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Geometry of Certain Curvature Tensors of Almost Contact Metric Manifold

A Thesis

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By

Mohammed Yousif Abass

Supervised by

Prof. Dr. Habeeb Mtashar Abood

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

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Recommendation of the Supervisor

I certify that the thesis entitled **"Geometry of Certain Curvature Tensors of Almost Contact Metric Manifold"**, which is being submitted by author was prepared under my supervision at the Department of Mathematics, College of Education for Pure Sciences, University of Basrah as partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

Signature:

Prof. Dr. Habeeb Mtashar Abood

Date: 7 / 9 / 2020

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In view of the available recommendations, I forward this thesis for debate by the examining committee.

Signature:

Prof. Dr. Mudhir A. Abdul Hussain

Head of Math. Dept.

Date: 7 / 9 /2020

Examine Committee Certification

We certify that we have read this thesis and as examining committee, we have examined the student in its content and in our opinion, it is adequate as a thesis for the degree of Doctor of Philosophy in Mathematics.

Chairman Signature: Prof. Dr. Habeeb K. Abdullah Date: 22 / 12 / 2020	Member Signature: Prof. Dr. Mudhir A. Abdul Hussain Date: 22 / 12 / 2020
Member Signature: Assist. Prof. Dr. Ali J. Mohammad Date: 22 / 12 / 2020	Member Signature: Assist. Prof. Dr. Taghreed H. Majeed Date: 22 / 12 / 2020
Member Signature: Assist. Prof. Dr. Qusay S. A. Al – Zamil Date: 22 / 12 / 2020	Member and Supervisor Signature: Prof. Dr. Habeeb M. Abood Date: 22 / 12 / 2020

Approved for the University Committee on Graduate Studies:

Signature: F Al-Khud

Prof. Dr. Falih H. Al – Khudair

Dean of the College of Education for Pure Sciences, University of Basrah

Date: 23 / 12 / 2020

Dedication

To the prophet of Islam and his household

The martyrs of Iraq

My father and mother

My brothers and sisters

My wife

All whom I love

Mohammed

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Praise and thanks goto Allah who gave me the power to complete this work. Greetings and peace be upon the Prophet Mohammed and his household; they always tried to guide us to the light of faith and knowledge.

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Contents

Abstract

In this thesis, we characterized a new class of almost contact metric ($ACR-$) manifolds and establish the equivalent conditions that characterize its identity in sense of Kirichenko's tensors. We demonstrate that the Kenmotsu manifold proves that the mentioned class, that is, the new class, can be decomposed into a direct sum of the Kenmotsu manifold and other classes. We prove that the manifold of dimension 3 coincides with the Kenmotsu manifold and provide an example of the new manifold of dimension 5, which is not the Kenmotsu manifold. Moreover, we establish that the Cartan's structure equations, components of Riemannian curvature tensor, and the Ricci tensor of the class should be kept under consideration. Further, the conditions required for the mentioned class to be an Einstein manifold have been determined. We called the aforementioned characterized class the class of the Kenmotsu type.

Furthermore, in this thesis, we provide an example of the class of Kenmotsu type as a warped product of the Hermitian manifold by the real line. The conditions required for the mentioned class to be of constant pointwise $\Phi-$holomorphic sectional curvature tensor are obtained on the associated $G-$structure space. We classify new classes of $ACR-$manifolds according to their curvature tensors and ascertain their relationships with our class. Moreover, we investigate the conditions that make our class satisfy the generalized Sasakian space forms, new classes, and Einstein manifolds.

The present thesis studies the generalized $\Phi-$recurrent manifold of the Kenmotsu type. The aim of this study is to determine the components of the covariant derivative of the Riemannian curvature tensor. Moreover, the conditions make a manifold of Kenmotsu type a locally symmetric or generalized $\Phi-$recurrent have been established. We concluded that the locally symmetric manifold of the Kenmotsu type is generalized $\Phi-$recurrent under suitable conditions and vice versa. Furthermore, the study shows the relationship between Einstein manifolds and locally symmetric manifolds of the Kenmotsu type.

For the same class, we determine the components of the generalized curvature tensor and establish that the mentioned class is $\eta-$Einstein manifold in the flatness of the generalized curvature tensor; the converse holds under suitable conditions. Moreover, we introduced the notion of generalized $\Phi-$holomorphic sectional curvature tensor. Thus, we find the necessary and sufficient condition that makes the aforementioned notion constant for the class of Kenmotsu type. In addition, the notion of the Φ-generalized semi-symmetric is introduced and its relationship with the class of Kenmotsu type and the $\eta-$Einstein manifold is established. Furthermore, we generalize the notion of the manifold of constant curvature where the structure is almost contact and we identify its relationship with the mentioned ideas. Finally, we show that the class of Kenmotsu type exists as a hypersurface of the Hermitian manifold and derive a relation between the components of the Riemannian curvature tensors of the almost Hermitian manifold and its hypersurfaces.

This thesis also discusses the geometry of the $ACR-$manifolds of class C_{12}. In particular, it determines the structure equations, the components of curvature and Ricci tensors on the associated $G-$structure space. It also studies some curvature identities of this class. Moreover, this thesis investigates the (κ, μ)-nullity distribution of the class C_{12} and establishes the sufficient and necessary conditions for the mentioned class to have (κ, μ)-nullity distribution and satisfy the $\eta-$Einstein criterion. Finally, an example of a 3-dimensional manifold of class C_{12} has been constructed.

Symbols and Abbreviations

Characters	Description
M^n	The smooth manifold of dimension n
g	The Riemannian metric
g_{ij}	The components of g
g^{ij}	The components of g^{-1}
(M^n, g)	The Riemannian manifold of dimension n
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The Euclidean space
$\mathbb{C}$	The set of complex numbers
$\mathbb{C}^n$	The complex Euclidean space
$C^\infty(M)$	The set of all smooth functions $f : M \to \mathbb{R}$
$X(M)$	The set of all vector fields over M
$T_p(M)$	The tangent space over M at the point $p \in M$
$AG-$structure space	Associated $G-$structure space
$ACR-$manifold	Almost contact metric manifold
ξ	The characteristic vector field of $ACR-$manifold
η	The 1-form of $ACR-$manifold
Φ	The tensor of type (1, 1) for $ACR-$manifold
$(M^{2n+1}, \xi, \eta, \Phi, g)$	The $ACR-$manifold of dimension $2n+1$
$\Phi HS-$curvature	$\Phi-$Holomorphic sectional curvature
$G\Phi HS-$curvature	Generalized $\Phi HS-$curvature
$\Phi GS-$symmetric	Φ-Generalized semi-symmetric

Characters	**Description**
$GS-$space forms	Generalized Sasakian space forms
$M(f_1, f_2, f_3)$	The $GS-$space forms
$\Omega(X, Y)$	$g(X, \Phi Y); \quad \forall\, X, Y \in X(M)$
$lcQ\mathcal{S}-$manifold	Locally conformally quasi-Sasakian manifold
$[X, Y]$	$XY - YX$
V^*	The dual space of V
$r-$form	The tensor of type $(r, 0)$
$\bigoplus$	The direct sum operation
$\otimes$	The tensor product operation
$\mathcal{T}_r^s(V)$	The set of all tensors of type (r, s) on V
$\mathcal{T}_r(V)$	The set of all $r-$forms on V
$\Sigma_r(V)$	The set of all symmetric $r-$forms on V
$\Lambda_r(V)$	The set of all alternating $r-$forms on V
$\Lambda(V)$	The Grassmann algebra
$Symbol(M)$	$Symbol(X(M))$
$(Symbol)_p(M)$	$Symbol(T_p(M))$
$\varphi \wedge \psi$	The exterior product of φ and ψ
δ_{ij} or δ_j^i	The Krönecker delta
$\widetilde{\delta}_{bc}^{ad}$	$\delta_b^a\ \delta_c^d + \delta_c^a\ \delta_b^d$
$B \times_f F$	The warped product of Riemannian manifolds B and F with smooth map $f : B \longrightarrow B$
$X^C(M)$	$\mathbb{C} \otimes X(M)$
∇	The Riemannian connection
$\nabla_X(\Phi)Y$	$\nabla_X \Phi(Y) - \Phi(\nabla_X Y)$
θ	The 1-form of ∇
θ_j^i	The components of θ on $AG-$structure space
$\hat{a}$	$a + n$
A-frame	$(p; \xi, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$
ω^k	The dual of A-frame with $k = 0, 1, ..., 2n$

Characters	**Description**
R	The Riemannian curvature tensor
R^i_{jkl}	The components of R of type (3, 1)
R_{ijkl}	The components of R of type (4, 0)
∇R	The covariant derivative of R
$B^{ab}{}_c$, $B_{ab}{}^c$	The components of first structure tensor B
B^{abc}, B_{abc}	The components of second structure tensor C
B^{ab}, B_{ab}	The components of third structure tensor D
$B^a{}_b$, $B_a{}^b$	The components of fourth structure tensor E
C^{ab}, C_{ab}	The components of fifth structure tensor F
C^a, C_a	The components of sixth structure tensor G
AH-manifold	Almost Hermitian manifold
$W_3 \oplus W_4$	The Hermitian class of AH-manifolds
J	The complex structure of AH-manifold
$\sigma_{\alpha\beta}$	The components of the second fundamental (quadratic) form σ
(..)	The symmetric operator of its interior
(.|.|.)	The symmetric operator of its interior except |.|
[..]	The alternating operator of its interior
[.|.|.]	The alternating operator of its interior except |.|
$\overline{T^i}$	The complex conjugate of $T^i (= T^{\hat{i}})$
Q	The Ricci operator
$\widetilde{B}$	The generalized curvature tensor
P	The projective curvature tensor
$\widetilde{C}$	The concircular curvature tensor

Introduction

Introduction

The establishment of modern differential geometry is attributed to Chern [32], who introduced the algebraic structures of the almost contact manifolds in 1953. In 1958, Boothby and Wang [28] discussed the regular and homogeneous contact manifolds and deduced their relationships with tangent sphere bundles. On the other hand, in 1959, Gray [55] gave some examples of $ACR-$manifolds. In the 1960s, Sasaki [101] published lecture notes on the $ACR-$manifolds and characterized the special class was later called Sasakian manifolds. Blair [16] studied the quasi-Sasakian structure; Blair and Ludden [22] considered the hypersurfaces on almost contact manifolds, whereas the concept of almost cosymplectic manifolds was first introduced by Goldberg and Yano [52].

In 1971, the nearly cosymplectic structure was established by Blair [17], while Blair and Yano [25] generalized the results that appeared in [17]. In 1972, Kenmotsu [63] defined a class of $ACR-$manifolds, which was not Sasakian. Later, this manifold bore the name of Kenmotsu manifolds. In 1973, Chen [30] concentrated on the geometry of submanifolds. In 1974, Blair and Showers [23] applied some of Gray's conclusions [53] on nearly Kähler manifolds to nearly cosymplectic manifolds. In 1976, Blair [18] discussed contact manifolds where the normal contact manifolds were Sasakian manifolds, whereas Blair et al. [24] highlighted nearly Sasakian structures. In 1980, Vaisman [112] investigated the conformal transformation of $ACR-$manifolds. In 1981, Olszak [90] gave examples of almost cosymplectic manifolds and studied their existence with non-zero constant curvature, while Janssens and Vanhecke [61] decomposed the $ACR-$manifolds that satisfied some curvature tensors into irreducible components.

In 1983, Kirichenko [66] and [67] investigated the geometry of nearly Sasakian spaces and almost cosymplectic manifolds that satisfy the axiom of planes with $\Phi-$holomorphic. In 1984, the axiom of $\Phi-$holomorphic planes on the contact metric geometry was studied by Kirichenko [68]. In 1985, Oubiña [92] determined new classes of $ACR-$manifolds. In 1986, Kirichenko [69] demonstrated an interesting method to determine contact geometry from generalized Hermitian geometry. Locally conformal almost cosymplectic manifolds were discovered in 1989 by Olszak [91]. In 1990, $ACR-$manifolds were classified according to their structure group into a direct sum of twelve irreducible classes by Chinea and Gonzalez [34]. In 1992, Tshikuna-Matamba [109] defined new classes of $ACR-$manifolds, which generalized the Kenmotsu class, such as nearly Kenmotsu manifolds, quasi-Kenmotsu manifolds, and so on. In 1994, Rustanov [98] discussed the geometry of quasi-Sasakian manifolds. In 1995, Chinea and et al. [35] studied almost contact submersions where the locally conformal total space is a cosymplectic manifold. In 1997, the author Volkova [115] studied normal manifolds of the Killing type, which satisfy the special curvature identities.

In 2000, Boeckx [26] classified the contact manifolds that satisfy (κ, μ)-nullity conditions. In 2001, Kirichenko [70] constructed a Kenmotsu manifold using a conformal transformation of cosymplectic manifold, while in [105], Stepanova and Banaru extracted $ACR-$manifolds from quasi-Kählerian manifolds as hypersurface. In 2002, the geometry of Kenmotsu manifold and some of its interesting generalizations were discussed by Umnova [111], whereas Volkova [116] investigated the normal manifolds of the Killing type, which satisfy the axiom of Φ-holomorphic planes. On the other hand, Blair [19] studied the geometry of special Riemannian manifolds that are contact and symplectic manifolds, while in [108], Terlizzi and Pastore investigated the $\mathcal{K}-$manifolds with the quasi-Sasakian manifold as a special case of it, defined an $f-$structure on a hypersurface of the $\mathcal{K}-$manifold, and provided an example of the $\mathcal{K}-$manifold. In 2003, Kirichenko [71] introduced a separate study of the differential geometric structures on the Riemannian manifolds by using the method of associated $G-$structure space (briefly, $AG-$structure space).

In 2004, Alegre et al. [5] generalized the idea of Sasakian-space-forms, whereas Falcitelli et al. [47] focused on Riemannian submersions and associated them with theoretical physics and the Einstein theory by providing examples. In 2005, Jun et al. [62] studied certain curvature conditions such as semi-symmetric and Weyl semi-symmetric of the Kenmotsu manifold. Moreover, they studied the transformation that saves the invariant of the Ricci tensor. However, in [44], Endo investigated nearly cosymplectic manifolds that had constant $\Phi-$sectional curvature. In 2006, Kirichenko and Dondukova [74] discussed the geodesic transformation of Kenmotsu manifolds and proved there is only a trivial transformation, while Falcitelli and Pastore [48] discussed the curvature properties of the Kenmotsu $f.pk-$manifolds.

In 2007, Kirichenko and Polkina [77] showed that on the quasi-Sasakian structures there are no non-trivial contact-geodesic metric transformations. They also proved that the normal regular locally conformally quasi-Sasakian (normal regular $lcQ\mathcal{S}-$) structures allow nontrivial contact-geodesic metric transformations. Moreover, the second author studied the analogs of Gray identities (see [54]) on $ACR-$ and $lcQ\mathcal{S}-$structures in [96], while Kirichenko and Baklashova [73] derived Ikuta's theorem on $ACR-$manifolds. In particular, they proved that the locally conformally cosymplectic manifold had closed contact form if and only if it is a normal regular $lcQ\mathcal{S}-$manifold. The normal regular $lcQ\mathcal{S}-$manifold is a Kenmotsu manifold if and only if its contact Lee form and the contact form are the same. At the same time, Pitiş [95] studied the geometry of Kenmotsu manifolds in detail. On the other hand, Dileo and Pastore [41] deduced the necessary and sufficient conditions for almost Kenmotsu manifolds to be locally symmetric. Falcitelli and Pastore [49] introduced and studied the notion of almost Kenmotsu $f.pk-$manifold.

In 2008, Kirichenko and Uskorev [80] described Kirichenko's tensors of $ACR-$manifold under conformal transformations, while Falcitelli [45] studied the $\Phi-$ sectional curvature of manifolds with locally conformal cosymplectic structures. Additionally, Alegre and Carriazo [6] studied the trans-Sasakian manifolds that satisfy the conditions of generalized Sasakian-space-forms ($GS-$space forms), and some general outcomes for dimension ≥ 5 and special cases for 3-dimensional were determined. In 2009, Dileo and Pastore [42] described the Riemannian geometry and Riemann submanifolds of almost Kenmotsu manifolds that satisfy some geometric conditions. They also characterized the CR-integrable almost Kenmotsu, classified almost

Kenmotsu manifolds based on certain nullity conditions, completely described the 3-dimensional case, and gave examples in [43]. Furthermore, Alegre and Carriazo [7] investigated the geometry of submanifolds in $GS-$space forms, while Kirichenko and Pol'kina [78] detected necessary and sufficient conditions for the quasi-Sasakian manifold to happen in a Fialkow space.

In 2010, Chinea [33] studied the harmonicity of special maps between $ACR-$manifolds. In 2011, Kirichenko and Kusova [76] classified weakly cosymplectic manifolds that satisfy contact analog curvature identities. At the same time, Dileo [40] analyzed the geometry of almost $\alpha-$Kenmotsu manifolds. She also focused on local symmetries and certain vanishing conditions for the Riemannian curvature. Parallelly, Ignatochkina [58], Ignatochkina and Morozov [60] and Nikiforova and Ignatochkina [89] studied the $ACR-$manifolds induced from almost Hermitian (AH-) manifolds by conformal transformations. On the other hand, Kharitonova [64] ascertained necessary and sufficient conditions for an $ACR-$manifold to be an almost $C(\lambda)-$manifold.

