The Estimate Function of Schrödinger Operator

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Abstract In this paper, as a basic step in arriving at a self-adjoint operator proof, we must find an estimate function and a continuation of our previous work in proof of convergence and smoothness of a function $\Psi(t,x)$. We structure the function $\xi(t,x)$, and we show the function $\xi(t,x)$ in $L^2(\mathbb{R}^n, dx)$ and get the relevance among this function and the function $\Psi(t,x)$ by using the characteristics of the first function. Generally previous function for any smooth potential $W(x)$ and we have the same results with the function $\xi(t,x)$. The advantages of $\xi(t,x)$ are investigated. Finally, we evidence that the function $\xi^{(W)}(t,x)$ converges to $\xi(t,x)$ in $L^2(\mathbb{R}^n, dx, dV)$ as $r \to \infty$.

Keywords Schrödinger operator · Electric potential · Magnetic potential · Infinitely differentiable

1 Introduction

The equation of the self-adjoint operator is major in the quantum machine (the Dirac–von Neumann formulation of quantum mechanics, physical observables such as position, momentum, and angular momentum).

Ismagilov [1, 2] showed the Schrödinger operator (denoted by Sch.O.) is essentially self-adjoint (denoted by E.S.A.). After this study, Simon [6] investigated the Sch.O. $-\Delta + q$ on $L^2(\mathbb{R}^m)$, where q is an operator of multiplication by a real-valued measurable function, $q(x)$, on $\mathbb{R}^n$. Many studies interested Essentiality self -adjoint operator refer to [3–5, 7, 9, 10].

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In this article, we define function $\xi(t,x)$ as

$$\xi(t,x) = \frac{\varphi(x)}{\sqrt{t}\Gamma\left(\frac{1}{2}\right)} + e^{ts}(2\pi)^{-n}\int_{-\infty}^{\infty} e^{it\sigma}\frac{\Psi\left(\frac{1}{s+i\sigma},x\right)\varphi(x)-\varphi(x)}{\sqrt{s+i\sigma}}d\sigma,$$

we study its properties for this function and investigate the relationship with the function $\Psi(t,x)$ of the variable t which is defined as [7]:

$$\Psi(t,x) = \int d\mu_x^t(w)\left\{\exp\left[-i\int_0^t b(\omega(s))d\omega \right.\right.$$
$$\left.\left. -\frac{1}{2}i\int_0^t \operatorname{div} b(\omega(s))ds - \int_0^t V(\omega(s))ds\right]\right\}\varphi(\omega(t)) \quad (1)$$

for the electromagnetic Schrödinger operator

$$H = \sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j + b_j(x)\right)^2 + V(x),$$

in $L^2(\mathbb{R}^n)$ where $b_j(x)$, $j = 1, 2, \ldots n$ and $V(x)$ are real-valued functions on $\mathbb{R}^n$, $V \in L^1_{loc}(\mathbb{R}^n)$, $b \in C^2(\mathbb{R}^n)$, $\partial_j = \frac{\partial}{\partial x_j}$, and $i = \sqrt{-1}$.

2 The Main Results

We investigate the function $\xi(t,x)$ and their relationship with $\Psi(t,x)$.

Proposition 2.1 *Let $V \in L^2(\mathbb{R}^n)$, $\varphi \in C_0^{\infty}$. Hence, $\xi(t,x)$ is found and has the characteristics*

i. $\xi(t,x) \in L^2\left(\mathbb{R}^n, dx\right) \forall t, s > 0$

ii. $\int_0^{\infty} e^{-ts}\xi(t,x)_{L^2(\mathbb{R}^n,dx)}dt < +\infty$ *for all* $t, s > 0$

iii. $\xi(t,x)$ *is finite* ($\xi(t,x) = 0$, *if* $d(x, \operatorname{supp}\varphi) > 2\sqrt{t}$)

iv. $\frac{1}{\sqrt{s}}\Psi\left(\frac{1}{s},x\right) = \int_0^{\infty} e^{-ts}\xi(t,x)dt$

v. $\xi(t,x) = \frac{\varphi(x)}{\sqrt{t}\Gamma(\frac{1}{2})} + e^{ts}(2\pi)^{-n} \int\limits_{-\infty}^{\infty} e^{it\sigma} \frac{\Psi(\frac{1}{s+i\sigma},x)\varphi(x)-\varphi(x)}{\sqrt{s+i\sigma}} d\sigma$, *$\Gamma(x)$ is Γ-function,* $\mathrm{Re}\sqrt{s+i\sigma} > 0$.

