcal{E}_{2,2}(37) \cup \mathcal{E}^{\star}_{2,2}(37)$	36	complete	$\mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{3,2}(37) = \mathcal{E}_{3,2}(37) \cup \mathcal{E}^{\star}_{3,2}(37)$	44	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{4,2}(37) = \mathcal{E}_{4,2}(37) \cup \mathcal{E}^{\star}_{4,2}(37)$	40	complete	$\mathbb{Z}/2\mathbb{Z}$
$\mathscr{C}_{5,2}(37) = \mathcal{E}_{5,2}(37) \cup \mathcal{E}^{\star}_{5,2}(37)$	48	complete	$\mathbb{Z}/3\mathbb{Z}$
$\mathscr{C}_{6,2}(37) = \mathcal{E}_{6,2}(37) \cup \mathcal{E}^{\star}_{6,2}(37)$	28	incomplete	$\mathbb{Z}/6\mathbb{Z}$

3.2. Proof Corollary 1

Theorem 2.1 tell us that there exists a projective $[k, n, d]_q$-code if and only if there exists an $(k, k-d)$-arc in $PG(n-1, q)$. So our result comes immediately from Theorem 1.1.

According to Theorem 2.2 and Theorem 1.1, we get the following Griesmer codes:

$[12, 3, 9]_7$, $[16, 3, 13]_{11}$, $[20, 3, 17]_{13}$, $[24, 3, 21]_{17}$, $[24, 3, 21]_{19}$, $[28, 3, 25]_{19}$, $[28, 3, 25]_{23}$, $[32, 3, 29]_{23}$, $[36, 3, 33]_{29}$, $[40, 3, 37]_{29}$, $[36, 3, 33]_{31}$, $[40, 3, 37]_{31}$, $[44, 3, 41]_{37}$ and $[48, 3, 45]_{37}$.

References

[1] S. Ball and A. Blokhuis, On the incompleteness of (k,n)-arcs in desarguesian planes of order q where n divides q, *Geometriae Dedicata*, **47**(3), (1999), 325-332.

[2] D. Bartoli, S. Marcugini, and F. Pambianco, The The non-existence of some NMDS codes and the extremal sizes of complete (n,3)-arcs in PG (2, 16), *Designs, Codes and Cryptography*, **72**(1), (2014), 129-134.

[3] D. Bartoli, S. Marcugini, and F. Pambianco, On the completeness of plane cubic curves over finite fields, *Designs, Codes and Cryptography*, **83**(2), (2017), 233-267.

[4] M. A. De Boer, Almost MDS codes, *Designs, Codes and Cryptography*, **9**(2), (1996), 143-155.

[5] Griesmer, J.H., A bound for error-correcting codes, *IBM Journal of Research and Development*, **4**(5), (1960), pp.532-542.

[6] Hill, R. and Newton, D.E., Optimal ternary linear codes, *Designs, Codes and Cryptography*, **2**(2), (1992), pp.137-157.

[7] J. Hirschfeld, *Projective geometries over finite fields*, in Oxford University Press, 1998.

[8] J. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces, *Journal of Statistical Planning and Inference*, **72**(1-2), (1998), 355-380.

[9] J. W. Hirschfeld and J. F. Voloch, The characterization of elliptic curves over finite fields, *Journal of the Australian Mathematical Society*, **45**(2), (1988), 275-286.

[10] J. W. Hirschfeld, G. Korchmaros, F. Torres and F. E. T. Orihuela, *Algebraic curves over a finite field*, Princeton University Press, 2008.

[11] Maruta, T., Extension theorems for linear codes over finite fields, *Journal of Geometry*, **101**(1-2), (2011), pp.173-183.

[12] Maruta, T., Griesmer bound for linear codes over finite fields, Online table.

[13] B. Segre, Ovals in a finite projective plane, *Canadian Journal of Mathematics*, **7**, (1955), 414-416.

[14] Solomon, G. and Stiffler, J.J., Algebraically punctured cyclic codes. *Information and Control*, **8**(2), (1965), pp.170-179.

[15] H. Stichtennoth, *Algebraic function fields and codes*, Springer Science, Business Media, 2009.

[16] T. Szonyi, On the embedding of (k, p)-arcs in maximal arcs, *Designs, Codes and Cryptography*, **18**(1-3), (1999), 235-246.