In 2012, Kirichenko and Kharitonova [75] determined the full group of structure equations, components of the Riemannian curvature tensor, components of the Ricci tensor, and components of the Weyl tensor on the $AG-$structure space for locally conformal manifolds with almost cosymplectic structures. Additionally, Falcitelli [46] studied the class of $ACR-$manifolds considered twisted product manifolds and derived theorems describing the aforementioned class with $GS-$space forms.

In 2013, Rehman [97] discussed the harmonic maps and morphisms between Kenmotsu manifolds and an AH-manifold. Moreover, she studied the spectral theory of these maps. Perrone [93] determined necessary and sufficient conditions for the Reeb vector field of 3-dimensional almost cosymplectic manifold to be minimal. Markellos and Tsichlias [85] constructed a new group of contact metric structures on $\mathbb{S}^3$. In 2014, Banaru [9] discussed the necessary and sufficient conditions for the $ACR-$manifold to be the hypersurface with type number 0 or 1 of the 6-dimensional Kähler submanifold of Cayley algebra. Kim et al. [65] characterized quasi-contact metric manifolds while De and Ghosh [36] studied $E-$Bochner curvature tensors that satisfy certain conditions of the $N(k)-$contact metric manifold of dimension n.

In 2015, Banaru and Kirichenko [13] derived the structure equations of $ACR-$manifold on a hypersurface of AH-manifold. They determined sufficient and necessary conditions for the Kenmotsu manifold on a hypersurface of the W_3-manifold (see Gray and Hervella [56]) to be minimal. Ghosh [50] examined contact metric manifolds with quasi-Einstein metrics, and he proved that every quasi-Einstein Sasakian manifold is an Einstein manifold. In 2016, Kirichenko and Pol'kina [79] were studied the concircular geometry of $lcQS-$manifold according to its contact Lie form. Banaru [11] showed that 2-hypersurfaces in a Kählerian manifold admit $ACR-$structures of a non-cosymplectic type. Wang [117] showed that a CR-integrable almost Kenmotsu manifold of a dimension of > 3 with certain conditions has constant sectional curvature of -1 if and only if it is conformally flat.

In 2017, Banaru [12] proved that hypersurfaces with type number 0 or 1 are identical in the Hermitian submanifold of dimension 6 in Cayley algebra. Nicola et al. [87] proved that each nearly Sasakian manifold with a dimension of > 5 is Sasakian as well as classified the nearly cosymplectic manifolds with a dimension of > 5. Loiudice [83] evaluated a class of contact manifolds of dimension $4n + 1$ and deduced that this class should have a dimension of 5 if it has constant sectional curvature. Alegre et al. [8] introduced a class of trans$-S-$manifolds that included special classes that were studied previously and they presented examples that supported their study. Petrov [94] studied the total space of the T^1-principal fiber bundle with almost Hermitian structures and flat connection over some classes of $ACR-$manifolds. Nikiforova [88] assessed some generalizations of conformal transformations for $ACR-$manifolds and discussed the invariance of six structure tensors (Kirichenko's tensors) under these transformations. In [59], Ignatochkina studied the transformation of the AH-manifold induced by a linear extension of $ACR-$manifolds having a conformal transformation. In 2018, Stepanova et al. [106] established certain theorems on the geometry of quasi-Sasakian manifolds as hypersurfaces of the Kählerian manifold. Rustanov et al. [99] regarded the contact formulae of Gray identities for $ACR-$manifolds of the class NC_{10}.

Siddiqui et al. [104] proved certain inequalities for bi-slant submanifolds of nearly trans-Sasakian manifolds and they found that the conditions of equality held. Additionally, they provided some related examples. Uddin et al. [110] studied semi-slant submanifolds and warped product semi-slant submanifolds of Kenmotsu manifolds. They obtained some characterizations and generalized the sharp inequality of the special form for such submanifolds and supported their work by providing significant examples. Hui et al. [57] explained using an example of the existence of special warped products and studied some inequalities of that warped product submanifolds.

On the other hand, Abood and Mohammed [4] studied the geometric properties of projective curvature tensor on $AG-$structure space of manifolds with nearly cosymplectic structures. Additionally, on the $AG-$structure space, Abood and Al-Hussaini [1] studied the geometry of conharmonic curvature tensors with $\Phi-$holomorphic sectional on manifolds having structures whose locally conformal transformation is an almost cosymplectic structure. In 2019, Blair [20] discussed his conjecture that a related metric to a given contact form for a contact manifold of dimension ≥ 5 must have some positive curvature. Abood and Al-Hussaini [2] determined the sufficient and necessary conditions for the manifold whose locally conformal transformation is almost cosymplectic manifold to be of constant curvature. Cabrera [29] proved the non-existence of 132 Chinea and González-Dávila classes for connected $ACR-$manifolds with a dimension of > 3. Zengin and Bektaş [119] determined various properties of W_2-curvature tensor on almost pseudo Ricci symmetric manifolds and explained using an example of the existence of these manifolds with certain conditions. Shanmukha and Venkatesha [103] studied the projective curvature tensor of generalized $(k;\mu)-$space forms. Mandal and Makhal [84] studied $*-$gradient Ricci solitons and $*-$Ricci solitons on 3-dimensional normal $ACR-$manifolds. Deszcz et al. [39] investigated hypersurfaces on space forms that satisfy certain conditions.

In 2020, Mohammed and Abood [86] constructed the generalized projective curvature tensor and studied its flatness on nearly cosymplectic manifolds. Additionally, they proved that the nearly cosymplectic manifold is a generalized Einstein manifold under suitable conditions, and conversely, Abood and Al-Hussaini [3] studied the flatness of the conharmonic curvature tensor on the locally conformal manifold for almost cosymplectic structure. They determined whether these manifolds are normal or $\eta-$Einstein manifolds.

The present thesis consists of five chapters. Chapter One includes the fundamental concepts related to our work, particularly, the construction of the smooth manifold, the $ACR-$manifold, the curvature tensors on the $ACR-$manifold, and the hypersurfaces on the AH-manifold.

In Chapter Two, we characterize the manifold of Kenmotsu type on the $AG-$structure space and we construct an example for the aforementioned manifold. Moreover, for the manifold of Kenmotsu type, we determine the Cartan's structure equations, the components of Riemannian curvature tensor, and the components of Ricci tensor, along with their applications on the $AG-$structure space.

Chapter Three is devoted to studying some curvature identities on the manifold of Kenmotsu type as an analog to Gray identities on the AH-manifold. Moreover, we determine the conditions that make the manifold of Kenmotsu type $GS-$space forms, and we discuss the covariant derivative of Riemannian curvature tensor for the manifold of Kenmotsu type. Thus, we investigate whether the manifold of Kenmotsu type is locally symmetric or generalized $\Phi-$recurrent.

Chapter Four discusses the generalized curvature tensor of the manifold of Kenmotsu type from several aspects, such as its components on the $AG-$structure space and its relationships with the other tensors. Moreover, we establish the manifold of Kenmotsu type as being a hypersurface of Hermitian manifold.

Chapter Five determines Cartan's structure equations of the $ACR-$ manifolds of the class C_{12} with examples on these manifolds of dimension 3. Moreover, we set down the components of Riemannian curvature tensor and Ricci tensor. Finally, the (κ, μ)-nullity conditions and Einstein situation of class C_{12} are investigated.

Chapter One

Basic Definitions and Theorems

Chapter 1

Basic Definitions and Theorems

This chapter focuses on the preliminaries closely related to the subject of our study in this thesis.

1.1 Smooth Manifolds

In this section, we recall the definitions related to the construction of smooth manifold.

Definition 1.1.1 [82] *A topological space M is called a topological $n-$manifold or a topological manifold of dimension n if M possesses the following properties:*

(i) *M is a Hausdorff space;*

(ii) *M is second-countable;*

(iii) *Every point of M has a neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.*

Definition 1.1.2 [82] *The pair (U, φ) is called a chart on a topological $n-$manifold M if $U \subseteq M$ is open and $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism.*

Definition 1.1.3 [82] *If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F : U \to V$ is said to be smooth if each of its component functions has continuous partial derivatives of all orders.*

Definition 1.1.4 [82] *Suppose that* $U \subseteq \mathbb{R}^n$ *and* $V \subseteq \mathbb{R}^m$ *are open subsets. A map* $F : U \to V$ *is called a diffeomorphism if* F *bijective, smooth and possesses the smooth inverse map.*

Definition 1.1.5 [82] *Two charts* $(U, \varphi), (V, \psi)$ *on a topological* $n-$*manifold* M *are called smoothly compatible if either* $U \cap V = \phi$ *or the map* $\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$ *is a diffeomorphism.*

Definition 1.1.6 [82] *A family of charts* $\{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ *on a topological* $n-$*manifold* M *is called an atlas if* $\bigcup_{\alpha\in\Lambda} U_\alpha = M$*. Moreover, a smooth atlas is an atlas* $\mathcal{A}$ *such that every two charts of it are smoothly compatible.*

Definition 1.1.7 [82] *A smooth atlas* $\mathcal{A}$ *on a topological* $n-$*manifold* M *is called a maximal or a complete if it is not properly contained in any other smooth atlas. The maximal smooth atlas* $\mathcal{A}$ *is called a smooth structure on* M*.*

Definition 1.1.8 [82] *The pair* $(M, \mathcal{A})$ *is called a smooth* $n-$*manifold or a smooth manifold of dimension* n *and denoted by* M^n *if* M *is a topological* $n-$*manifold and* $\mathcal{A}$ *is a smooth structure on* M*.*

Remark 1.1.1 The readers can return to the citation [82] for examples about the smooth manifolds.

1.2 Tensor Analysis

This section introduces a brief part of the tensor analysis that makes the reader surrounds by the subject.

Definition 1.2.1 [82] *Suppose that* M *is a smooth* n*-manifold,* k *is a nonnegative integer, and* $f : M \to \mathbb{R}^k$ *is any function. We say that* f *is a smooth function if for every* $p \in M$*, there exists a smooth chart* (U, φ) *for* M *whose domain contains* p *and such that the composite function* $f \circ \varphi^{-1}$ *is smooth on the open subset* $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$*. Moreover, the set of all smooth functions* $f : M \longrightarrow \mathbb{R}$ *is denoted by* $C^\infty(M)$*.*

Definition 1.2.2 [14] *A vector field on a smooth manifold M is an operator $X : C^\infty(M) \longrightarrow C^\infty(M)$ satisfies the following conditions:*

(i) $X(af + bg) = aX(f) + bX(g)$;

(ii) $X(fg) = X(f)g + fX(g)$,

for all $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$.

We denote $X(M)$ to set of all vector fields on the smooth manifold M.

Definition 1.2.3 [27] *A tangent vector on a smooth manifold M at the point $p \in M$ is a mapping $X_p : C^\infty(M) \longrightarrow \mathbb{R}$ satisfies the following conditions:*

(i) $X_p(af + bg) = aX_p(f) + bX_p(g)$;

(ii) $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$,

for all $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$. Moreover, the set of all tangent vectors on M at p is called a tangent space on M at p and denote by $T_p(M)$.

Remark 1.2.1 [27] We can also define the vector field $X \in X(M)$ as a map that assigns for every point $p \in M$ a tangent vector $X_p \in T_p(M)$, such that $X(f)(p) = X_p(f)$ for all $f \in C^\infty(M)$.

Definition 1.2.4 [27] *For every vector fields $X, Y \in X(M)$, we can define a new vector field on $X(M)$ by $[X, Y] = XY - YX$. The vector field $[X, Y]$ is called a product of X and Y or a Lie bracket of them. In addition, the tangent vector $[X, Y]_p$ is given by:*

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)); \quad f \in C^\infty(M); \quad p \in M.$$

Definition 1.2.5 [81] *Suppose that V is a real vector space of finite dimension. A tensor of type (r, s) on V is a map $F : \underbrace{V \times ... \times V}_{r\ copies} \times \underbrace{V^* \times ... \times V^*}_{s\ copies} \longrightarrow \mathbb{R}$ which is linear in each argument, where V^* is the dual space of V. Moreover, a tensor of type $(r, 0)$ on V is called $r-$form.*

Definition 1.2.6 [81] *Suppose that V is a real vector space of finite dimension. A multilinear map* $F : \underbrace{V \times ... \times V}_{r\ copies} \times \underbrace{V^* \times ... \times V^*}_{s\ copies} \longrightarrow V$ *is a tensor of type* $(r, s+1)$ *on* V.

Definition 1.2.7 [81] *Suppose that F and G are tensors on V of types (p, q) and (r, s) respectively. A tensor product $F \otimes G$ is a tensor of type $(p + r, q + s)$ on V defined by:*

$$\begin{aligned} &F \otimes G(X_1, ..., X_{p+r}, \theta^1, ..., \theta^{q+s}) \\ &\quad = F(X_1, ..., X_p, \theta^1, ..., \theta^q)G(X_{p+1}, ..., X_{p+r}, \theta^{q+1}, ..., \theta^{q+s}), \end{aligned}$$

where $X_1, ..., X_{p+r} \in V$ and $\theta^1, ..., \theta^{q+s} \in V^*$.

Remark 1.2.2 [72] We denote $\mathcal{T}^s_r(V)$ the set of all tensors of type (r, s) on V and $\mathcal{T}_r(V)$ to the set of all $r-$forms on V.

Definition 1.2.8 [81] *The trace or contraction operator* $tr : \mathcal{T}^{s+1}_{r+1}(V) \longrightarrow \mathcal{T}^s_r(V)$ *is defined by:*

$$\begin{aligned} tr(F)(X_1, ..., X_r, \theta^1, ..., \theta^s) &= F(X_1, ..., X_r, \cdot, \theta^1, ..., \theta^s, \cdot); \\ &= \sum_{k=1}^{n} F(X_1, ..., X_r, \xi_k, \theta^1, ..., \theta^s, \eta^k), \end{aligned}$$

where $F \in \mathcal{T}^{s+1}_{r+1}(V)$, $tr(F) \in \mathcal{T}^s_r(V)$, $X_i \in V$, $\theta^j \in V^*$ for all i and j, such that $\{\xi_1, ..., \xi_n\}$ is a basis of V with $\eta^k(\xi_l) = \delta^k_l$. Moreover, for any basis of the spaces $\mathcal{T}^s_r(V)$ and $\mathcal{T}^{s+1}_{r+1}(V)$, we can define the components of $tr(F)$ in this basis by

$$(trF)^{j_1...j_s}_{i_1...i_r} = F^{j_1...j_s k}_{i_1...i_r k},$$

where all indices take the values of $\{1, ..., n\}$.

Definition 1.2.9 [27] *A form $\tau \in \mathcal{T}_r(V)$ is called a symmetric if for all $1 \leq i, j \leq r$, we have*

$$\tau(X_1, ..., X_i, ..., X_j, ..., X_r) = \tau(X_1, ..., X_j, ..., X_i, ..., X_r).$$

Whereas, if for all $1 \leq i, j \leq r$, we have

$$\tau(X_1, ..., X_i, ..., X_j, ..., X_r) = -\tau(X_1, ..., X_j, ..., X_i, ..., X_r),$$

then τ is called a skew or antisymmetric or alternating.

Remark 1.2.3 [72] We denote $\Sigma_r(V)$ to the set of all symmetric $r-$forms on V and $\Lambda_r(V)$ to the set of all alternating $r-$forms on V. Moreover, the Grassmann algebra given by $\Lambda(V) = \bigoplus_{r=0}^{\infty} \Lambda_r(V)$.

Definition 1.2.10 [27] *The transformations on $\mathcal{T}_r(V)$, $Sym : \mathcal{T}_r(V) \longrightarrow \mathcal{T}_r(V)$ and $Alt : \mathcal{T}_r(V) \longrightarrow \mathcal{T}_r(V)$ are called respectively symmetrizing mapping and alternating mapping which are defined by the following formulas:*

$$Sym(F)(X_1, ..., X_r) = \frac{1}{r!} \sum_{\sigma \in S_r} F(X_{\sigma(1)}, ..., X_{\sigma(r)});$$
$$Alt(F)(X_1, ..., X_r) = \frac{1}{r!} \sum_{\sigma \in S_r} sgn(\sigma) F(X_{\sigma(1)}, ..., X_{\sigma(r)}),$$

where S_r is the group of all permutations of r letters and $sgn(\sigma)$ is $+1$ if σ even and -1 if σ odd.

Definition 1.2.11 [27] *Suppose that $\varphi \in \Lambda_r(V)$ and $\psi \in \Lambda_s(V)$. The exterior product $\varphi \wedge \psi \in \Lambda_{r+s}(V)$ is defined by:*

$$\varphi \wedge \psi = \frac{(r+s)!}{r!s!} Alt(\varphi \otimes \psi).$$

Remark 1.2.4 In this thesis, we take $V = X(M)$ or $T_p(M)$. So, the above symbols given by $Symbol(X(M)) \equiv Symbol(M)$ and $Symbol(T_p(M)) \equiv (Symbol)_p(M)$.

Lemma 1.2.1 [82] Suppose that M is a smooth manifold, then there exists a unique operator $d : \Lambda(M) \to \Lambda(M)$, satisfies the following properties:

1. d is linear over $\mathbb{R}$.

2. $d(\Lambda_k(M)) \subset \Lambda_{k+1}(M)$.

3. $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \, \omega_1 \wedge d\omega_2$, where $\omega_1 \in \Lambda_k(M)$; $\omega_2 \in \Lambda_l(M)$.

4. $d \circ d = 0$.

5. For $f \in C^\infty(M)$ and $X \in X(M)$, then $df(X) = X(f)$.

Lemma 1.2.2 [27] Suppose that $\omega \in \Lambda_1(M)$ and $X, Y \in X(M)$. Then the following equality holds:

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

1.3 Almost Contact Metric Manifolds

In this section, we recall the basic ideas about $ACR-$Manifolds and their characterization in $AG-$structure space.