Proof From (1.1), we have

$$\Psi_\nu(t,x) = \frac{(-1)^\nu}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} (-i)^j b(\omega(s))^j_{L^2[0,t]} \int\limits_0^t ds_1 \int\limits_{s_1}^t ds_2 \ldots \int\limits_{s_{\nu-j-1}}^t ds_{\nu-j} \int\limits_{\mathbb{R}^1} \ldots$$
$$\int\limits_{\mathbb{R}^1} p(x,x_1,s_1)\ldots p(x_{\nu-j-1},x_{\nu-j},s_{\nu-j}-s_{\nu-j-1})\,p(x_{\nu-j},y,t-s_{\nu-j})$$
$$\left(-\frac{i}{2}\mathrm{div}\, b(x_1) - V(x_1)\right)\left(-\frac{i}{2}\mathrm{div}\, b(x_2) - V(x_2)\right)\ldots$$
$$\left(-\frac{i}{2}\mathrm{div}\, b(x_{\nu-j}) - V(x_{\nu-j})\right)\varphi(y)dx_1dx_2\ldots dx_{\nu-j}dy$$

We have

$$\Psi(t,x) = \sum_{\nu=0}^{\infty} \Psi_\nu(t,x)$$

Now we can write the function

$$\Psi_\nu(t,x) = \frac{(-1)^\nu}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} (-i)^j b(\omega(s))^j_{L^2[0,t]} \int\limits_0^t ds_1 \int\limits_{s_1}^t ds_2 \ldots \int\limits_{s_{\nu-j-1}}^t ds_{\nu-j} \int\limits_{\mathbb{R}^1}$$
$$\ldots \int\limits_{\mathbb{R}^1} p(x,x_1,s_1)\,p(x_{\nu-j-1},x_{\nu-j},s_{\nu-j}-s_{\nu-j-1})\,p(x_{\nu-j},y,t-s_{\nu-j})$$
$$\left(-\frac{i}{2}\mathrm{div} b(x_1) - V(x_1)\right)\left(-\frac{i}{2}\mathrm{div} b(x_2) - V(x_2)\right)\ldots$$
$$\left(-\frac{i}{2}\mathrm{div} b(x_{\nu-j}) - V(x_{\nu-j})\right)$$
$$\varphi(y)dx_1dx_2\ldots dx_{\nu-j}dy \quad (2)$$

where $p(x,y,t) = (2\pi t)^{-\frac{1}{2}} \exp\left(\frac{-(x-y)^2}{2t}\right)$. By Fourier transformation

$$\widehat{\Psi_\nu}(t,\lambda) = \frac{(-1)^\nu}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} (-i)^j t^{\nu-j} b(\omega(s))^j_{L^2[0,t]}$$

$$\times \int\limits_{\sigma_1+\sigma_2+cldots+\sigma_{\nu-j}<1} \cdots \int d\sigma_1 \dots d\sigma_{\nu-j} \int\limits_{\mathbb{R}^1}$$
$$\times \int\limits_{\mathbb{R}^1} d\lambda_1 d\lambda_2 \dots d\lambda_{\nu-j} e^{-t\sigma_1\lambda^2} e^{-t\sigma_2\lambda_1^2} \dots e^{-t(1-\sigma_1-\sigma_2-\dots-\sigma_{\nu-j})\lambda_{\nu-j}^2}$$
$$\times \left(-\frac{i}{2}\widehat{\operatorname{div}} b(\lambda-\lambda_1) - \hat{V}(\lambda-\lambda_1)\right)\left(-\frac{i}{2}\widehat{\operatorname{div}} b(\lambda_1-\lambda_2) - \hat{V}(\lambda_1-\lambda_2)\right)\dots$$
$$\times \left(-\frac{i}{2}\widehat{\operatorname{div}} b\big(\lambda_{\nu-j-1}-\lambda_{\nu-j}\big) - \hat{V}\big(\lambda_{\nu-j-1}-\lambda_{\nu-j}\big)\right)\hat{\varphi}\big(\lambda_{\nu-j}\big).$$

We put $t = \frac{1}{s}$ and use the Laplace transformation; then, we have

$$\frac{1}{\sqrt{s}}\widehat{\Psi_\nu}\left(\frac{1}{s},\lambda\right) = \frac{(-1)^\nu}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-i)^j \frac{1}{s^{\frac{\nu-j}{2}}} b(\omega(s))^j_{L^2\left[0,\frac{1}{s}\right]}$$
$$\int\limits_{\sigma_1+\sigma_2+\dots+\sigma_{\nu-j}<1} \cdots \int d\sigma_1 \dots d\sigma_{\nu-j} \int_0^\infty e^{-s\tau} d\tau \int\limits_{\mathbb{R}} \cdots \int\limits_{\mathbb{R}^1} d\lambda_1 d\lambda_2 \dots d\lambda_{\nu-j}$$
$$\left(\frac{\cos(2\sqrt{\sigma_1\tau_1}\lambda}{\sqrt{\pi\tau_1}}\right) * \left(\frac{\cos(2\sqrt{\sigma_2\tau_2}\lambda_1}{\sqrt{\pi\tau_2}}\right) * \dots * \left(\frac{\cos(2\sqrt{\sigma_{\nu-j}\tau_{\nu-j}}\lambda_{\nu-j-1}}{\sqrt{\pi\tau_{\nu-j}}}\right) *$$
$$\left(\frac{\cos(2\sqrt{(1-\sigma_1-\sigma_2-\dots-\sigma_{\nu-j})\tau_{\nu-j+1}}\lambda_{\nu-j}}{\sqrt{\pi\tau_{\nu-j+1}}}\right)\left(-\frac{i}{2}\widehat{\operatorname{div}}\, b(\lambda-\lambda_1) - \widehat{V}(\lambda-\lambda_1)\right)$$
$$\left(-\frac{i}{2}\widehat{\operatorname{div}}\, b(\lambda_1-\lambda_2) - \widehat{V}(\lambda_1-\lambda_2)\right)\dots\left(-\frac{i}{2}\widehat{\operatorname{div}}\, b\big(\lambda_{\nu-j-1}-\lambda_{\nu-j}\big)\right.$$
$$\left.-\widehat{V}\big(\lambda_{\nu-j-1}-\lambda_{\nu-j}\big)\right)\widehat{\varphi}\big(\lambda_{\nu-j}\big)$$

where $f * g$ is convolution. If $\Delta_q f(x) = 2^{-1}(f(x+q) + f(x-q))$, thus

$$\mathcal{F}\Delta_q\mathcal{F}^{-1} f(x) = \mathcal{F}\left(\mathcal{F}^{-1}\left(\frac{1}{2}f(x+q)\right) + \mathcal{F}^{-1}\left(\frac{1}{2}f(x-q)\right)\right) = \Delta_q f(x)$$

Therefore,

$$\frac{1}{\sqrt{s}}\Psi_\nu\left(\frac{1}{s},x\right) = \frac{(-1)^\nu}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-i)^j \frac{1}{s^{\frac{\nu-j}{2}}\pi^{\frac{\nu-j+1}{2}}} b(\omega(s))^j_{L^2[0,\frac{1}{s}]}$$
$$\int\limits_{\sigma_1+\sigma_2+\dots+\sigma_{\nu-j}\le 1} \cdots \int d\sigma_1 \dots d\sigma_{\nu-j} \int_0^\infty e^{-s\tau} d\tau \int\limits_{\tau_1+\tau_2+\dots+\tau_{\nu-j}\le\tau} \cdots$$
$$\int d\tau_1 \dots d\tau_{\nu-j} \frac{1}{\sqrt{\tau_1}} . \frac{1}{\sqrt{\tau_2}} \cdots \frac{1}{\sqrt{\tau-\tau_1-\tau_2-\dots-\tau_{\nu-j}}} \Delta_{2\sqrt{\sigma_1\tau_1}}$$

$$\left(-\frac{i}{2}\text{div } b(\tau_1) - V(\tau_1)\right) \ldots \Delta_{2\sqrt{\sigma_{\nu-j}\tau_{\nu-j}}}$$
$$\left(-\frac{i}{2}\text{div } b(\tau_{\nu-j}) - V(\tau_{\nu-j})\right)\Delta_{2\sqrt{(1-\sigma_1-\sigma_2-\ldots-\sigma_{\nu-j})(\tau-\tau_1-\tau_2-\ldots-\tau_{\nu-j})}}\varphi(x).$$