Definition 1.3.1 [27] *A bilinear form $g : X(M) \times X(M) \longrightarrow \mathbb{R}$ is said to be a Riemannian metric on M if g is symmetric and positive definite.*

Definition 1.3.2 [27] *A smooth manifold M with the Riemannian metric g on M is called a Riemannian manifold and denote it by the pair (M, g) or (M^n, g) if M of dimension n.*

Example 1.3.1 [27] *An example on the Riemannian manifold is $(M = \mathbb{R}^n, g)$ such that $g(e_i, e_j) = \delta_{ij}$ and $e_j = \frac{\partial}{\partial x_j}, \quad i, j = 1, 2, ..., n$. In addition, for any $X \in X(M)$ we have $X = \sum_{i=1}^{n} \alpha_i \; e_i$ and $\alpha_i \in \mathbb{R}$.*

Definition 1.3.3 [15] *Suppose that (B, g_B) and (F, g_F) are Riemannian manifolds and $f : B \to B$ is a positive smooth function. The Riemannian manifold $(B \times F, g)$ is called a warped product manifold and denoted by $B \times_f F$, if $g(X, Y) = g_B(\pi_*(X), \pi_*(Y)) + f^2(\pi(p))g_F(\psi_*(X), \psi_*(Y))$ for all X, Y belong to the tangent space $T_p(M)$, where $M = B \times F$, and $\pi : M \to B$, $\psi : M \to F$ are projections. Moreover, π_* and ψ_* are the differential maps of π and ψ respectively.*

Definition 1.3.4 [72] *A Riemannian manifold (M^{2n+1}, g) is said to be an $ACR-$manifold if it is furnished by a structure of triple (ξ, η, Φ), where ξ is a characteristic vector field, η is a 1-form and Φ is a tensor of type (1, 1) over $X(M)$, such that*

$$\Phi(\xi) = 0; \quad \eta(\xi) = 1; \quad \eta \circ \Phi = 0; \quad \Phi^2 = -\mathrm{id} + \eta \otimes \xi;$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y); \quad \forall \; X, Y \in X(M).$$

We denote $(M^{2n+1}, \xi, \eta, \Phi, g)$ to the $ACR-$manifold.

Remark 1.3.1 [72] If $(M^{2n+1}, \xi, \eta, \Phi, g)$ is an $ACR-$manifold, then in $X(M)$ there are two complementary projections $l = -\Phi^2$ and $m = \eta \otimes \xi$ such that $X(M) = \mathcal{L} \oplus \mathcal{M}$, where $\mathcal{L} = Im(l) = Im(\Phi) = \ker(\eta)$ and $\mathcal{M} = Im(m) = \ker(\Phi)$. Then

$\dim(\mathcal{L}) = 2n$ and $\dim(\mathcal{M}) = 1$. On the other hand, we have an almost Hermitian structure on $\mathcal{L}$ with almost complex structure $J = \Phi|_{\mathcal{L}}$.

Now, we take the complexification $X^C(M) = \mathbb{C} \otimes X(M)$ of $X(M)$. That is every elment of $X^C(M)$ written as follows:

$$\sum_i z_i X_i; \quad z_i \in \mathbb{C}; \quad X_i \in X(M).$$

Therefore, $X^C(M) = D \oplus \overline{D} \oplus \mathcal{M}$, where D and $\overline{D}$ are given respectively by the image of the following complementary projections on $\mathcal{L}^C$:

$$\sigma = \frac{1}{2}(id - \sqrt{-1}\Phi^c); \quad \overline{\sigma} = \frac{1}{2}(id + \sqrt{-1}\Phi^c),$$

where $\Phi^c(\sum_i z_i X_i) = \sum_i z_i \Phi(X_i)$. Also, there are another two projections from $X^C(M)$ into D and $\overline{D}$ respectively defined by

$$\Pi = -\frac{1}{2}\{(\Phi^c)^2 + \sqrt{-1}\Phi^c\}; \quad \overline{\Pi} = \frac{1}{2}\{-(\Phi^c)^2 + \sqrt{-1}\Phi^c\}.$$

Kirichenko [72] defined a new frame $(p; \varepsilon_0 = \xi, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$ called A-frame from the standard frame $(p; e_0 = \xi, e_1, ..., e_n, e_{\hat{1}}, ..., e_{\hat{n}})$ which satisfies $g(e_i, e_j) = \delta_{ij}$, where $p \in M$, $\{e_0 = \xi, e_1, ..., e_n, e_{\hat{1}}, ..., e_{\hat{n}}\}$ is a basis of $X(M)$, $\varepsilon_a = \sqrt{2}\ \sigma(e_a)$, $\varepsilon_{\hat{a}} = \sqrt{2}\ \overline{\sigma}(e_a)$, $a = 1, 2, ..., n$, $\hat{a} = a + n$ and $i, j = 0, 1, ..., 2n$.

Definition 1.3.5 [72] *The set of all A-frames on $ACR-$manifold M^{2n+1} is called an $AG-$structure space of M^{2n+1}.*

Definition 1.3.6 [72] *For an $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$, the Riemannian metric g and the tensor Φ given by the following formulas on the $AG-$structure space:*

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & O & I_n \\ 0 & I_n & O \end{pmatrix}; \qquad (\Phi^i_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & O \\ 0 & O & -\sqrt{-1}I_n \end{pmatrix},$$

where I_n is $n \times n$ identity matrix.

Definition 1.3.7 [27] *Suppose that M is a smooth manifold. A mapping $\nabla : X(M) \times X(M) \longrightarrow X(M)$ defined by $\nabla : (X, Y) \longrightarrow \nabla_X Y$ is called a connection on M and it has the following properties:*

(1) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$;

(2) $\nabla_X(fY + gZ) = f\nabla_X Y + g\nabla_X Z + X(f)Y + X(g)Z$,

for all $f, g \in C^\infty(M)$ *and* $X, Y, Z \in X(M)$.

Lemma 1.3.1 [27] Suppose that $X, Y \in X(M)$ and ∇ is a connection on M. If $X = 0$, or $Y = 0$ then $\nabla_X Y = 0$.

Definition 1.3.8 [27] *Suppose that* (M, g) *is a Riemannian manifold. A Riemann connection on* M *is a connection which has the following properties:*

(i) $\nabla_X Y - \nabla_Y X = [X, Y]$;

(ii) $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

for all $X, Y, Z \in X(M)$.

Theorem 1.3.1 [27] (The Fundamental Theorem of Riemannian Geometry) *If* (M, g) *is a Riemannian manifold then there exists on* M *a unique connection which is Riemannian connection.*

Theorem 1.3.2 [72] *Suppose that* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *is an* $ACR-$*manifold,* ∇ *is the Riemannian connection on* M *and* θ *is the 1-form of* ∇ *on* $AG-$*structure space with components* θ^i_j. *Then on* $AG-$*structure space, we have:*

$$dg_{ij} - g_{ik}\ \theta^k_j - g_{kj}\ \theta^k_i = 0;$$
$$d\Phi^i_j - \Phi^i_k\ \theta^k_j + \Phi^k_j\ \theta^i_k = \Phi^i_{j,k}\ \omega^k,$$

where $i, j, k = 0, 1, ..., 2n$ *and* ω^k *are the dual of an A-frame (1-forms), with* $\omega^0 = \omega$.

Regarding the above theorem, we have the following corollary:

Corollary 1.3.1 [74] *Suppose that* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *is an* $ACR-$*manifold. The components of the 1-form* θ *on* $AG-$*structure space are given by*

$$\begin{array}{lllllllll}
\theta^a_{\hat{b}} & = & \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},k}\ \omega^k; & \theta^{\hat{a}}_b & = & -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,k}\ \omega^k; & \Phi^a_{b,k} & = & 0; \\
\theta^0_{\hat{a}} & = & \sqrt{-1}\Phi^0_{\hat{a},k}\ \omega^k; & \theta^0_a & = & -\sqrt{-1}\Phi^0_{a,k}\ \omega^k; & \Phi^{\hat{a}}_{\hat{b},k} & = & 0; \\
\theta^{\hat{a}}_0 & = & -\sqrt{-1}\Phi^{\hat{a}}_{0,k}\ \omega^k; & \theta^a_0 & = & \sqrt{-1}\Phi^a_{0,k}\ \omega^k; & \Phi^0_{0,k} & = & 0.
\end{array}$$

Moreover, $\theta^i_j + \theta^{\hat{j}}_{\hat{i}} = 0$; $\theta^0_0 = 0$; $\Phi^i_{j,k} = -\Phi^{\hat{j}}_{\hat{i},k}$, *where* $a, b = 1, 2, ..., n$, $\hat{a} = a + n$, $\hat{0} = 0$, *and* $\hat{\hat{i}} = i$.

1.4 Curvature Tensors

In this section, we recall the most important curvature tensors which are studied in this thesis.

Definition 1.4.1 [27] *Suppose that* (M, g) *is a Riemannian manifold. A tensor* $R : X(M) \times X(M) \times X(M) \longrightarrow X(M)$ *of type (3, 1) is said to be a Riemannian curvature tensor of type (3, 1) if*

$$R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z,$$

for all $X, Y, Z \in X(M)$, *where* ∇ *is the Riemannian connection on* M. *Moreover, the formula* $R(X, Y, Z, W) = g(R(Z, W)Y, X)$ *is a Riemannian curvature tensor of type (4, 0).*

Lemma 1.4.1 [81] Suppose that $(M^{2n+1}, \xi, \eta, \Phi, g)$ is an $ACR-$manifold and R its Riemannian curvature tensor of type (4, 0) with components R_{ijkl} on $AG-$structure space. Then R satisfies the following:

(1) $R_{ijkl} = -R_{jikl}$;

(2) $R_{ijkl} = -R_{ijlk}$;

(3) $R_{ijkl} = R_{klij}$;

(4) $R_{ijkl} + R_{iljk} + R_{iklj} = 0$,

where $i, j, k, l = 0, 1, ..., 2n$.

Theorem 1.4.1 [27] (Cartan's structure equations) *Suppose that* (M^n, g) *is the Riemannian manifold and* θ *is 1-form of the Riemannian connection, while* R *is the Riemannian curvature tensor of type (3, 1) and* $\{\omega^1, ..., \omega^n\}$ *is the dual frame to the basis frame* $\{E_1, ..., E_n\}$ *of* $X(M)$. *Then we have*

(1) $d\omega^i = -\theta^i_j \wedge \omega^j$; *(first group)*

(2) $d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \frac{1}{2}R^i_{jkl}\ \omega^k \wedge \omega^l$, *(second group)*

where θ^i_j *and* R^i_{jkl} *are the components of* θ *and* R *respectively, whereas,* $i, j, k, l = 1, ..., n$.

Definition 1.4.2 [5] *An* $ACR-$*manifold* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *is called a* $GS-$*space forms if there exist three functions* f_1, f_2, f_3 *on* M *such that*

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\Phi Z)\Phi Y - g(Y,\Phi Z)\Phi X + 2g(X,\Phi Y)\Phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}; \end{aligned}$$

for all $X, Y, Z \in X(M)$, *where* R *is the Riemannian curvature tensor of* M. *Such manifold is denoted by* $M(f_1, f_2, f_3)$.

Definition 1.4.3 [81] *A Ricci tensor of* $ACR-$*manifold is a tensor* r *of type (2, 0) which is the contracting of the Riemannian curvature tensor* R *of type (3, 1) as follows:*

$$r_{ij} = -R^k_{ijk} = -g^{kl}R_{kijl},$$

where r_{ij} *and* g^{kl} *are the components of* r *and* g^{-1} *on* $AG-$*structure space respectively. Whereas,* R^k_{ijk} *and* R_{kijl} *are the components of* R *of type (3, 1) and (4, 0) respectively. Moreover,* $r_{ij} = r_{ji}$ *that is* r *symmetric tensor.*

Definition 1.4.4 [63] $(M^{2n+1}, \xi, \eta, \Phi, g)$ *is called an* $\eta-$*Einstein manifold if its Ricci tensor* r *satisfies the equation*

$$r = \alpha g + \beta \eta \otimes \eta,$$

where α *and* β *are suitable smooth functions. Moreover, if* $\beta = 0$, *then* M *is called an Einstein manifold.*

Definition 1.4.5 [72] *The Ricci operator* Q *of* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *is a tensor of type (1, 1), such that* $r(X,Y) = g(QX,Y)$ *for all* $X, Y \in X(M)$, *where* r *is the Ricci tensor of type (2, 0).*

Definition 1.4.6 [72] *$(M^{2n+1}, \xi, \eta, \Phi, g)$ is said to have $\Phi-$invariant Ricci tensor if $\Phi \circ Q = Q \circ \Phi$.*

Lemma 1.4.2 [72] An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ has $\Phi-$invariant Ricci tensor if and only if, on $AG-$structure space, we have $Q_0^{\hat{a}} = Q_b^{\hat{a}} = 0$, or equivalently, $r_{a0} = r_{ab} = 0$, where $a, b = 1, 2, ..., n$ and $\hat{a} = a + n$.

Definition 1.4.7 [102] *The projective and concircular curvature tensors of type (4, 0) on $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ are defined by the following formulas respectively:*

$$P(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2n}\{g(X,Z)r(Y,W) - g(X,W)r(Y,Z)\};$$
$$\widetilde{C}(X,Y,Z,W) = R(X,Y,Z,W) - \frac{s}{2n(2n+1)}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\},$$

for all $X, Y, Z, W \in X(M)$, where $s = g^{ij}r_{ij}$, r and R are the scalar curvature, the Ricci tensor and the Riemannian curvature tensor, respectively.

We can rewrite the above tensors on $AG-$structure space as follows:

$$P_{ijkl} = R_{ijkl} - \frac{1}{2n}\{g_{ik}\ r_{jl} - g_{il}\ r_{jk}\}; \tag{1.4.1}$$
$$\widetilde{C}_{ijkl} = R_{ijkl} - \frac{s}{2n(2n+1)}\{g_{ik}\ g_{jl} - g_{il}\ g_{jk}\}, \tag{1.4.2}$$

where $i, j, k, l = 0, 1, ..., 2n$.

Definition 1.4.8 [38] *The conharmonic curvature tensor $\mathcal{H}$ of type (3, 1) on $ACR-$ manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is defined by the following formula:*

$$\begin{aligned}\mathcal{H}(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\{&r(Y,Z)X - r(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\},\end{aligned}$$

for all $X, Y, Z \in X(M)$, where r is the Ricci tensor and $r(X,Y) = g(QX,Y)$.

Definition 1.4.9 [102] *The generalized curvature tensor $\widetilde{B}$ of type (4, 0) on $ACR-$ manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is defined by the following formula:*

$$\begin{aligned}\widetilde{B}(X,Y,Z,W) = a_0R(X,Y,Z,W) &+ a_1\{g(X,Z)r(Y,W) - g(X,W)r(Y,Z) \\ &+ r(X,Z)g(Y,W) - r(X,W)g(Y,Z)\} \\ &+ 2a_2s\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\},\end{aligned}$$

for all $X, Y, Z, W \in X(M)$, where s is the scalar curvature and a_0, a_1, a_2 are scalars.

Definition 1.4.10 [75] *An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is called a locally symmetric, if $\nabla_X(R)(Y,Z)W = 0$, for all $X, Y, Z, W \in X(M)$, where R is the Riemann curvature tensor of M.*

Definition 1.4.11 [114] *An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is called a generalized $\Phi-$recurrent, if there are nonzero 1-forms ρ and λ such that the following hold for all $X, Y, Z, W \in X(M)$:*

$$\Phi^2(\nabla_X(R)(Y,Z)W) = \rho(X)R(Y,Z)W + \lambda(X)\{g(Z,W)Y - g(Y,W)Z\},$$

where R is the Riemannian curvature tensor of M.

On the other hand, Kirichenko [71] introduced six tensors called the first structure tensor B, ..., and sixth structure tensor G on $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ which are described as follow:

$$\begin{aligned}
B(X,Y) &= -\frac{1}{8}\{\Phi \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi^2 X) + \Phi \circ \nabla_{\Phi Y}(\Phi)(\Phi X) + \Phi^2 \circ \nabla_{\Phi Y}(\Phi)(\Phi^2 X) \\
&\quad - \Phi^2 \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi X)\}; \\
C(X,Y) &= -\frac{1}{8}\{-\Phi \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi^2 X) + \Phi \circ \nabla_{\Phi Y}(\Phi)(\Phi X) + \Phi^2 \circ \nabla_{\Phi Y}(\Phi)(\Phi^2 X) \\
&\quad + \Phi^2 \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi X)\}; \\
D(X) &= \frac{1}{4}\{2\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi - 2\Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi - \Phi \circ \nabla_{\xi}(\Phi)(\Phi^2 X) \\
&\quad + \Phi^2 \circ \nabla_{\xi}(\Phi)(\Phi X)\}; \\
E(X) &= -\frac{1}{2}\{\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi + \Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi\}; \\
F(X) &= \frac{1}{2}\{\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi - \Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi\}; \\
G &= \Phi \circ \nabla_{\xi}(\Phi)\xi.
\end{aligned}$$

The above structure tensors have components on $AG-$structure space of M respectively are described below.

$$
\begin{array}{lcllcl}
B^{ab}{}_{c} & = & -\frac{1}{2}\sqrt{-1}\Phi^{a}_{\hat{b},c}; & B_{ab}{}^{c} & = & \frac{1}{2}\sqrt{-1}\Phi^{\hat{a}}_{b,\hat{c}}; \\
B^{abc} & = & \frac{1}{2}\sqrt{-1}\Phi^{a}_{[\hat{b},\hat{c}]}; & B_{abc} & = & -\frac{1}{2}\sqrt{-1}\Phi^{\hat{a}}_{[b,c]}; \\
B^{ab} & = & \sqrt{-1}(\Phi^{a}_{0,\hat{b}} - \frac{1}{2}\Phi^{a}_{\hat{b},0}); & B_{ab} & = & -\sqrt{-1}(\Phi^{\hat{a}}_{0,b} - \frac{1}{2}\Phi^{\hat{a}}_{b,0}); \\
B^{a}{}_{b} & = & \sqrt{-1}\Phi^{a}_{0,b}; & B_{a}{}^{b} & = & -\sqrt{-1}\Phi^{\hat{a}}_{0,\hat{b}}; \\
C^{ab} & = & \sqrt{-1}\Phi^{0}_{[\hat{a},\hat{b}]}; & C_{ab} & = & -\sqrt{-1}\Phi^{0}_{[a,b]}; \\
C^{a} & = & -\sqrt{-1}\Phi^{0}_{\hat{a},0}; & C_{a} & = & \sqrt{-1}\Phi^{0}_{a,0};
\end{array}
$$

and all other components of these tensors being zero, where $a, b, c = 1, ..., n$, $\hat{a} = a+n$ and $[.,.]$ denotes the alternating operator of their indexes.