We note that by Cauchy inequality we have

$$\left(\sum_{k=1}^{\nu-j} 2\sqrt{\sigma_k\tau_k} + 2\sqrt{(1-\sigma_1-\sigma_2-\cdots-\sigma_{\nu-j})}\sqrt{(\tau-\tau_1-\tau_2-\cdots-\tau_{\nu-j})}\right)^2$$
$$\leq \sum_{k=1}^{\nu-j} 4\sigma_k\tau_k + 4(1-(\sigma_1+\sigma_2+\cdots+\sigma_{\nu-j}))(\tau-(\tau_1+\tau_2+\cdots+\tau_{\nu-j})) \leq 4\tau.$$

We get

$$\sum_{k=1}^{\nu-j} 2\sqrt{\sigma_k\tau_k} + 2\sqrt{(1-\sigma_1-\sigma_2-\cdots-\sigma_{\nu-j})}\sqrt{(\tau-\tau_1-\tau_2-\cdots-\tau_{\nu-j})} \leq 2\sqrt{\tau}.$$

So,

$$\Delta_{2\sqrt{\sigma_1\tau_1}}\left(-\frac{i}{2}\widehat{\text{div}\, b}(\tau_1) - \widehat{V}(\tau_1)\right) \ldots \Delta_{2\sqrt{\sigma_{\nu-j}\tau_{\nu-j}}}\left(-\frac{i}{2}\widehat{\text{div}\, b}(\tau_{\nu-j}) - \widehat{V}(\tau_{\nu-j})\right)$$
$$\Delta_{2\sqrt{(1-\sigma_1-\sigma_2-\cdots-\sigma_{\nu-j})(\tau-\tau_1-\tau_2-\cdots-\tau_{\nu-j})}}\varphi(x) = 0$$

If $\mathrm{d}(x, \text{supp}\varphi) > 2\sqrt{\tau}$.
Thus,

$$\sum_{\nu=0}^{\infty} \frac{1}{\sqrt{s}}\Psi_\nu\left(\frac{1}{s}, x\right) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} (-i)^j \frac{1}{s^{\frac{\nu-i}{2}}\pi^{\frac{\nu-j+1}{2}}} b(\omega(s))^j_{L^2[0,\frac{1}{s}]}$$
$$\int_{\sigma_1+\sigma_2+\ldots+\sigma_{\nu-j}\leq 1} \ldots \int \mathrm{d}\sigma_1 \ldots \int \mathrm{d}\sigma_{\nu-j} \int_0^\infty \mathrm{e}^{-s\tau}\,\mathrm{d}\tau$$
$$\int_{\tau_1+\tau_2+\ldots+\tau_{\nu-j}\leq\tau} \ldots \int \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{\nu-j} \frac{1}{\sqrt{\tau_1}} \cdot \frac{1}{\sqrt{\tau_2}}$$
$$\ldots \frac{1}{\sqrt{\tau-\tau_1-\tau_2-\cdots-\tau_{\nu-j}}}$$
$$\Delta_{2\sqrt{\sigma_1\tau_1}}\left(-\frac{i}{2}\widehat{\text{div}\, b}(\tau_1) - V(\tau_1)\right) \ldots \Delta_{2\sqrt{\sigma_{\nu-j}\tau_{\nu-j}}} \Delta_{2\sqrt{(1-\sigma_1-\sigma_2-\ldots-\sigma_{\nu-j})(\tau-\tau_1-\tau_2-\ldots-\tau_{\nu-j})}}\varphi(x) \tag{3}$$

Thus, we have

$$\frac{1}{\sqrt{s}}\Psi\left(\frac{1}{s},x\right)\varphi(x)=\int_0^\infty \mathrm{e}^{-s\tau}\xi(\tau,x)\mathrm{d}\tau$$

By using Proposition (1.1) in [7], the function $\xi(t,x)$ satisfied the properties (i) and (ii).

Moreover, we can compose (iii) by replacing s by $s+i\sigma$ in Proposition (2.1) in [7]. Hence, we have

$$\frac{1}{\sqrt{s+i\sigma}}\left(\Psi\left(\frac{1}{s+i\sigma},x\right)-\varphi(x)\right)=\int_0^\infty \mathrm{e}^{-s\tau-i\sigma\tau}\left(\xi(\tau,x)-\frac{1}{\Gamma\left(\frac{1}{2}\right)\sqrt{\tau}}\varphi(x)\right)\mathrm{d}\tau$$

$$\frac{1}{\sqrt{s+i\sigma}}\Psi\left(\frac{1}{s+i\sigma},x\right)-\varphi(x)_{L^2(\mathbb{R}^n,dx)}\leq\frac{1}{|s+i\sigma|^{-3/2}}\text{const}$$

Hence, by inverse Fourier transformation

$$\xi(t,x)=\frac{1}{\Gamma\left(\frac{1}{2}\right)\sqrt{t}}\varphi(x)+\frac{\mathrm{e}^{s\tau}}{(2\pi)^n}\int_{-\infty}^{\infty}\mathrm{e}^{i\sigma t}\frac{1}{\sqrt{s+i\sigma}}\left(\Psi\left(\frac{1}{s+i\sigma},x\right)-\varphi(x)\right)d\sigma$$

By the Proposition (2.1), we can easily prove the next proposition.

Proposition 2.2 *Suppose that* $\mathrm{W}(x)$ *is a smooth function with compact support, suppose* $\varphi\in C_0^\infty(\mathbb{R}^n)$*, let*

$$\Psi(t,x)=\int \mathrm{d}\mu_x^t(w)\left\{\exp\left[-i\int_0^t b(\omega(s))\mathrm{d}\omega\right.\right.$$
$$\left.\left.-\frac{1}{2}i\int_0^t \mathrm{div}b(\omega(s))\mathrm{d}s-\int_0^t W(\omega(s))\mathrm{d}s\right]\right\}\varphi(\omega(t))$$

Then, $\exists$ a family $\xi^{(W)}(t,x)\ni$

i. $\xi^{(W)}(t,x)\in L^2\left(\mathbb{R}^n,\mathrm{d}x\right)$,

ii. $\int_{-\infty}^{\infty}\mathrm{e}^{-ts}\xi^{(W)}(t,x)_{L^2(\mathbb{R}^n,\mathrm{d}x)}\mathrm{d}t\,\langle+\infty$ for any $s,t\rangle 0$

iii. $\xi(t,x)$ is finite $\xi(t,x)=0$, if $\mathrm{d}(x,\operatorname{supp}\varphi)>2\sqrt{t}$

iv. $\frac{1}{\sqrt{s}}\Psi^{(W)}\left(\frac{1}{s}, x\right) = \int\limits_0^\infty e^{-ts}\xi^{(W)}(t, x)\mathrm{d}t$

v. $\xi^{(W)}(t, x) = \frac{\varphi(x)}{\sqrt{t}\Gamma\left({}^1\!/_2\right)} + e^{ts}(2\pi)^{-n}\int\limits_{-\infty}^{\infty} e^{it\sigma}\frac{\Psi^{(w)}\left(\frac{1}{s+i\sigma}, x\right) - \varphi(x)}{\sqrt{s+i\sigma}}\mathrm{d}\sigma,$

where $\Psi^{(W)}(t + is, x)$ is an analytic expansion of $\Psi(t, x)$ into a domain $\mathrm{Re}(t + is) > 0$ and an account of a quadrate root is possessed in a right-hand half-plane for any s> 0.

Suppose $V(x)$ have the form

$$v_{j-1,m}(x - (a_{j-1,m1}, a_{j-1,m2}, \ldots .a_{j-1,mn}))\xi_{j-1,m} + v_{j,m}(x - (a_{j,m1}, a_{j,m2}, \ldots, a_{j,mn}))\xi_{j,m}$$

For $r \in Z$, put $v_{j,m}(x) = 0$ for $|m| > r$. Thus, the random potential conformable to $v_{j,m}$ is a smooth finite function of x. Stand for W and stratify to the proposition.