Remark 1.4.1 [72] If T is a tensor with components T^{i} then $T^{\hat{i}} = T_{i}$ and its complex conjugate is $\overline{T^{i}} = T^{\hat{i}}$.

Definition 1.4.12 [21] *A (κ, μ)-nullity distribution of $(M^{2n+1}, \xi, \eta, \Phi, g)$ with the Riemannian curvature tensor R and $(\kappa, \mu) \in \mathbb{R}^2$ is*

$$
\begin{aligned}
N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \{Z \in T_p(M) : R(X, Y)Z = \kappa[g(Y, Z)X \\
- g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\},
\end{aligned}
$$

for all $X, Y \in T_p(M)$, where $h = \frac{1}{2}\mathfrak{L}_{\xi}(\Phi)$ and $\mathfrak{L}$ is the Lie derivative. Moreover,

$$
h(X) = \frac{1}{2}\{\nabla_{\xi}(\Phi)X - \nabla_{\Phi X}\xi + \Phi(\nabla_X \xi)\}; \quad \forall\ X \in X(M).
$$

Definition 1.4.13 [75] *A $\Phi-$holomorphic sectional ($\Phi HS-$) curvature of $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ in the direction of X ($X \neq 0$; $X \in \ker(\eta)$) is defined by*

$$
H(X) = \frac{g(R(X, \Phi X)X, \Phi X)}{g(X, X)^2},
$$

where R is the Riemannian curvature tensor of M. Moreover, M is called of a pointwise constant $\Phi HS-$curvature if $H(X) = \gamma$, where $\gamma \in C^{\infty}(M)$ and γ does not depend on X.

Theorem 1.4.2 [75] *An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ has pointwise constant $\Phi HS-$curvature if and only if, on $AG-$ structure space, the Riemannian curvature* *tensor R of M satisfies*

$$R^{(a\ d)}_{(bc)} = \frac{\gamma}{2}\widetilde{\delta}^{ad}_{bc} = \frac{\gamma}{2}(\delta^a_b\delta^d_c + \delta^a_c\delta^d_b),$$

where $(\cdot\cdot)$ denotes the symmetric operator of the including indexes.

Definition 1.4.14 [63] *An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is called a Kenmotsu manifold if*

$$\nabla_X(\Phi)Y = -g(X, \Phi Y)\xi - \eta(Y)\Phi X; \quad \forall\ X, Y \in X(M),$$

where ∇ is the Riemannian connection on M.

Theorem 1.4.3 [72] *Suppose that $(M^{2n+1}, \xi, \eta, \Phi, g)$ is Kenmotsu manifold. Then the following are equivalent:*

(1) $\nabla_X(\Phi)Y = -g(X, \Phi Y)\xi - \eta(Y)\Phi X; \quad \forall\ X, Y \in X(M)$*;*

(2) $B = C = D = F = G = 0, \quad E = \mathrm{id}$*;*

(3) *On $AG-$structure space, we have the following:*

$$B^{ab}{}_c = B^{abc} = B^{ab} = 0;$$
$$B_{ab}{}^c = B_{abc} = B_{ab} = 0;$$
$$B^a{}_b = B_b{}^a = \delta^a_b;$$
$$C^{bc} = C_{bc} = C^b = C_b = 0.$$

Theorem 1.4.4 [95] *The $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is normal if and only if,*

$$\Phi(\nabla_X(\Phi)Y) - \nabla_{\Phi X}(\Phi)Y - (\nabla_X(\eta)Y)\xi = 0; \quad \forall\ X, Y \in X(M).$$

Remark 1.4.2 [34] The normal $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ has the following class:

$$C_3 \oplus C_4 \oplus C_5 \oplus C_6 \oplus C_7 \oplus C_8,$$

where

Classes	Defining conditions
C_3	$\nabla_X(\Omega)(Y,Z) - \nabla_{\Phi X}(\Omega)(\Phi Y, Z) = 0; \quad \delta\Omega = 0$
C_4	$\nabla_X(\Omega)(Y,Z) = -\frac{1}{2(n-1)}[g(\Phi X, \Phi Y)\delta\Omega(Z) - g(\Phi X, \Phi Z)\delta\Omega(Y)$ $-\Omega(X,Y)\delta\Omega(\Phi Z) + \Omega(X,Z)\delta\Omega(\Phi Y)]; \quad \delta\Omega(\xi) = 0$
C_5	$\nabla_X(\Omega)(Y,Z) = \frac{1}{2n}[\Omega(X,Z)\eta(Y) - \Omega(X,Y)\eta(Z)]\delta\eta$
C_6	$\nabla_X(\Omega)(Y,Z) = \frac{1}{2n}[g(X,Z)\eta(Y) - g(X,Y)\eta(Z)]\delta\Omega(\xi)$
C_7	$\nabla_X(\Omega)(Y,Z) = \eta(Z)\nabla_Y(\eta)\Phi X + \eta(Y)\nabla_{\Phi X}(\eta)Z; \quad \delta\Omega = 0$
C_8	$\nabla_X(\Omega)(Y,Z) = -\eta(Z)\nabla_Y(\eta)\Phi X + \eta(Y)\nabla_{\Phi X}(\eta)Z; \quad \delta\eta = 0$

for all $X, Y, Z \in X(M)$, *such that* $\Omega(X,Y) = g(X, \Phi Y)$, *and* $\delta\eta$, $\delta\Omega$ *are the coderivatives of* η *and* Ω *respectively.*

Theorem 1.4.5 [74] *Suppose that* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *is an* $ACR-$*manifold and* $\{\omega^0 = \omega, \omega^1, ..., \omega^{2n}\}$ *is the dual of A-frame on* M. *Then the first group of Cartan's structure equations on* $AG-$*structure space is given by*

(1) $d\omega^a = -\theta^a_b \wedge \omega^b + B^{ab}{}_c\ \omega^c \wedge \omega_b + B^{abc}\ \omega_b \wedge \omega_c + B^a{}_b\ \omega \wedge \omega^b + B^{ab}\ \omega \wedge \omega_b$;

(2) $d\omega_a = \theta^b_a \wedge \omega_b + B_{ab}{}^c\ \omega_c \wedge \omega^b + B_{abc}\ \omega^b \wedge \omega^c + B_a{}^b\ \omega \wedge \omega_b + B_{ab}\ \omega \wedge \omega^b$;

(3) $d\omega = C_{bc}\ \omega^b \wedge \omega^c + C^{bc}\ \omega_b \wedge \omega_c + C^b_c\ \omega^c \wedge \omega_b + C_b\ \omega \wedge \omega^b + C^b\ \omega \wedge \omega_b,$

where $C^b_c = B^b{}_c - B_c{}^b$.

Theorem 1.4.6 [72] (The Fundamental Theorem of Tensor Analysis) *Suppose that* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *is an* $ACR-$*manifold. If* T *is a tensor of type* (r,s) *on* M *and* ∇ *is the Riemannian connection on* M *with components* $T^{j_1...j_s}_{i_1...i_r}$ *and* θ^i_j *respectively on* $AG-$*structure space, then the following equality holds:*

$$\begin{aligned}\nabla T^{j_1...j_s}_{i_1...i_r} &= dT^{j_1...j_s}_{i_1...i_r} - T^{j_1...j_s}_{ki_2...i_r}\ \theta^k_{i_1} - ... - T^{j_1...j_s}_{i_1...i_{r-1}k}\ \theta^k_{i_r} \\ &+ T^{kj_2...j_s}_{i_1...i_r}\ \theta^{j_1}_k + ... + T^{j_1...j_{s-1}k}_{i_1...i_r}\ \theta^{j_s}_k = T^{j_1...j_s}_{i_2...i_r,k}\ \omega^k,\end{aligned}$$

where ∇T *is a tensor of type* $(r+1, s)$ *on* M *with components* $T^{j_1...j_s}_{i_1...i_r,k}$. *Note that all indexes run from 0 to* $2n$.

1.5 Hypersurface on Almost Hermitian Manifolds

In this section, we recall an almost contact structure on a hypersurface of AH-manifold and derive its basic relations.

Definition 1.5.1 [72] *A Riemannian manifold (N^{2n}, h) is said to be an almost Hermitian (AH-) manifold if it is furnished by a tensor J of type (1, 1) over $X(N)$, such that $J^2 = -\mathrm{id}; \quad h(JX, JY) = h(X, Y); \quad \forall\ X, Y \in X(N)$. Tensor J is called an almost complex structure.*

Now, if M^{2n-1} is a hypersurface of (N^{2n}, J, h) then we can define an almost contact structure on M as follows [13]:

$$\xi = J(n_0); \quad \eta(X) = h(\xi, X); \quad \Phi(X) = J \circ \Pi_3(X); \quad g(X, Y) = h(X, Y);$$

where $X, Y \in X(M)$, $(n_0)_p$ is a unit tangent vector which form a basis of

$$T_p^{\perp}(M) = \{X \in T_p(N) \mid h(X, Y) = 0; \quad \forall\ Y \in T_p(M)\},$$

for all $p \in M$, $\Pi_3 = id - \overline{n_3}$, $\overline{n_3} = \overline{n_1} + \overline{n_2}$, $\overline{n_2} = \eta \otimes \xi$, $\overline{n_1} = \zeta \otimes n_0$ and

$$\zeta(X) = h(n_0, X); \quad X \in X(N).$$

Theorem 1.5.1 [13] *An $ACR-$manifold $(M^{2n-1}, \xi, \eta, \Phi, g)$ which is a hypersurface of an AH-manifold (N^{2n}, J, h) has the following first family of the Cartan structure equations:*

$$\begin{aligned}
d\omega^\alpha &= \theta^\alpha_\beta \wedge \omega^\beta + C^{\alpha\beta}_\gamma\ \omega^\gamma \wedge \omega_\beta + C^{\alpha\beta\gamma}\ \omega_\beta \wedge \omega_\gamma + (\sqrt{2}C^{\alpha n}_\beta + \sqrt{-1}\sigma^\alpha_\beta)\omega^\beta \wedge \omega \\
&\quad + (\sqrt{-1}\sigma^{\alpha\beta} - \sqrt{2}\widetilde{C}^{n\alpha\beta} - \frac{1}{\sqrt{2}}C^{\alpha\beta}_n - \frac{1}{\sqrt{2}}\widetilde{C}^{\alpha\beta n})\omega_\beta \wedge \omega; \\
d\omega_\alpha &= -\theta^\beta_\alpha \wedge \omega_\beta + C^\gamma_{\alpha\beta}\ \omega_\gamma \wedge \omega^\beta + C_{\alpha\beta\gamma}\ \omega^\beta \wedge \omega^\gamma + (\sqrt{2}C^\beta_{\alpha n} - \sqrt{-1}\sigma^\beta_\alpha)\omega_\beta \wedge \omega \\
&\quad - (\sqrt{-1}\sigma_{\alpha\beta} + \sqrt{2}\widetilde{C}_{n\alpha\beta} + \frac{1}{\sqrt{2}}C^n_{\alpha\beta} + \frac{1}{\sqrt{2}}\widetilde{C}_{\alpha\beta n})\omega^\beta \wedge \omega; \\
d\omega &= \sqrt{2}C_{n\alpha\beta}\ \omega^\alpha \wedge \omega^\beta + \sqrt{2}C^{n\alpha\beta}\ \omega_\alpha \wedge \omega_\beta + (\sqrt{2}C^{n\alpha}_\beta - \sqrt{2}C^\alpha_{n\beta} \\
&\quad - 2\sqrt{-1}\sigma^\alpha_\beta)\omega^\beta \wedge \omega_\alpha + (\widetilde{C}_{n\beta n} + C^n_{n\beta} + \sqrt{-1}\sigma_{n\beta})\omega \wedge \omega^\beta \\
&\quad + (\widetilde{C}^{n\beta n} + C^{n\beta}_n - \sqrt{-1}\sigma^\beta_n)\omega \wedge \omega_\beta,
\end{aligned}$$

where

$$\begin{array}{rclcrcl}
\widetilde{C}^{abc} & = & \frac{\sqrt{-1}}{2} J^{a}_{\hat{b},\hat{c}} \ ; & & \widetilde{C}_{abc} & = & -\frac{\sqrt{-1}}{2} J^{\hat{a}}_{b,c} \ ; \\
C^{abc} & = & \widetilde{C}^{a[bc]} \ ; & & C_{abc} & = & \widetilde{C}_{a[bc]} \ ; \\
C^{ab}_{c} & = & -\frac{\sqrt{-1}}{2} J^{a}_{\hat{b},c} \ ; & & C^{c}_{ab} & = & \frac{\sqrt{-1}}{2} J^{\hat{a}}_{b,\hat{c}} \ ,
\end{array}$$

and $\sigma : X(M) \times X(M) \longrightarrow X(M)$ *is the second fundamental (quadratic) form which is symmetric* ($\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$) *such that* $\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ *with* $\widetilde{\nabla}$ *and* ∇ *are the Riemannian connections of* N *and* M *respectively (see [31]). Further,* $\alpha, \beta, \gamma = 1, 2, ..., n-1$*, while* $a, b, c = 1, 2, ..., n$ *and* $\omega = \omega^n = \omega_n$.

Chapter Two

The Geometry on the Manifold of Kenmotsu Type

Chapter 2

The Geometry on the Manifold of Kenmotsu Type

In this chapter, we generalize the Kenmotsu manifold to a new manifold called a manifold of Kenmotsu type. Moreover, the characterization identity, the Cartan's structure equations, and another discussion of the manifold of Kenmotsu type are written in more detail. In particular, we introduce a theoretical Physical application for the mentioned manifold.

2.1 The Manifold of Kenmotsu Type

In this section, we introduce a new class of $ACR-$manifolds with the class of Kenmotsu manifolds as a subclass. We called it a manifold of Kenmotsu type. Moreover, we discuss its characterization on $AG-$structure space.

Definition 2.1.1 *An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is said to be a manifold of Kenmotsu type if its Riemannian connection ∇ satisfies the following identity:*

$$\nabla_X(\Phi)Y - \nabla_{\Phi X}(\Phi)\Phi Y = -\eta(Y)\Phi X; \quad \forall\ X, Y \in X(M).$$

Now, the manifold of Kenmotsu type can be characterized on the $AG-$structure space by the following identity:

$$(\Phi^i_{j,k} - \Phi^i_{l,t}\ \Phi^t_k\ \Phi^l_j)X^k\ Y^j\ \varepsilon_i = -\eta_j\ \Phi^i_k\ X^k\ Y^j\ \varepsilon_i; \tag{2.1.1}$$

where $i, j, k, l, t = 0, a, \hat{a}$; $a = 1, ..., n$; $\hat{a} = a + n$. Then we can rewrite the identity (2.1.1) as follows:

$$\Phi^i_{j,k} - \Phi^i_{l,t}\ \Phi^t_k\ \Phi^l_j + \eta_j\ \Phi^i_k = 0. \tag{2.1.2}$$

Theorem 2.1.1 *On the $AG-$structure space, the manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ of Kenmotsu type verifies the following conditions:*

$$\Phi^i_{j,0} = \Phi^i_{a,b} = 0; \quad \Phi^i_{0,a} = -\sqrt{-1}\delta^i_a,$$

and their complex conjugate, where $i, j = 0, 1, ..., 2n$ and $a, b = 1, ..., n$.