We secure random functions $\Psi^{(W)}(t, x)$ and $\xi^{(W)}(t, x)$ together the advantages (i)–(v). From the outcomes of Proposition (2.1) and Lemma (2.2) in [7], we acquire

$$\Psi^{(W)}(t, x) \text{ converge } \Psi(t, x) \text{ in } L^2(\mathbb{R}^n, \mathrm{d}x, \mathrm{d}V) \text{if } r \to \infty.$$

converge $\Psi(t, x)\mathrm{in}L^2(\mathbb{R}^n, \mathrm{d}x, \mathrm{d}V)\mathrm{if}\ r \to \infty$.

Proposition 2.3 *For $t = \tau + i\gamma$, the assessment holds:*

$$\Psi(t, x) - \varphi(x) + tH\varphi(x) - \frac{t^2}{2!}H^2\varphi(x) + \cdots (-1)^{m+1}\frac{t^m}{m!}H^m\varphi(x)_{L^2(\mathbb{R}^n, \mathrm{d}x, \mathrm{d}V)} \le C_{m+1}\frac{t^{m+1}}{m!} \quad (4)$$

where C_{m+1} is a constant.

Proof As it was shown

$$\frac{\partial^m}{\partial t^m}\int\limits_{\mathbb{R}^n} E\left(\Psi(t, x)\sum_{k,l} C_{k,l}h_k(x)\theta_l(V)\right)\mathrm{d}x = (-1)^m\int\limits_{\mathbb{R}^n} E\left(\widetilde{\Psi}_m(t, x)\sum_{k,l} C_{k,l}h_k(x)\theta_l(V)\right)\mathrm{d}x, \quad (5)$$

where $\widetilde{\Psi}_m(t, x) \equiv H^m\varphi$. By Taylor's formula,

$$\int_{\mathbb{R}^n} E\left(\Psi(t,x)\sum_{k,l} C_{k,l}h_k(x)\theta_l(V)\right)\mathrm{d}x = \sum_{j=0}^{n}\frac{(t-t_1)^j}{j!}.\frac{\partial^j}{\partial t^j}$$

$$\int_{\mathbb{R}^n} E(\Psi(t,x)\sum_{k,l} C_{k,l}h_k(x)\theta_l(V))\,\mathrm{d}x|_{t=t_1} + \int_{t_1}^{t}\mathrm{d}\tau_1\int_{t_1}^{\tau_1}\mathrm{d}\tau_2\ldots\int_{t_1}^{\tau_n}\mathrm{d}\tau_{n+1}\frac{\partial^{n+1}}{\partial t^{n+1}}$$

$$\int_{\mathbb{R}^n} E\left(\Psi(t,x)\sum_{k,l} C_{k,l}h_k(x)\theta_l(V)\right)\mathrm{d}x|_{t=\tau_n} \tag{6}$$

substitute, by formula (6), the derivations into (5). According to: [8]

$$\frac{\partial\Psi}{\partial t} = \sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j + b_j(x)\right)^2\Psi - V\Psi$$

A .e. V, we acquire

$$\lim_{t_1\to 0}\widetilde{\Psi}_m(t_1,x) = H^m\varphi(x).$$

From (6),

$$\int_{\mathbb{R}^n} E\left((\Psi(t,x) - \sum_{j=0}^{m}\frac{(-t)^j}{j!}H^j\varphi(x))\sum_{k,l} C_{k,l}h_k(x)\theta_l(V)\right)\mathrm{d}x$$

$$= (-1)^{m+1}\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2\ldots\int_0^{\tau_m} d\tau_{m+1}\int_{R^n} E\left(\widetilde{\Psi}_{m+1}(t,x)\times\sum_{k,l} C_{k,l}h_k(x)\theta_l(V)\right)\mathrm{d}x \tag{7}$$

by the estimates,

$$\left|\int_{R^n} E\left(\Psi(t,x)\sum_{k,l} C_{k,l}h_k(x)\theta_l(V)\right)\mathrm{d}x\right| \leq \text{const.}\sum_{k,l} C_{k,l}h_k(x)\theta_l(V)_{L^2(\mathbb{R}^n,dx,dV)}. \tag{8}$$

Now (4) follows from (7) and (8). ■

We return to $\xi^{(W)}(t,x)$ and (v). By the proposition,

$$\frac{1}{|\sqrt{s+i\sigma}|}\Psi^{(w)}\left(\frac{1}{s+i\sigma},x\right) - \varphi(x)_{L^2(\mathbb{R}^n,dx,dV)} \leq C_1\frac{1}{|s+i\sigma|^{3/2}}, \tag{9}$$

$$\frac{1}{|\sqrt{s+i\sigma}|}\Psi\left(\frac{1}{s+i\sigma},x\right)-\varphi(x)_{L^2(\mathbb{R}^n,dx,dV)} \leq C_1\frac{1}{|s+i\sigma|^{3/2}} \tag{10}$$

Since (9) does not depend on r, then by (v) of Proposition (2.3) that $\xi^{(W)}(t,x)$ have limits in $L^2(\mathbb{R}^n, \mathrm{d}x, \mathrm{d}V)$, and denote by $\xi(t,x)$. In addition,

$$\xi(t,x)=\frac{\varphi(x)}{\sqrt{t}\Gamma\left(1/2\right)}+\mathrm{e}^{ts}(2\pi)^{-n}\int_{-\infty}^{\infty}\mathrm{e}^{It\sigma}\frac{\Psi\left(\frac{1}{s+I\sigma},x\right)-\varphi(x)}{\sqrt{s+i\sigma}}\mathrm{d}\sigma. \tag{11}$$

We note for $\xi(t,x)$ the following:

i. $\xi(t,x)\in L^2(\mathbb{R}^n,\mathrm{d}x,\mathrm{d}V)$.

ii. $\int_0^\infty \mathrm{e}^{-ts}\xi(t,x)_{L^2(\mathbb{R}^n,\mathrm{d}x,\mathrm{d}V)}\mathrm{d}t\langle+\infty$ for any $s,t\rangle 0$

iii. $\xi(t,x)$ is finite $\xi(t,x)=0$, if $\mathrm{d}(x,\operatorname{supp}\varphi)>2\sqrt{t}$

iv. $\frac{1}{\sqrt{s}}\Psi\left(\frac{1}{s},x\right)=\int_{-\infty}^{\infty}\mathrm{e}^{-ts}\xi(t,x)\mathrm{d}t,$

which is convergent in $L^2(\mathbb{R}^n,\mathrm{d}x,\mathrm{d}V)$.

These advantages can be found by the limit the conformable advantages of $\xi^{(W)}(t,x)$.

By depending on scalar product in $L^2(\mathbb{R}^n,dx,dV)$ of (11) and $\sum_{k,l}C_{k,l}h_k(x)\theta_l(V)$, thus we acquire a. e. V, the $\xi(t,x)$ is smooth and

$$H\xi(t,x)=\xi^{(1)}(t,x)$$

where $\xi^{(1)}(t,x)$ is found by (11) and $\varphi(x)$ is displaced by $H\varphi(x)$. This pursues from $H\Psi(t,x)=\Psi^{(1)}(t,x)$. Suppose that evidences our proposition.

From (11) and the results in [8],

$$\left|E\int_{\mathbb{R}^n}\Psi(t,x)\sum_{k,l=1}^{p}C_{k,l}\sum_{j=1}^{n}2^{-1}\left(i\partial_j+b_j(x)\right)^2h_k(x)\theta_l(V)\mathrm{d}x\right|$$
$$\leq \text{const}\sum_{k,l=1}^{p}C_{k,l}h_k(x)\theta_l(V)_{L^2(\mathbb{R}^n,dx,dV)}$$

we get

$$E\left(\int_{\mathbb{R}^n} \xi(t,x) \sum_{k,l} C_{k,l}\left(\sum_{j=1}^{n} \frac{1}{2}\left(i\partial_j + b_j(x)\right)^2\right) h_k(x)\theta_l(V)\mathrm{d}x\right)$$

$$= \frac{1}{\sqrt{t}\Gamma\left(^1/_2\right)} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{n} \frac{1}{2}\left(i\partial_j + b_j(x)\right)^2\right) \varphi(x) E\left(\sum_{k,l} C_{k,l} h_k(x)\theta_l(V)\right)\mathrm{d}x$$

$$+ (2\pi)^{-n} \mathrm{e}^{ts} \int_{-\infty}^{\infty} \mathrm{e}^{it\sigma} \frac{1}{\sqrt{s+i\sigma}} E\left[\int_{\mathbb{R}^n} \mathrm{d}x \left(\left(\sum_{j=1}^{n} \frac{1}{2}\left(i\partial_j + b_j(x)\right)^2 \Psi\left(\frac{1}{s+i\sigma}, x\right)\right.\right.\right.$$