Proof: Regarding the identity (2.1.2) and the Definition 1.3.6, we have $\Phi^i_{j,0} = 0$ if we put $k = 0$ in (2.1.2). Moreover, if we put $j = 0$ and $k = a$ in (2.1.2), we obtain $\Phi^i_{0,a} = -\sqrt{-1}\delta^i_a$, while if $j = a$ and $k = b$, we get that $\Phi^i_{a,b} = 0$. Notice that $\eta_j = g_{0j}$. □

Now, from the above theorem and the components of Kirichenko's tensors in chapter 1, we can deduce the following corollary:

Corollary 2.1.1 *If $(M^{2n+1}, \xi, \eta, \Phi, g)$ is the manifold of Kenmotsu type, then the conditions below are equivalents.*

(1) $\nabla_X(\Phi)Y - \nabla_{\Phi X}(\Phi)\Phi Y = -\eta(Y)\Phi X; \quad \forall\ X, Y \in X(M).$

(2) $C = D = F = G = 0;\ E = id.$

(3) *On the $AG-$structure space appears that*

$$B^{ab}{}_c = -B^{ba}{}_c; \qquad B_{ab}{}^c = -B_{ba}{}^c;$$
$$B^{abc} = B^{ab} = C^{ab} = C^a = 0;$$
$$B_{abc} = B_{ab} = C_{ab} = C_a = 0;$$
$$B^a{}_b = B_b{}^a = \delta^a_b.$$

Theorem 2.1.2 *There is no 3-dimensional manifold of Kenmotsu type.*

Proof: Suppose that M is a manifold of Kenmotsu type with dimension $2n+1 = 3$. Then $n = 1$ and $a = b = c = 1$. Moreover, the components of the first structure tensor B are $B^{ab}{}_c = B^{11}{}_1$ and $B_{ab}{}^c = B_{11}{}^1$. But from the Corollary 2.1.1; item (4), we have $B^{11}{}_{1} = -B^{11}{}_{1}$ and $B_{11}{}^{1} = -B_{11}{}^{1}$ and this implies that $B^{ab}{}_{c} = B_{ab}{}^{c} = 0$. Then according to the Theorem 1.4.3; item (4), we conclude that M is Kenmotsu manifold. $\square$

Next, we construct an interesting example for a manifold of Kenmotsu type which is not Kenmotsu.

Example 2.1.1 Suppose that $(M^5, \xi, \eta, \Phi, g)$ is an $ACR-$manifold of dimension 5, such that

$$M = \{(x, y, z, u, v) \in \mathbb{R}^5 : xzv \neq 0\};$$

and $\{e_0 = \xi, e_1, e_2, e_3, e_4\}$ is a basis of $X(M)$, given by

$$e_0 = \frac{\partial}{\partial v}; \quad e_1 = \exp(-v)\frac{\partial}{\partial x}; \quad e_2 = \exp(-(v + x + z))\frac{\partial}{\partial y}; \quad e_3 = \exp(-v)\frac{\partial}{\partial z};$$
$$e_4 = \exp(-(v + x + z))\frac{\partial}{\partial u}.$$

Then we have the following Lie brackets:

$$[e_1, e_0] = e_1; \quad [e_4, e_1] = \exp(-v)e_4; \quad [e_3, e_0] = e_3; \quad [e_4, e_0] = e_4;$$
$$[e_1, e_3] = 0; \quad [e_2, e_0] = e_2; \quad [e_2, e_3] = \exp(-v)e_2; \quad [e_2, e_4] = 0;$$
$$[e_2, e_1] = \exp(-v)e_2; \quad [e_4, e_3] = \exp(-v)e_4.$$

Moreover, if we have the following:

$$\Phi(e_0) = 0; \quad \Phi(e_1) = e_3; \quad \Phi(e_2) = e_4; \quad \Phi(e_3) = -e_1; \quad \Phi(e_4) = -e_2;$$
$$\eta(e_0) = 1; \quad \eta(e_1) = \eta(e_2) = \eta(e_3) = \eta(e_4) = 0;$$
$$g(e_i, e_j) = \begin{cases} 1, & i = j; \\ 0, & i \neq j; \end{cases}$$

where $i, j = 0, 1, 2, 3, 4$. Then from the Koszul's formula that given in [37] as follows:

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z])$$
$$+ g(Z, [X, Y]); \quad \forall\, X, Y, Z \in X(M).$$

We deduce the following values for the Riemannian connection ∇ of the metric g:

$$\begin{array}{llllll}
\nabla_{e_0}e_0 = 0; & \nabla_{e_0}e_1 = 0; & \nabla_{e_0}e_2 = 0; \\
\nabla_{e_1}e_0 = e_1; & \nabla_{e_1}e_1 = -e_0; & \nabla_{e_1}e_2 = 0; \\
\nabla_{e_2}e_0 = e_2; & \nabla_{e_2}e_1 = \exp(-v)e_2; & \nabla_{e_2}e_2 = -\exp(-v)(e_1+e_3)-e_0; \\
\nabla_{e_3}e_0 = e_3; & \nabla_{e_3}e_1 = 0; & \nabla_{e_3}e_2 = 0; \\
\nabla_{e_4}e_0 = e_4; & \nabla_{e_4}e_1 = \exp(-v)e_4; & \nabla_{e_4}e_2 = 0;
\end{array}$$

$$\begin{array}{ll}
\nabla_{e_0}e_3 = 0; & \nabla_{e_0}e_4 = 0; \\
\nabla_{e_1}e_3 = 0; & \nabla_{e_1}e_4 = 0; \\
\nabla_{e_2}e_3 = \exp(-v)e_2; & \nabla_{e_2}e_4 = 0; \\
\nabla_{e_3}e_3 = -e_0; & \nabla_{e_3}e_4 = 0; \\
\nabla_{e_4}e_3 = \exp(-v)e_4; & \nabla_{e_4}e_4 = -\exp(-v)(e_1+e_3)-e_0.
\end{array}$$

To clarify the above result, we apply Koszul's formula only for $\nabla_{e_2}e_3$ and then similarly for the rest.

$$2g(\nabla_{e_2}e_3, e_0) = -g(e_2, [e_3, e_0]) - g(e_3, [e_2, e_0]) + g(e_0, [e_2, e_3]) = 0;$$

$$2g(\nabla_{e_2}e_3, e_1) = -g(e_2, [e_3, e_1]) - g(e_3, [e_2, e_1]) + g(e_1, [e_2, e_3]) = 0;$$

$$2g(\nabla_{e_2}e_3, e_2) = -g(e_2, [e_3, e_2]) - g(e_3, [e_2, e_2]) + g(e_2, [e_2, e_3]) = 2g(\exp(-v)e_2, e_2);$$

$$2g(\nabla_{e_2}e_3, e_3) = -g(e_2, [e_3, e_3]) - g(e_3, [e_2, e_3]) + g(e_3, [e_2, e_3]) = 0;$$

$$2g(\nabla_{e_2}e_3, e_4) = -g(e_2, [e_3, e_4]) - g(e_3, [e_2, e_4]) + g(e_4, [e_2, e_3]) = 0.$$

Then $\nabla_{e_2}e_3 = \exp(-v)e_2$ and regarding the above discussion, we deduce that M is the manifold of Kenmotsu type, but M is not Kenmotsu manifold because if $X = e_4$ and $Y = e_1$, then

$$\begin{aligned}
\nabla_X(\Phi)Y &= \nabla_X\Phi(Y) - \Phi(\nabla_X Y) \\
&= \exp(-v)(e_2+e_4) \neq 0 = -g(X, \Phi Y)\xi - \eta(Y)\Phi X.
\end{aligned}$$

2.2 The Structure Equations of the Manifold of Kenmotsu Type

In this section, we establish Cartan's structure equations for the manifold of Kenmotsu type.

Theorem 2.2.1 *If $(M^{2n+1}, \xi, \eta, \Phi, g)$ is the manifold of Kenmotsu type with Riemannian connection ∇, then the components of the connection form θ on the AG−structure space are given by:*

$$\theta^a_{\hat{b}} = -B^{ab}{}_c\,\omega^c; \quad \theta^{\hat{a}}_b = \overline{\theta^a_{\hat{b}}}; \qquad \theta^0_0 = 0;$$
$$\theta^0_{\hat{a}} = -\omega^a; \qquad \theta^0_a = \overline{\theta^0_{\hat{a}}}; \quad \theta^i_j + \theta^{\hat{j}}_{\hat{i}} = 0.$$

where $\overline{B^{ab}{}_c} = B_{ab}{}^c$; $\overline{\omega^a} = \omega_a$; $\overline{\omega_a} = \omega^a$.

Proof: According to the Corollary 1.3.1, the components of Kirichenko's tensors and the Theorem 2.1.1, we have

$$\begin{aligned}
\theta^a_{\hat{b}} &= \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},k}\,\omega^k;\\
&= \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},0}\,\omega^0 + \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},c}\,\omega^c + \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},\hat{c}}\,\omega^{\hat{c}};\\
&= \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},c}\,\omega^c;\\
&= -B^{ab}{}_c\,\omega^c,
\end{aligned}$$

and similarly for the remaining components. □

Theorem 2.2.2 *The manifold of Kenmotsu type has the following Cartan's structure equations (first group):*

(1) $d\omega = 0$;

(2) $d\omega^a = -\theta^a_b \wedge \omega^b + B^{ab}{}_c\,\omega^c \wedge \omega_b - \omega^a \wedge \omega$;

(3) $d\omega_a = \theta^b_a \wedge \omega_b + B_{ab}{}^c\,\omega_c \wedge \omega^b - \omega_a \wedge \omega$.

Proof: Regarding the Theorem 1.4.5 and the Corollary 2.1.1; item (4), yield the results. □

Theorem 2.2.3 *The second family of Cartan's structure equations for the manifold of Kenmotsu type is given by:*

(1) $d\theta^a_b = -\theta^a_c \wedge \theta^c_b + A^{ad}_{bc}\,\omega^c \wedge \omega_d + A^a_{bcd}\,\omega^c \wedge \omega^d + A^{acd}_b\,\omega_c \wedge \omega_d$;

(2) $dB^{ab}{}_c = B^{ab}{}_d\,\theta^d_c - B^{db}{}_c\,\theta^a_d - B^{ad}{}_c\,\theta^b_d + B^{ab}{}_{cd}\,\omega^d + B^{abd}{}_c\,\omega_d - B^{ab}{}_c\,\omega$;

(3) $dB_{ab}{}^{c} = -B_{ab}{}^{d}\,\theta^c_d + B_{db}{}^{c}\,\theta^d_a + B_{ad}{}^{c}\,\theta^d_b + B_{ab}{}^{cd}\,\omega_d + B_{abd}{}^{c}\,\omega^d - B_{ab}{}^{c}\,\omega,$

where $A^a_{[bcd]} = A^{[bcd]}_a = 0$ *and*

$$A^{ad}_{[bc]} - B^{ad}{}_{[cb]} - B^{ah}{}_{[b}\,B_{|h|c]}{}^{d} = 0; \quad A^b_{acd} + B_{a[cd]}{}^{b} - B_{a[c}{}^{h}B_{|h|d]}{}^{b} = 0;$$
$$A^{[bc]}_{ad} + B_{ad}{}^{[cb]} + B_{ah}{}^{[b}\,B^{|h|c]}{}_{d} = 0; \quad A^{acd}_b - B^{a[cd]}{}_{b} + B^{a[c}{}_{h}B^{|h|d]}{}_{b} = 0.$$

Proof: By applying the exterior differentiation operator d on the Theorem 2.2.2; item (2) and using the Lemma 1.2.1, we get

$$\begin{aligned}0 = &-d\theta^a_b \wedge \omega^b + \theta^a_b \wedge (-\theta^b_c \wedge \omega^c + B^{bc}{}_{d}\,\omega^d \wedge \omega_c - \omega^b \wedge \omega)\\ &+ dB^{ab}{}_{c} \wedge \omega^c \wedge \omega_b + B^{ab}{}_{c}(-\theta^c_d \wedge \omega^d + B^{cd}{}_{h}\,\omega^h \wedge \omega_d - \omega^c \wedge \omega) \wedge \omega_b\\ &- B^{ab}{}_{c}\,\omega^c \wedge (\theta^d_b \wedge \omega_d + B_{bd}{}^{h}\,\omega_h \wedge \omega^d - \omega_b \wedge \omega)\\ &- (-\theta^a_b \wedge \omega^b + B^{ab}{}_{c}\,\omega^c \wedge \omega_b - \omega^a \wedge \omega) \wedge \omega.\end{aligned}$$

Then after changing the indexes of some terms, we obtain the following:

$$\begin{aligned}0 = &-(d\theta^a_b + \theta^a_c \wedge \theta^c_b) \wedge \omega^b - B^{a[c}{}_{h}\,B^{|h|d]}{}_{b}\,\omega^b \wedge \omega_c \wedge \omega_d\\ &+ (dB^{ab}{}_{c} + B^{db}{}_{c}\,\theta^a_d + B^{ad}{}_{c}\,\wedge \theta^b_d - B^{ab}{}_{d}\,\theta^d_c) \wedge \omega^c \wedge \omega_b\\ &- B^{ab}{}_{c}\,\omega^c \wedge \omega \wedge \omega_b + B^{ah}{}_{[b}\,B_{|h|c]}{}^{d}\,\omega^b \wedge \omega^c \wedge \omega_d.\end{aligned} \tag{2.2.3}$$

Since $d\theta^a_b + \theta^a_c \wedge \theta^c_b$ is a 2-form then it can be written according to the family of basis for 2-forms on $AG-$structure space:

$$\{\theta^c_d \wedge \theta^f_h, \theta^c_d \wedge \omega^h, \theta^c_d \wedge \omega_h, \theta^c_d \wedge \omega, \omega^c \wedge \omega^d, \omega^c \wedge \omega_d, \omega^c \wedge \omega, \omega_c \wedge \omega_d, \omega_c \wedge \omega\}$$

as follows:

$$\begin{aligned}d\theta^a_b + \theta^a_c \wedge \theta^c_b = &A^{adh}_{bcf}\,\theta^c_d \wedge \theta^f_h + A^{ad}_{bch}\,\theta^c_d \wedge \omega^h + A^{adh}_{bc}\,\theta^c_d \wedge \omega_h\\ &+ A^{ad}_{bc0}\,\theta^c_d \wedge \omega + A^a_{bcd}\,\omega^c \wedge \omega^d + A^{ad}_{bc}\,\omega^c \wedge \omega_d\\ &+ A^a_{bc0}\,\omega^c \wedge \omega + A^{acd}_b\,\omega_c \wedge \omega_d + A^{ac0}_b\,\omega_c \wedge \omega,\end{aligned}$$

where $\{A^{adh}_{bcf}, A^{ad}_{bch}, A^{adh}_{bc}, A^{ad}_{bc0}, A^a_{bcd}, A^{ad}_{bc}, A^a_{bc0}, A^{acd}_b, A^{ac0}_b\}$ are suitable family of smooth functions and all indexes are run from 1 to n.

In the same manner, $dB^{ab}{}_{c} + B^{db}{}_{c}\,\theta^a_d + B^{ad}{}_{c}\,\theta^b_d - B^{ab}{}_{d}\,\theta^d_c$ can be written according to the 1-forms family of basis on $AG-$structure space $\{\theta^d_h, \omega^d, \omega_d, \omega\}$ as follows:

$$dB^{ab}{}_{c} + B^{db}{}_{c}\,\theta^a_d + B^{ad}{}_{c}\,\theta^b_d - B^{ab}{}_{d}\,\theta^d_c = B^{abh}{}_{cd}\,\theta^d_h + B^{ab}{}_{cd}\,\omega^d + B^{abd}{}_{c}\,\omega_d + B^{ab0}{}_{c}\,\omega,$$

where also $\{B^{abh}{}_{cd}, B^{ab}{}_{cd}, B^{abd}{}_{c}, B^{ab0}{}_{c}\}$ are suitable family of smooth functions. Then the equation (2.2.3) becomes

$$
\begin{aligned}
&- A^{adh}_{bcf}\ \theta^c_d \wedge \theta^f_h \wedge \omega^b - A^{ad}_{[b|c|h]}\ \theta^c_d \wedge \omega^h \wedge \omega^b - A^{adh}_{bc}\ \theta^c_d \wedge \omega_h \wedge \omega^b - A^{ad}_{bc0}\ \theta^c_d \wedge \omega \wedge \omega^b \\
&- A^a_{[bcd]}\ \omega^c \wedge \omega^d \wedge \omega^b - A^{ad}_{[bc]}\ \omega^c \wedge \omega_d \wedge \omega^b - A^a_{[bc]0}\ \omega^c \wedge \omega \wedge \omega^b - A^{acd}_b\ \omega_c \wedge \omega_d \wedge \omega^b \\
&- A^{ac0}_b\ \omega_c \wedge \omega \wedge \omega^b + B^{abh}{}_{cd}\ \theta^d_h \wedge \omega^c \wedge \omega_b + B^{ab}{}_{[cd]}\ \omega^d \wedge \omega^c \wedge \omega_b \\
&+ B^{ab0}{}_c\ \omega \wedge \omega^c \wedge \omega_b - B^{a[c}{}_h\ B^{|h|d]}_b\ \omega^b \wedge \omega_c \wedge \omega_d - B^{ab}{}_c\ \omega^c \wedge \omega \wedge \omega_b \\
&+ B^{ah}{}_{[b}\ B_{|h|c]}{}^d\ \omega^b \wedge \omega^c \wedge \omega_d + B^{a[bd]}_c\ \omega_d \wedge \omega^c \wedge \omega_b = 0.
\end{aligned}
$$

Then from the above discussion, we get

$$
\begin{aligned}
A^{adh}_{bcf} = A^{ad}_{[b|c|h]} = A^{ad}_{bc0} = A^a_{[bcd]} &= 0; \\
A^{ad}_{[bc]} - B^{ad}{}_{[cb]} - B^{ah}{}_{[b}\ B_{|h|c]}{}^d &= 0; \\
A^{acd}_b - B^{a[cd]}_b + B^{a[c}{}_h B^{|h|d]}_b &= 0; \\
A^{ac0}_b - B^{ac0}{}_b - B^{ac}{}_b &= 0; \\
A^{adh}_{bc} + B^{ahd}{}_{bc} = 0; \quad A^a_{[bc]0} &= 0,
\end{aligned}
\tag{2.2.4}
$$

where [.|.|.] denotes the alternating operator of its indexes except |.|, while [..] just the alternating operator of its indexes.