$$\left.\left.\left. - \left(\sum_{j=1}^{n} \frac{1}{2}\left(i\partial_j + b_j(x)\right)^2 \varphi(x)\right) \sum_{k,l} C_{k,l} h_k(x)\theta_l(V)\right]\mathrm{d}\sigma$$

$$\left(\sum_{j=1}^{n} \frac{1}{2}\left(i\partial_j + b_j(x)\right)^2 \Psi\left(\frac{1}{s+i\sigma}, x\right) - \left(\sum_{j=1}^{n} \frac{1}{2}\left(i\partial_j + b_j(x)\right)^2 \varphi(x)\right.\right. \tag{12}$$

$$= H\varphi(x) - H\Psi\left(\frac{1}{s+i\sigma}, x\right) + V\Psi\left(\left(\frac{1}{s+i\sigma}, x\right) - \varphi(x)\right)$$

$$= \widetilde{\varphi}_1(x) - \widetilde{\Psi}_1\left(\frac{1}{s+i\sigma}, x\right) + V\left(\Psi\left(\frac{1}{s+i\sigma}, x\right) - \varphi(x)\right), \tag{13}$$

where $\widetilde{\varphi}_1 = H\varphi(x)$ and $\widetilde{\Psi}_1(t,x)$ corresponds to $\widetilde{\varphi}_1(x)$.

Note that the Proposition (2.2) remains satisfied for $\widetilde{\Psi}_1$ and $\widetilde{\varphi}_1$. In particular,

$$\widetilde{\Psi}_1 - \widetilde{\varphi}_1{}_{L^2(\mathbb{R}^n, \mathrm{d}x, \mathrm{d}V)} \leq \text{const}\frac{1}{|s+i\sigma|}. \tag{14}$$

If we multiply the integrand in (5) by $V(x)$, then (4) remains satisfied

$$V(\Psi(t,x) - \sum_{j=0}^{n}(-1)^{j+1}\frac{(t+1)^j}{j!}\varphi(x)\Bigg)_{L^2(\mathbb{R}^n, \mathrm{d}x, \mathrm{d}V)} \leq C_{n+1}\frac{t^{n+1}}{n!}, \tag{15}$$

where C_{n+1} is a constant.

$$V\left(\Psi\left(\frac{1}{s+i\sigma}, x\right) - \varphi(x)\right)_{L^2(\mathbb{R}^n, \mathrm{d}x, \mathrm{d}V)} \leq \frac{\text{const}}{|s+i\sigma|} \tag{16}$$

substitute (13)–(16) into (11). After the Fourier transform, we get

$$\left| E\left(\int_{\mathbb{R}^n} \hat{\xi}(t,q)\left(\frac{-n}{2}|q|^2 - \frac{1}{2}\sum_{j=1}^{n} i\partial_j \hat{b}_j(q) - \frac{1}{2}\sum_{j=1}^{n} i\hat{b}_j(q)|q| + \frac{1}{2}\sum_{j=1}^{n} \hat{b}_j^2(q)\right)\right.\right.$$

$$\sum_{k,l} C_{k,l}\hat{h}_k(q)\theta_l(V)\Bigg)\mathrm{d}q\Bigg| \leq \mathrm{const}\left(\frac{1}{\sqrt{t}} + B_s \mathrm{e}^{ts}\right)$$
$$\sum_{k,l} C_{k,l} h_k(x)\theta_l(V)_{L^2(\mathbb{R}^n, \mathrm{d}x, \mathrm{d}V)}, \quad (17)$$

where A, B_s are constants. The estimate (17) implies $\xi(t, x) \in W_1$ a. e. V, also, $\xi^2_{W_1}$ an integrable of V. Further, we can return above by the same method as in [8]. Thus, we get the stated assertion.

3 Conclusion

W construct the function $\xi(t, x)$ and note that the function$\xi^{(W)}(t, x)$ converges to $\xi(t, x)$ in $L^2(R^n, \mathrm{d}x, \mathrm{d}V)$ as $r \to \infty$. So, the same properties of the $\Psi(t, x)$ are satisfied.

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