Now, applying the same argument above on the Theorem 2.2.2; item (3), we have

$$
\begin{aligned}
0 = d\theta^b_a \wedge \omega_b &- \theta^b_a \wedge (\theta^d_b \wedge \omega_d + B_{bd}{}^h\ \omega_h \wedge \omega^d - \omega_b \wedge \omega) \\
&+ dB_{ab}{}^c \wedge \omega_c \wedge \omega^b + B_{ab}{}^c\ (\theta^d_c \wedge \omega_d + B_{cd}{}^h\ \omega_h \wedge \omega^d - \omega_c \wedge \omega) \wedge \omega^b \\
&- B_{ab}{}^c\ \omega_c \wedge (-\theta^b_d \wedge \omega^d + B^{bd}{}_h\ \omega^h \wedge \omega_d - \omega^b \wedge \omega) \\
&- (\theta^b_a \wedge \omega_b + B_{ab}{}^c\ \omega_c \wedge \omega^b - \omega_a \wedge \omega) \wedge \omega.
\end{aligned}
$$

Rearrangement the above equation and interchanging some indexes, we get

$$
\begin{aligned}
0 = (d\theta^b_a - \theta^d_a \wedge \theta^b_d) \wedge \omega_b &- B_{ah}{}^{[c}\ B^{|h|b]}{}_d\ \omega_b \wedge \omega_c \wedge \omega^d \\
&+ (dB_{ab}{}^c - B_{db}{}^c\ \theta^d_a - B_{ad}{}^c\ \theta^d_b + B_{ab}{}^d\ \theta^c_d) \wedge \omega_c \wedge \omega^b \\
&+ B_{ab}{}^c\ \omega_c \wedge \omega^b \wedge \omega + B_{a[b}{}^h\ B_{|h|c]}{}^d\ \omega_d \wedge \omega^c \wedge \omega^b.
\end{aligned}
\tag{2.2.5}
$$

Since $\theta^b_a = -\overline{\theta^a_b}$ and $B_{ab}{}^c = \overline{B^{ab}{}_c}$, then from the equation (2.2.4), we get

$$
\begin{aligned}
d\theta^b_a - \theta^d_a \wedge \theta^b_d = A^{bch}_{ad}\ \theta^d_c \wedge \omega_h &+ A^{bc}_{adh}\ \theta^d_c \wedge \omega^h + A^{bcd}_a\ \omega_c \wedge \omega_d + A^{bc}_{ad}\ \omega_c \wedge \omega^d \\
&+ A^{bc0}_a\ \omega_c \wedge \omega + A^b_{acd}\ \omega^c \wedge \omega^d + A^b_{ac0}\ \omega^c \wedge \omega,
\end{aligned}
$$

and

$$dB_{ab}{}^{c} - B_{db}{}^{c}\,\theta_a^d - B_{ad}{}^{c}\,\theta_b^d + B_{ab}{}^{d}\,\theta_d^c = B_{abh}{}^{cd}\,\theta_d^h + B_{ab}{}^{cd}\,\omega_d + B_{abd}{}^{c}\,\omega^d + B_{ab0}{}^{c}\,\omega.$$

If we substitute the above equations in the equation (2.2.5), then we establish the following:

$$\begin{aligned} 0 = {} & A_{ad}^{[b|c|h]}\,\theta_c^d \wedge \omega_h \wedge \omega_b + A_{adh}^{bc}\,\theta_c^d \wedge \omega^h \wedge \omega_b + A_a^{[bcd]}\,\omega_c \wedge \omega_d \wedge \omega_b \\ & + A_{ad}^{[bc]}\,\omega_c \wedge \omega^d \wedge \omega_b + A_a^{[bc]0}\,\omega_c \wedge \omega \wedge \omega_b + A_{acd}^{b}\,\omega^c \wedge \omega^d \wedge \omega_b \\ & + A_{ac0}^{b}\,\omega^c \wedge \omega \wedge \omega_b + B_{abh}{}^{cd}\,\theta_d^h \wedge \omega_c \wedge \omega^b + B_{ab}{}^{[cd]}\,\omega_d \wedge \omega_c \wedge \omega^b \\ & + B_{a[bd]}{}^{c}\,\omega^d \wedge \omega_c \wedge \omega^b + B_{ab0}{}^{c}\,\omega \wedge \omega_c \wedge \omega^b + B_{ab}{}^{c}\,\omega_c \wedge \omega^b \wedge \omega \\ & + B_{a[b}{}^{h}\,B_{|h|c]}{}^{d}\,\omega_d \wedge \omega^c \wedge \omega^b - B_{ah}{}^{[c}\,B^{|h|b]}{}_{d}\,\omega_b \wedge \omega_c \wedge \omega^d. \end{aligned}$$

So, the above equation produce the following relations:

$$\begin{aligned} A_{ad}^{[b|c|h]} = A_a^{[bcd]} = 0; \quad A_{adh}^{bc} - B_{ahd}{}^{bc} = 0; \\ A_{ad}^{[bc]} + B_{ad}{}^{[cb]} + B_{ah}{}^{[b}\,B^{|h|c]}{}_{d} = 0; \\ A_{acd}^{b} + B_{a[cd]}{}^{b} - B_{a[c}{}^{h}B_{|h|d]}{}^{b} = 0; \\ A_{ac0}^{b} + B_{ac0}{}^{b} + B_{ac}{}^{b} = 0; \quad A_a^{[bc]0} = 0. \end{aligned} \tag{2.2.6}$$

Regarding Corollary 2.1.1; item (4), we have $B^{[ab]}{}_{c} = B^{ab}{}_{c}$ and $B_{[ab]}{}^{c} = B_{ab}{}^{c}$. Therefore, all the components of their exterior differentiation have the same property. Then from this fact, the equations (2.2.4) and (2.2.6), we deduce the required results. □

2.3 Curvature Tensors Components on Manifold of Kenmotsu Type

This section establishes the components of Riemannian Curvature tensor and Ricci tensor on the $AG-$structure space for the manifold of Kenmotsu type.

Theorem 2.3.1 *On the $AG-$structure space, the components of Riemannian curvature tensor R for the manifold of Kenmotsu type are given by*

(1) $R^a_{0c0} = -\delta^a_c$;

(2) $R^a_{bcd} = 2A^a_{bcd}$;

(3) $R^a_{bc\hat{d}} = A^{ad}_{bc} - B^{ah}{}_c\, B_{bh}{}^d - \delta^a_c\ \delta^d_b$;

(4) $R^a_{\hat{b}cd} = 2(B^{ab}{}_{[cd]} - \delta^a_{[c}\ \delta^b_{d]})$;

(5) $R^a_{\hat{b}c\hat{d}} = B^{abd}{}_c - B^{ab}{}_h\ B^{hd}{}_c$,

and the other components are identical to zero or given by the properties of R in Lemma 1.4.1*, or the complex conjugate to the above components.*

Proof: Regarding the Cartan's structure equations (second group) in the Theorem 1.4.1; item (2), we conclude the following:

$$d\theta^i_j + \theta^i_k \wedge \theta^k_j = \frac{1}{2} R^i_{jkl}\ \omega^k \wedge \omega^l;$$

$$d\theta^i_j + \theta^i_0 \wedge \theta^0_j + \theta^i_c \wedge \theta^c_j + \theta^i_{\hat{c}} \wedge \theta^{\hat{c}}_j = R^i_{jc0}\ \omega^c \wedge \omega + R^i_{j\hat{c}0}\ \omega_c \wedge \omega + \frac{1}{2} R^i_{jcd}\ \omega^c \wedge \omega^d$$
$$+ R^i_{jc\hat{d}}\ \omega^c \wedge \omega_d + \frac{1}{2} R^i_{j\hat{c}\hat{d}}\ \omega_c \wedge \omega_d. \qquad (2.3.7)$$

Moreover, we take $i, j, k, l = 0, 1, ..., 2n$ and $a, b, c, d = 1, 2, ..., n$. So, there are several cases regarding the values of $i, j = 0, a, \hat{a}$. These cases are designing as the following:

Case (1). If we put $i = j = 0$ in the equation (2.3.7), then the Theorem 2.2.1 produces the following:

$$R^0_{0c0} = R^0_{0\hat{c}0} = R^0_{0cd} = R^0_{0c\hat{d}} = R^0_{0\hat{c}\hat{d}} = 0.$$

Case (2). If we set $i = a$ and $j = 0$ in the equation (2.3.7), then the Theorem 2.2.1 gives us the following:

$$d\omega^a + \theta^a_c \wedge \omega^c - B^{ac}{}_d\ \omega^d \wedge \omega_c = R^a_{0c0}\ \omega^c \wedge \omega + R^a_{0\hat{c}0}\ \omega_c \wedge \omega + \frac{1}{2} R^a_{0cd}\ \omega^c \wedge \omega^d$$
$$+ R^a_{0c\hat{d}}\ \omega^c \wedge \omega_d + \frac{1}{2} R^a_{0\hat{c}\hat{d}}\ \omega_c \wedge \omega_d.$$

Regarding the Theorem 2.2.2; item (2), then the above equation reduces to the following:

$$-\delta^a_c\ \omega^c \wedge \omega = R^a_{0c0}\ \omega^c \wedge \omega + R^a_{0\hat{c}0}\ \omega_c \wedge \omega + \frac{1}{2} R^a_{0cd}\ \omega^c \wedge \omega^d$$
$$+ R^a_{0c\hat{d}}\ \omega^c \wedge \omega_d + \frac{1}{2} R^a_{0\hat{c}\hat{d}}\ \omega_c \wedge \omega_d.$$

So, we have

$$R^a_{0c0} = -\delta^a_c; \quad R^a_{0\hat{c}0} = R^a_{0cd} = R^a_{0c\hat{d}} = R^a_{0\hat{c}\hat{d}} = 0.$$

Case (3). If we assign $i = a$ and $j = b$ in the equation (2.3.7) and using the Theorem 2.2.1, then we conclude that

$$d\theta^a_b - \omega^a \wedge \omega_b + \theta^a_c \wedge \theta^c_b + B^{ac}{}_d\, B_{cb}{}^h\, \omega^d \wedge \omega_h = R^a_{bc0}\, \omega^c \wedge \omega + R^a_{b\hat{c}0}\, \omega_c \wedge \omega$$
$$+ \frac{1}{2} R^a_{bcd}\, \omega^c \wedge \omega^d + R^a_{bc\hat{d}}\, \omega^c \wedge \omega_d + \frac{1}{2} R^a_{b\hat{c}\hat{d}}\, \omega_c \wedge \omega_d.$$

Interchanging the indexes of the fourth term on the left side for the above equation by the permutation (chd), then it can be written as follows:

$$d\theta^a_b - \delta^a_c\, \delta^d_b\, \omega^c \wedge \omega_d + \theta^a_c \wedge \theta^c_b - B^{ah}{}_c\, B_{bh}{}^d\, \omega^c \wedge \omega_d = R^a_{bc0}\, \omega^c \wedge \omega + R^a_{b\hat{c}0}\, \omega_c \wedge \omega$$
$$+ \frac{1}{2} R^a_{bcd}\, \omega^c \wedge \omega^d + R^a_{bc\hat{d}}\, \omega^c \wedge \omega_d + \frac{1}{2} R^a_{b\hat{c}\hat{d}}\, \omega_c \wedge \omega_d.$$

Then taking into account the Theorem 2.2.3; item (1), we have

$$(A^{ad}_{bc} - B^{ah}{}_c\, B_{bh}{}^d \;-\delta^a_c\, \delta^d_b\,)\omega^c \wedge \omega_d + A^a_{bcd}\, \omega^c \wedge \omega^d + A^{acd}_b\, \omega_c \wedge \omega_d = R^a_{bc0}\, \omega^c \wedge \omega$$
$$+ R^a_{b\hat{c}0}\, \omega_c \wedge \omega + \frac{1}{2} R^a_{bcd}\, \omega^c \wedge \omega^d + R^a_{bc\hat{d}}\, \omega^c \wedge \omega_d + \frac{1}{2} R^a_{b\hat{c}\hat{d}}\, \omega_c \wedge \omega_d.$$

Thus we conclude that

$$R^a_{bc0} = R^a_{b\hat{c}0} = 0; \quad R^a_{bcd} = 2A^a_{bcd}; \quad R^a_{bc\hat{d}} = A^{ad}_{bc} - B^{ah}{}_c\, B_{bh}{}^d \;- \delta^a_c\, \delta^d_b;$$

$$R^a_{b\hat{c}\hat{d}} = 2A^{acd}_b.$$

Case (4). If we determine $i = a$, and $j = \hat{b}$ in the equation (2.3.7) and applying the Theorem 2.2.1, we get

$$d(-B^{ab}{}_d\, \omega^d) - \delta^a_{[c}\, \delta^b_{d]}\, \omega^c \wedge \omega^d - B^{cb}{}_d\, \theta^a_c \wedge \omega^d + B^{ac}{}_d\, \omega^d \wedge \theta^b_c = R^a_{\hat{b}c0}\, \omega^c \wedge \omega$$
$$+ R^a_{\hat{b}\hat{c}0}\, \omega_c \wedge \omega + \frac{1}{2} R^a_{\hat{b}cd}\, \omega^c \wedge \omega^d + R^a_{\hat{b}c\hat{d}}\, \omega^c \wedge \omega_d + \frac{1}{2} R^a_{\hat{b}\hat{c}\hat{d}}\, \omega_c \wedge \omega_d.$$

Regarding the Lemma 1.2.1; item (3), then the above equation becomes

$$- dB^{ab}{}_d \wedge \omega^d - B^{ab}{}_c\, d\omega^c - \delta^a_{[c}\, \delta^b_{d]}\, \omega^c \wedge \omega^d - B^{cb}{}_d\, \theta^a_c \wedge \omega^d + B^{ac}{}_d\, \omega^d \wedge \theta^b_c$$
$$= R^a_{\hat{b}c0}\, \omega^c \wedge \omega + R^a_{\hat{b}\hat{c}0}\, \omega_c \wedge \omega + \frac{1}{2} R^a_{\hat{b}cd}\, \omega^c \wedge \omega^d + R^a_{\hat{b}c\hat{d}}\, \omega^c \wedge \omega_d + \frac{1}{2} R^a_{\hat{b}\hat{c}\hat{d}}\, \omega_c \wedge \omega_d.$$

Then according to the Theorem 2.2.2; item (2) and the Theorem 2.2.3; item (2), we have

$$d\omega^c = -\theta^c_d \wedge \omega^d + B^{cd}{}_h\, \omega^h \wedge \omega_d - \omega^c \wedge \omega;$$
$$dB^{ab}{}_d = B^{ab}{}_c\, \theta^c_d - B^{cb}{}_d\, \theta^a_c - B^{ac}{}_d\, \theta^b_c + B^{ab}{}_{dc}\, \omega^c + B^{abc}{}_d\, \omega_c - B^{ab}{}_d\, \omega.$$

So, the substitution of them in the above equation and interchanging the indices of certain terms as needed to get that

$$(B^{ab}{}_{[cd]} - \delta^a_{[c}\, \delta^b_{d]})\omega^c \wedge \omega^d + (B^{abd}{}_c - B^{ab}{}_h\, B^{hd}{}_c)\omega^c \wedge \omega_d = R^a_{\hat{b}c0}\, \omega^c \wedge \omega$$
$$+ R^a_{\hat{b}\hat{c}0}\, \omega_c \wedge \omega + \frac{1}{2}R^a_{\hat{b}cd}\, \omega^c \wedge \omega^d + R^a_{\hat{b}c\hat{d}}\, \omega^c \wedge \omega_d + \frac{1}{2}R^a_{\hat{b}\hat{c}\hat{d}}\, \omega_c \wedge \omega_d.$$

So, we get

$$R^a_{\hat{b}c0} = R^a_{\hat{b}\hat{c}0} = R^a_{\hat{b}\hat{c}\hat{d}} = 0; \quad R^a_{\hat{b}cd} = 2(B^{ab}{}_{[cd]} - \delta^a_{[c}\, \delta^b_{d]}); \quad R^a_{\hat{b}c\hat{d}} = B^{abd}{}_c - B^{ab}{}_h\, B^{hd}{}_c.$$

This complete the proof. □

Corollary 2.3.1 *The Riemannian curvature tensor R of the manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ satisfies the following:*

(1) $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$*;*

(2) $R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X$*,*

for all $X, Y \in X(M)$.

Proof: On the A-frame $(p; \xi, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$ of $AG-$structure space and regarding the Theorem 2.3.1, we have

$$\begin{aligned} R(X,Y)\xi &= R^i_{0jk}\, X^j Y^k \varepsilon_i; \\ &= R^i_{00b}\, X^0 Y^b \varepsilon_i + R^i_{00\hat{b}}\, X^0 Y^{\hat{b}} \varepsilon_i + R^i_{0b0}\, X^b Y^0 \varepsilon_i + R^i_{0\hat{b}0}\, X^{\hat{b}} Y^0 \varepsilon_i; \\ &= \delta^i_b\, X^0 Y^b \varepsilon_i + \delta^i_{\hat{b}}\, X^0 Y^{\hat{b}} \varepsilon_i - \delta^i_b\, X^b Y^0 \varepsilon_i - \delta^i_{\hat{b}}\, X^{\hat{b}} Y^0 \varepsilon_i; \\ &= \eta(X)Y - \eta(Y)X. \end{aligned}$$

$$\begin{aligned} R(X,\xi)Y &= R^i_{jk0}\, X^k Y^j \varepsilon_i; \\ &= R^i_{0b0}\, X^b Y^0 \varepsilon_i + R^i_{0\hat{b}0}\, X_b Y^0 \varepsilon_i + R^i_{b\hat{c}0}\, X_c Y^b \varepsilon_i + R^i_{\hat{b}c0}\, X^c Y_b \varepsilon_i; \\ &= -\delta^i_b\, X^b Y^0 \varepsilon_i - \delta^i_{\hat{b}}\, X_b Y^0 \varepsilon_i + \delta^i_0\, \delta^c_b\, X_c Y^b \varepsilon_i + \delta^i_0\, \delta^b_c\, X^c Y_b \varepsilon_i; \\ &= g(X,Y)\xi - \eta(Y)X. \end{aligned}$$

$\square$

Theorem 2.3.2 *The components of the Ricci tensor r on the $AG-$structure space of the manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ are given by*

(1) $r_{00} = -2n;$

(2) $r_{a0} = 0;$

(3) $r_{ab} = -2A^{c}_{abc} + B_{cab}{}^{c} - B_{ca}{}^{h}\ B_{hb}{}^{c};$

(4) $r_{\hat{a}b} = -2(n\delta^{a}_{b} + B^{ca}{}_{[bc]}) + A^{ac}_{cb} - B^{ah}{}_{b}\ B_{ch}{}^{c},$

and the remaining components are given by the symmetric property or the complex conjugate to the above components. Take into consideration, all indexes have a range from 1 to n, except $\hat{a} = n+1, ..., 2n$.

Proof: Suppose that r is the Ricci tensor of type (2, 0), then

$$r(X,Y) = \sum_{i=0}^{2n} g(R(e_i, Y)X, e_i); \quad \forall\ X, Y \in X(M),$$

where $\{e_0 = \xi, e_1, ..., e_{2n}\}$ is orthonormal basis of $X(M)$.
Regarding Corollary 2.3.1; item (2), we have

$$\begin{aligned} r(X,\xi) &= \sum_{i=0}^{2n} g(R(e_i,\xi)X, e_i); \\ &= \sum_{i=0}^{2n} [g(e_i, X)g(\xi, e_i) - \eta(X)g(e_i, e_i)]; \\ &= \sum_{i=0}^{2n} [g(e_i, X)\eta(e_i) - \eta(X)\delta_{ii}]; \\ &= \eta(X) - (2n+1)\eta(X); \\ &= -2n\ \eta(X). \end{aligned}$$

The above result follows from the fact that

$$g(\xi, e_i) = \eta(e_i) = \begin{cases} 1, & \text{if } i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $r_{i0} = -2n\ \eta_i$ with $i = 0, 1, ..., 2n$ and since $\eta_i = g_{0i}$, then from the Definition 1.3.6, we determine the values of r_{00} and r_{a0} with $a = 1, 2, ..., n$.
After that, we compute the other components of the Ricci tensor on the $AG-$ structure space due to the following:

$$\begin{aligned} r_{ij} &= -R^k_{ijk}; \\ &= -R^0_{ij0} - R^c_{ijc} - R^{\hat{c}}_{ij\hat{c}}, \end{aligned}$$

where $i, j, k = 0, 1, ..., 2n$, $c = 1, ..., n$ and $\hat{c} = c + n$. If a and b have the same range of c, then according to the Theorem 2.3.1, we have

$$\begin{aligned} r_{ab} &= -R^0_{ab0} - R^c_{abc} - R^{\hat{c}}_{ab\hat{c}}; \\ &= -2A^c_{abc} + B_{cab}{}^c - B_{ca}{}^h\ B_{hb}{}^c. \\ r_{\hat{a}b} &= -R^0_{\hat{a}b0} - R^c_{\hat{a}bc} - R^{\hat{c}}_{\hat{a}b\hat{c}}; \\ &= -\delta^a_b - 2(B^{ca}{}_{[bc]} - \delta^c_{[b}\ \delta^a_{c]}) + A^{ac}_{cb} - B^{ah}{}_b\ B_{ch}{}^c - \delta^a_b\ \delta^c_c; \\ &= -2(n\delta^a_b + B^{ca}{}_{[bc]}) + A^{ac}_{cb} - B^{ah}{}_b\ B_{ch}{}^c. \end{aligned}$$

So, we conclude the requirement results. $\square$

The next theorem gives a theoretical Physical application for the manifold of Kenmotsu type.

Theorem 2.3.3 *The manifold of Kenmotsu type* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *is an Einstein manifold if and only if, the following conditions hold:*

$$\alpha = -2n; \quad A^c_{abc} = 0; \quad B_{cab}{}^c = B_{ca}{}^h\ B_{hb}{}^c; \quad B^{ca}{}_{[bc]} = 0; \quad A^{ac}_{cb} = B^{ah}{}_b\ B_{ch}{}^c.$$

Proof: Suppose that M is an Einstein manifold, then from the Definition 1.4.4, we have on $AG-$structure space the following:

$$r_{ij} = \alpha\ g_{ij},$$

where $i, j = 0, 1, ..., 2n$. Especially, $r_{00} = \alpha\ g_{00}$ then regarding Theorem 2.3.2 and the Definition 1.3.6, we have $\alpha = -2n$. Moreover, we must have $r_{ab} = 0$ and $r_{\hat{a}b} = -2n\ \delta^a_b$. This equivalent to the following equations:

$$-2A^c_{abc} + B_{cab}{}^c - B_{ca}{}^h\ B_{hb}{}^c = 0; \quad -2B^{ca}{}_{[bc]} + A^{ac}_{cb} - B^{ah}{}_b\ B_{ch}{}^c = 0.$$

From the fact that $B_{ab}{}^{c} = -B_{ba}{}^{c}$ and $B^{ab}{}_{c} = -B^{ba}{}_{c}$, we get

$$-2A^{c}_{abc} - B_{acb}{}^{c} + B_{ac}{}^{h}\ B_{hb}{}^{c} = 0; \quad 2B^{ac}{}_{[bc]} + A^{ac}_{cb} + B^{ah}{}_{b}\ B_{hc}{}^{c} = 0.$$

Since $R^{a}_{bcd} = 2A^{a}_{bcd} = -R^{a}_{bdc} = -2A^{a}_{bdc}$, then by taking the alternating operator of the indexes b and c of the above equations, we obtain

$$3A^{c}_{acb} - (A^{c}_{acb} + B_{a[cb]}{}^{c} - B_{a[c}{}^{h}\ B_{|h|b]}{}^{c}) = 0; \quad 3B^{ac}{}_{[bc]} + (A^{ac}_{[cb]} - B^{ac}{}_{[bc]} - B^{ah}{}_{[c}\ B_{|h|b]}{}^{c}) = 0.$$

Now, from Theorem 2.2.3, we deduce that $A^{c}_{acb} = B^{ac}{}_{[bc]} = 0$ and this gives the required conditions. Conversely, if the conditions hold then Theorem 2.3.2 gives that M is Einstein manifold. □

Corollary 2.3.2 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ is an Einstein manifold if and only if it has $\Phi-$invariant Ricci tensor and satisfies the following equations:*

$$\alpha = -2n; \quad B^{ca}{}_{[bc]} = 0; \quad A^{ac}_{cb} = B^{ah}{}_{b}\ B_{ch}{}^{c}.$$

Proof: The result follows from Theorem 2.3.3 and Lemma 1.4.2. □

Theorem 2.3.4 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ is $\eta-$Einstein manifold if and only if the following conditions hold:*

$$\alpha + \beta = -2n; \quad A^{c}_{abc} = 0; \quad B_{cab}{}^{c} = B_{ca}{}^{h}\ B_{hb}{}^{c};$$
$$B^{ca}{}_{[bc]} = \frac{\beta}{3}\delta^{a}_{b}; \quad A^{ac}_{cb} = B^{ah}{}_{b}\ B_{ch}{}^{c} - \frac{\beta}{3}\delta^{a}_{b}.$$

Proof: Suppose that M is an $\eta-$Einstein manifold, then regarding Definition 1.4.4, we have $r_{00} = \alpha + \beta$. So, Theorem 2.3.2 gives $\alpha + \beta = -2n$. Moreover, we must have $r_{ab} = 0$ and $r_{\hat{a}b} = \alpha\ g_{\hat{a}b} = (-2n - \beta)\delta^{a}_{b}$. Similar to the manner in the proof of Theorem 2.3.3, we get this theorem's conditions. The converse also true by simple calculations. □

Corollary 2.3.3 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ is $\eta-$Einstein manifold if and only if, M has $\Phi-$invariant Ricci tensor and satisfies the following equations:*

$$\alpha + \beta = -2n; \quad B^{ca}{}_{[bc]} = \frac{\beta}{3}\delta^{a}_{b}; \quad A^{ac}_{cb} = B^{ah}{}_{b}\ B_{ch}{}^{c} - \frac{\beta}{3}\delta^{a}_{b}.$$

Proof: The assertion of this corollary follows from the Theorem 2.3.4 and the Lemma 1.4.2. □

Remark 2.3.1 From the above discussion, it is clearly that α and β are scalars.

Chapter Three

The Curvature Identities and Curvature Derivation for the Manifold of Kenmotsu Type

Chapter 3

The Curvature Identities and Curvature Derivation for the Manifold of Kenmotsu Type

This chapter deals with two types of study, the first study devotes to the manifold of Kenmotsu type which satisfies the $GS-$space forms and (or) some curvature identities that similar to the Gary identities in the AH-manifolds [54]. Whereas, the second study concentrated on the covariant derivative of the Riemannian curvature tensor of the manifold of Kenmotsu type.

3.1 The Curvature Identities for the Manifold of Kenmotsu Type

We begin this section with an example on the manifold of Kenmotsu type and then discuss some curvature identities including $\Phi HS-$curvature.

Example 3.1.1 Suppose that (N^{2n}, J, h) is an AH-manifold of class $W_3 \oplus W_4$ (see Gray and Hervella [56]), then N satisfies the following identity:

$$D_X(J)Y - D_{JX}(J)JY = 0; \quad \forall\ X, Y \in X(N), \tag{3.1.1}$$

where D is the Riemannian connection of N with respect to the metric h. We take $M = \mathbb{R} \times_f N$, where $f(t) = e^t$ defined on $\mathbb{R}$. If N has local coordinates

$(x_1, x_2, ..., x_{2n})$, then M has local coordinates $(t, x_1, x_2, ..., x_{2n})$. Now, suppose that $\xi = \frac{\partial}{\partial t}$ is the Reeb vector field of M. Let $X \in X(M)$, then $X = X_0 + \eta(X)\xi$, where $X_0 \in X(N)$ and $\eta(X_0) = 0$. Then we define the endomorphism Φ on M by $\Phi(X) = J(X_0)$. Suppose that g is the Riemannian metric of M and ∇ is the Riemannian connection on M. Then from Goldberg [51] we obtain

$$\nabla_{X_0} Y_0 = D_{X_0} Y_0 - H(X_0, Y_0)\xi; \quad \forall\, X_0, Y_0 \in X(N) \tag{3.1.2}$$

where $H(X_0, Y_0) = g(\nabla_{X_0}\xi, Y_0)$. If we put $Y = \xi$ in the identity of the manifold of Kenmotsu type, we get

$$\nabla_X \xi = X - \eta(X)\xi = -\Phi^2(X).$$

Since $\eta(X_0) = 0$, then $H(X_0, Y_0) = g(X_0, Y_0)$ and the equation (3.1.2) becomes

$$\nabla_{X_0} Y_0 = D_{X_0} Y_0 - g(X_0, Y_0)\xi.$$

Moreover, for any $X, Y \in X(M)$ we have

$$\begin{aligned}
\nabla_X(\Phi)Y &= \nabla_X \Phi(Y) - \Phi(\nabla_X Y);\\
&= \nabla_{X_0+\eta(X)\xi}\Phi(Y) - \Phi(\nabla_{X_0+\eta(X)\xi}(Y_0 + \eta(Y)\xi));\\
&= \nabla_{X_0}\Phi(Y) + \eta(X)\nabla_\xi \Phi(Y) - \Phi\{\nabla_{X_0} Y_0 + \eta(X)\nabla_\xi Y_0\\
&+ (\nabla_{X_0}(\eta)Y)\xi + \eta(Y)\nabla_{X_0}\xi + \eta(X)(\nabla_\xi(\eta)Y)\xi + \eta(X)\eta(Y)\nabla_\xi\xi\};\\
&= \nabla_{X_0}\Phi(Y) + \eta(X)\nabla_\xi\Phi(Y) - \Phi(\nabla_{X_0}Y_0) - \eta(X)\Phi(\nabla_\xi Y_0)\\
&- \eta(Y)\Phi(\nabla_{X_0}\xi).
\end{aligned}$$

Since $\Phi(X) \in X(N)$, then $[\Phi(X), \xi] = 0$ and according to the equality $[X, Y] = \nabla_X Y - \nabla_Y X$, we get

$$\nabla_\xi \Phi(Y) = \nabla_{\Phi(Y)}\xi = \Phi(Y); \qquad \nabla_\xi Y_0 = \nabla_{Y_0}\xi = Y_0.$$

Then from the previous discussion and the fact $\Phi(X_0) = \Phi(X)$, we deduce that

$$\nabla_X(\Phi)Y = D_{X_0}(J)Y_0 - g(X, \Phi(Y))\xi - \eta(Y)\Phi(X). \tag{3.1.3}$$

Now, regarding the equation (3.1.3), we conclude that

$$\nabla_{\Phi(X)}(\Phi)\Phi(Y) = D_{J(X_0)}(J)J(Y_0) - g(X, \Phi(Y))\xi. \tag{3.1.4}$$

So, subtracting equation (3.1.4) from equation (3.1.3), and using equation (3.1.1), imply to attain the identity of the manifold of Kenmotsu type.

Theorem 3.1.1 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ has pointwise constant $\Phi HS-$curvature if and only if the following equality holds on the $AG-$structure space:*

$$A^{ad}_{bc} = B_{bc}{}^{[ad]} - B_{hb}{}^{a}\ B^{dh}{}_{c} + \frac{\gamma+1}{2}\widetilde{\delta}^{ad}_{bc}.$$

Proof: Suppose that M is the manifold of Kenmotsu type and has pointwise constant $\Phi HS-$curvature. According to Theorem 2.3.1, we have the Riemannian curvature tensor for the manifold of Kenmotsu type owned the following on the $AG-$structure space:

$$R^{a\ d}_{bc} = R^{a}_{bc\hat{d}} = A^{ad}_{bc} - B^{ah}{}_{c}\ B_{bh}{}^{d} - \delta^{a}_{c}\ \delta^{d}_{b}.$$

Regarding Theorem 1.4.2 and the above equation, the following equation holds on the $AG-$structure space:

$$A^{(ad)}_{(bc)} = B^{(a|h|}{}_{(c}\ B_{b)h}{}^{d)} + \frac{\gamma+1}{2}\widetilde{\delta}^{ad}_{bc},$$

where $|h|$ means the index h does not act by the symmetric operator $(..)$. Then using the fact $B_{bh}{}^{d} = -B_{hb}{}^{d}$ and the symmetric property of the indexes a and d, we get

$$\begin{aligned} A^{(ad)}_{(bc)} &= -B^{(d|h|}{}_{(c}\ B_{|h|b)}{}^{a)} + \frac{\gamma+1}{2}\widetilde{\delta}^{ad}_{bc}; \\ &= -B_{h(b}{}^{(a}\ B^{d)h}{}_{c)} + \frac{\gamma+1}{2}\widetilde{\delta}^{ad}_{bc}. \end{aligned}$$

Since $A^{ad}_{bc} = A^{[ad]}_{[bc]} + A^{[ad]}_{(bc)} + A^{(ad)}_{[bc]} + A^{(ad)}_{(bc)}$. Then regarding the Theorem 2.2.3, the alternating (symmetric) property and the Corollary 2.1.1; item (4), we get

$$\begin{aligned} A^{[ad]}_{[bc]} &= B_{bc}{}^{[ad]} - B_{h[b}{}^{[a}\ B^{d]h}{}_{c]}; \\ A^{[ad]}_{(bc)} &= -B_{h(b}{}^{[a}\ B^{d]h}{}_{c)}; \\ A^{(ad)}_{[bc]} &= -B_{h[b}{}^{(a}\ B^{d)h}{}_{c]}. \end{aligned}$$

From the above discussion, we have the required assertion. $\square$

Theorem 3.1.2 *If the manifold of Kenmotsu type is Einstein manifold and has pointwise constant $\Phi HS-$curvature γ, then $\gamma = -1$.*

Proof: Suppose that M is the manifold of Kenmotsu type and satisfies Einstein's criterion. Then Theorem 2.3.3 gives the following:

$$B^{ca}{}_{[bc]} = 0; \quad A^{ac}_{cb} = B^{ah}{}_{b}\, B_{ch}{}^{c}. \tag{3.1.5}$$

Since M has pointwise constant $\Phi HS-$curvature γ, then from Theorem 3.1.1, and the fact $\delta^c_c = n$, we get

$$A^{ac}_{cb} = B_{cb}{}^{[ac]} - B_{hc}{}^{a}\, B^{ch}{}_{b} + \frac{(\gamma+1)(n+1)}{2}\delta^a_b. \tag{3.1.6}$$

Now, combine equations (3.1.5) and (3.1.6), we obtain

$$B^{ah}{}_{b}\, B_{ch}{}^{c} - B_{ch}{}^{a}\, B^{ch}{}_{b} = \frac{(\gamma+1)(n+1)}{2}\delta^a_b.$$

Since $n > 1$, then by contracting the above equation with respect to the indexes a and c, we conclude the result. □

Corollary 3.1.1 *If the manifold of Kenmotsu type is Einstein manifold and has pointwise constant $\Phi HS-$curvature, then it is locally isometric to the warped product $\mathbb{R} \times_f \mathbb{C}^n$.*

Proof: The result follows from the above theorem and Tanno [107]. □

According to Vanhecke [113], we can define new classes of $ACR-$manifolds as the following:

Definition 3.1.1 *An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is called of class*
G_1 *if* $R(\Phi X, \Phi Y, \Phi Z, \Phi W) = R(X, Y, Z, W); \quad \forall\ X, Y, Z, W \in \ker(\eta)$;
G_2 *if* $R(X, Y, \Phi Z, \Phi W) = R(X, Y, Z, W); \quad \forall\ X, Y, Z, W \in \ker(\eta)$;
G_3 *if* $R(\Phi X, Y, Z, \Phi W) = R(X, Y, Z, W); \quad \forall\ X, Y, Z, W \in \ker(\eta)$;
G_4 *if* $R(\Phi^2 X, \Phi^2 Y, \Phi^2 Z, \Phi^2 W) = R(X, Y, Z, W); \quad \forall\ X, Y, Z, W \in X(M)$.
Moreover, $ACR-$manifold of class G_1 and G_4 can be called classes of $\Phi-$invariant and Φ^2-invariant Riemann curvature tensor respectively.

Theorem 3.1.3 *On the $AG-$structure space, the $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is of class*

1. G_1 *if and only if,* $R_{\widehat{a}bcd} = 0$;

2. G_2 *if and only if,* $R_{\widehat{a}\widehat{b}cd} = 0$;

3. G_3 *if and only if,* $R_{\widehat{a}bc\widehat{d}} = 0$;

4. G_4 *if and only if,* $R_{a0b0} = R_{\widehat{a}0b0} = R_{a0bc} = R_{\widehat{a}0bc} = R_{a0\widehat{b}c} = 0$.

Proof: Since $R(X,Y,Z,W) = R_{ijkl}\ X^i\ Y^j\ Z^k\ W^l$, where $i,j,k,l = 0,1,...,2n$ and for short, we set $i,j,k,l = 0,a,\widehat{a}$, where $a = 1,2,...,n$ and $\widehat{a} = a+n$. Then we have M of class G_1 if and only if,

$$R(\Phi X, \Phi Y, \Phi Z, \Phi W) = R(X,Y,Z,W); \quad \forall\ X,Y,Z,W \in \ker(\eta).$$

Then the above equation equivalent to

$$R_{rstu}\ (\Phi X)^r\ (\Phi Y)^s\ (\Phi Z)^t\ (\Phi W)^u = R_{ijkl}\ X^i\ Y^j\ Z^k\ W^l,$$

where i,j,k,l,r,s,t,u have the same range and do not vanish because $X,Y,Z,W \in \ker(\eta)$. Then the last equation simplifies to

$$R_{rstu}\ \Phi^r_i\ \Phi^s_j\ \Phi^t_k\ \Phi^u_l\ X^i\ Y^j\ Z^k\ W^l = R_{ijkl}\ X^i\ Y^j\ Z^k\ W^l.$$

Then $R_{rstu}\ \Phi^r_i\ \Phi^s_j\ \Phi^t_k\ \Phi^u_l = R_{ijkl}$ and regarding the values of the indexes and Φ in Definition 1.3.6, we attain the result. Similarly, if M of class G_2 or G_3.
Now, if M of class G_4, then we have

$$R(\Phi^2 X, \Phi^2 Y, \Phi^2 Z, \Phi^2 W) = R(X,Y,Z,W).$$

The above equation can be written in the following form:

$$\begin{aligned} 0 = {} & \eta(X)\eta(Z)R(\xi,Y,\xi,W) + \eta(X)\eta(W)R(\xi,Y,Z,\xi) + \eta(Y)\eta(Z)R(X,\xi,\xi,W) \\ & + \eta(Y)\eta(W)R(X,\xi,Z,\xi) - \eta(X)R(\xi,Y,Z,W) - \eta(Y)R(X,\xi,Z,W) \\ & - \eta(Z)R(X,Y,\xi,W) - \eta(W)R(X,Y,Z,\xi). \end{aligned}$$

If we replace X,Y,Z,W,ξ by the indexes $i,j,k,l,0$ respectively in the last equation, we get

$$\begin{aligned} 0 = {} & \eta_i\eta_k R_{0j0l} + \eta_i\eta_l R_{0jk0} + \eta_j\eta_k R_{i00l} + \eta_j\eta_l R_{i0k0} - \eta_i R_{0jkl} \\ & - \eta_j R_{i0kl} - \eta_k R_{ij0l} - \eta_l R_{ijk0}. \end{aligned}$$

So, if we take the values of i, j, k, l as above and use the properties of Riemannian curvature tensor, then we obtain the result. □

Corollary 3.1.2 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ can not be of class G_4.*

Proof: Suppose that M is the manifold of Kenmotsu type, then from Theorem 2.3.1, we have

$$R_{\hat{a}0b0} = R^{a}_{0b0} = -\delta^{a}_{b} \neq 0.$$

Therefore from Theorem 3.1.3, we arrive to the substance of this corollary. □

Corollary 3.1.3 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ belong to the class of*

1. *G_1 if and only if, $A^{a}_{bcd} = 0$; or equivalently $B_{bcd}{}^{a} = B_{bc}{}^{h} B_{hd}{}^{a}$;*
2. *G_2 if and only if, $B^{ab}{}_{[cd]} = \delta^{a}_{[c}\ \delta^{b}_{d]}$;*
3. *G_3 if and only if, $A^{ad}_{bc} = B^{ah}{}_{c}\ B_{bh}{}^{d} + \delta^{a}_{c}\ \delta^{d}_{b}$.*

Proof: The results follow from Theorems 2.3.1 and 3.1.3. □

Corollary 3.1.4 *If $(M^{2n+1}, \xi, \eta, \Phi, g)$ is the manifold of Kenmotsu type and of class G_3, then M is a manifold of class G_2.*

Proof: The assertion of the present corollary follows from the conditions of Theorem 2.2.3 and Corollary 3.1.3. □

Corollary 3.1.5 *If $(M^{2n+1}, \xi, \eta, \Phi, g)$ is an $ACR-$manifold of class G_3, then M posses vanishing $\Phi HS-$curvature tensor H.*

Proof: Suppose that M of class G_3, then for all $X \in \ker(\eta)$, we get

$$H(X) = \frac{R(\Phi X, X, X, \Phi X)}{(g(X,X))^2} = \frac{R(X, X, X, X)}{(g(X,X))^2} = 0.$$

□

3.2 The Generalized Sasakian Space Forms for the Manifold of Kenmotsu Type

In this section, we characterize the definition of $GS-$space forms on $AG-$structure space and we derive the conditions for the manifold of Kenmotsu type to be $GS-$space forms.

Remark 3.2.1 According to the Definition 1.4.2, the components of Riemannian curvature tensor of the $GS-$space forms $M(f_1, f_2, f_3)$ on the $AG-$structure space are given by

$$\begin{aligned} R_{ijkl} &= f_1\{g_{ik}\ g_{jl} - g_{il}\ g_{jk}\} + f_2\{\Omega_{il}\ \Omega_{kj} - \Omega_{lj}\ \Omega_{ik} + 2\Omega_{ij}\ \Omega_{kl}\} \\ &\quad + f_3\{\eta_j\ \eta_k\ g_{il} - \eta_j\ \eta_l\ g_{ik} + \eta_i\ \eta_l\ g_{jk} - \eta_i\ \eta_k\ g_{jl}\}, \end{aligned} \tag{3.2.7}$$

where $\Omega(X, Y) = g(X, \Phi Y)$ for all $X, Y \in X(M)$. Moreover, the components of Ω on the $AG-$structure space for any $ACR-$manifold are given by

$$\Omega_{00} = \Omega_{a0} = \Omega_{\hat{a}0} = \Omega_{ab} = \Omega_{\widehat{a}\widehat{b}} = 0; \quad \Omega_{\hat{a}b} = \sqrt{-1}\delta^a_b; \quad \Omega_{ij} = -\Omega_{ji}. \tag{3.2.8}$$

Theorem 3.2.1 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ is a $GS-$space forms if and only if, M attains the following on the $AG-$structure space:*

1. $f_3 = f_1 + 1; \quad A^a_{bcd} = 0;$
2. $A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d + (f_2 + f_3)\delta^a_c\ \delta^d_b + 2f_2\ \delta^a_b\ \delta^d_c;$
3. $B^{ad}{}_{[cb]} = (f_3 - f_2)\delta^a_{[c}\ \delta^d_{b]}; \quad B^{abd}{}_c = B^{ab}{}_h\ B^{hd}{}_c.$

Proof: Regarding Theorem 2.3.1, Definition 1.3.6 and equations (3.2.7) and (3.2.8), we get the requirements. For instance, if $(i, j, k, l) = (\hat{a}, 0, b, 0)$, then

$$\begin{aligned} R_{\hat{a}0b0} &= f_1\{g_{\hat{a}b}\ g_{00} - g_{\hat{a}0}\ g_{0b}\} + f_2\{\Omega_{\hat{a}0}\ \Omega_{b0} - \Omega_{00}\ \Omega_{\hat{a}b} + 2\Omega_{\hat{a}0}\ \Omega_{b0}\} \\ &\quad + f_3\{\eta_0\ \eta_b\ g_{\hat{a}0} - \eta_0\ \eta_0\ g_{\hat{a}b} + \eta_{\hat{a}}\ \eta_0\ g_{0b} - \eta_{\hat{a}}\ \eta_b\ g_{00}\}; \\ -\delta^a_b &= f_1\ \delta^a_b - f_3\ \delta^a_b. \end{aligned}$$

So, we have $f_3 = f_1 + 1$. Similarly for the others. □

Theorem 3.2.2 *The $GS-$space forms $M(f_1, f_2, f_3)$ has pointwise constant $\Phi HS-$curvature γ if and only if, $\gamma + f_1 + 3f_2 = 0$.*

Proof: $M(f_1, f_2, f_3)$ has pointwise constant $\Phi HS-$curvature γ if and only if,

$$R(\Phi X, X, X, \Phi X) = \gamma(g(X, X))^2; \quad \forall\ X \in \ker(\eta).$$

But the Riemann curvature tensor of $M(f_1, f_2, f_3)$ satisfies the following:

$$R(\Phi X, X, X, \Phi X) = -(f_1 + 3f_2)(g(X, X))^2; \quad \forall\ X \in \ker(\eta).$$

Then the subtracting of the above equations confirm the result. □

Theorem 3.2.3 *The $GS-$space forms $M(f_1, f_2, f_3)$ is of class*

1. *G_1 constantly;*

2. *G_2 if and only if, $n = 1$ or $f_1 = f_2$;*

3. *G_3 if and only if, $f_1 = f_2 = 0$;*

4. *G_4 if and only if, $f_1 = f_3$.*

Proof: Taking the equation (3.2.7) into account, we conclude that

$$\begin{aligned}
&R_{\hat{a}bcd} = 0;\\
&R_{\hat{a}\hat{b}cd} = (f_1 - f_2)\{\delta_c^a\delta_d^b - \delta_d^a\delta_c^b\};\\
&R_{\hat{a}bc\hat{d}} = (f_1 + f_2)\delta_c^a\delta_b^d + 2f_2\ \delta_b^a\delta_c^d;\\
&R_{a0b0} = R_{a0bc} = R_{\hat{a}0bc} = R_{a0\hat{b}\hat{c}} = 0; \quad R_{\hat{a}0b0} = (f_1 - f_3)\delta_b^a.
\end{aligned}$$

Compare the above equations with Theorem 3.1.3, we deduce the results. □

On the $AG-$structure space, we can determine the components of the Ricci tensor of $M(f_1, f_2, f_3)$ from the equation (3.2.7) as follows:

$$\begin{aligned}
r_{jk} &= -g^{il}R_{ijkl}\\
&= (2nf_1 + 3f_2 - f_3)g_{jk} - (3f_2 + (2n-1)f_3)\eta_j\ \eta_k;
\end{aligned}$$

where g^{il} are the components of g^{-1}. Then we deduce the following theorem:

Theorem 3.2.4 *The $GS-$space forms $M(f_1, f_2, f_3)$ is an $\eta-$Einstein manifold with $\alpha = 2nf_1 + 3f_2 - f_3$ and $\beta = -(3f_2 + (2n-1)f_3)$.*

Theorem 3.2.5 *If the manifold of Kenmotsu type is $GS-$space forms $M(f_1, f_2, f_3)$ and it has pointwise constant $\Phi HS-$curvature γ, then*

$$\gamma = \frac{1}{3}; \quad f_3 = \frac{n}{3(n-1)}; \quad f_2 = \frac{n-2}{9(n-1)}; \quad f_1 = \frac{-2n+3}{3(n-1)}.$$

Proof: Combine the value of A^{ad}_{bc} from Theorem 3.1.1 with its value in Theorem 3.2.1, we get

$$B_{bc}{}^{[ad]} - B_{hb}{}^{a}\, B^{dh}{}_{c} + \frac{\gamma+1}{2}\widetilde{\delta}^{ad}_{bc} = B^{ah}{}_{c}\, B_{bh}{}^{d} + (f_2+f_3)\delta^a_c\ \delta^d_b + 2f_2\ \delta^a_b\ \delta^d_c.$$

If we applying the symmetric operator on the indexes a and d of the above equation, then we deduce that

$$\frac{\gamma+1}{2}\widetilde{\delta}^{ad}_{bc} = (f_2+f_3)\widetilde{\delta}^{ad}_{bc} + 2f_2\ \widetilde{\delta}^{ad}_{bc}.$$

So, we have $\frac{\gamma+1}{2} = 3f_2 + f_3$. Regarding Theorems 3.2.1 and 3.2.2, directly, we get the value of γ.

Since $M(f_1, f_2, f_3)$ is $\eta-$Einstein manifold with $\beta = -(3f_2 + (2n-1)f_3)$, then the manifold of Kenmotsu type is an $\eta-$Einstein manifold with $\beta = -(3f_2 + (2n-1)f_3)$. But from Theorem 2.3.4, we have $B^{ca}{}_{[bc]} = \frac{\beta}{3}\delta^a_b$. So, regarding Theorem 3.2.1, we attain the values of $\{f_1, f_2, f_3\}$. □

3.3 The Covariant Derivative Curvature for the Manifold of Kenmotsu Type

In this section, we investigate the geometric properties of the covariant derivative for the Riemannian curvature tensor which denotes ∇R, on the manifold of Kenmotsu type by determining its components on the $AG-$structure space.

Theorem 3.3.1 *On the $AG-$structure space, the manifold of Kenmotsu type satisfies the following equations:*

$$\Delta A^a_{bcd} = A^a_{bcdh}\ \omega^h + A^{ah}_{bcd}\ \omega_h - 2A^a_{bcd}\ \omega; \tag{3.3.9}$$

$$\Delta A^{ad}_{bc} = \tilde{A}^{ad}_{bch}\ \omega^h + \tilde{A}^{adh}_{bc}\ \omega_h - 2A^{ad}_{bc}\ \omega; \tag{3.3.10}$$

$$\Delta B^{ab}{}_{cd} = B^{ab}{}_{cdh}\ \omega^h + B^{abh}{}_{cd}\ \omega_h - 2B^{ab}{}_{cd}\ \omega; \tag{3.3.11}$$

where $h = 1, ..., n$, and

$$\begin{aligned}
\Delta A^a_{bcd} &= dA^a_{bcd} + A^h_{bcd}\ \theta^a_h - A^a_{hcd}\ \theta^h_b - A^a_{bhd}\ \theta^h_c - A^a_{bch}\ \theta^h_d;\\
\Delta A^{ad}_{bc} &= dA^{ad}_{bc} + A^{hd}_{bc}\ \theta^a_h + A^{ah}_{bc}\ \theta^d_h - A^{ad}_{hc}\ \theta^h_b - A^{ad}_{bh}\ \theta^h_c;\\
\Delta B^{ab}{}_{cd} &= dB^{ab}{}_{cd} + B^{hb}{}_{cdh}\ \theta^a_h + B^{ah}{}_{cd}\ \theta^b_h - B^{ab}{}_{hd}\ \theta^h_c - B^{ad}{}_{ch}\ \theta^h_d.
\end{aligned}$$

Proof: If we differentiate the Cartan's second structure equations in Theorem 2.2.3 exteriorly, then on the $AG-$structure space, there are suitable smooth functions such that the target equations are attained. □

Now, we can establish the components of ∇R on $(M^{2n+1}, \xi, \eta, \Phi, g)$ from the following identity [75]:

$$dR_{ijkl} - R_{tjkl}\ \theta^t_i - R_{itkl}\ \theta^t_j - R_{ijtl}\ \theta^t_k - R_{ijkt}\ \theta^t_l = R_{ijkl,t}\ \omega^t; \tag{3.3.12}$$

where $R(X, Y, Z, W) = g(R(Z, W)Y, X)$, $R_{ijkl} = R^{\hat{i}}_{jkl}$, $t = 0, 1, ..., 2n$ and

$$R_{ijkl,t} = g(\nabla_{\varepsilon_t}(R)(\varepsilon_k, \varepsilon_l)\varepsilon_j, \varepsilon_i).$$

Theorem 3.3.2 *On the $AG-$structure space, the components of ∇R for the manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ are given by*

1. $R_{a0b0,0} = R_{a0b0,h} = R_{a0b0,\hat{h}} = 0$;
2. $R_{\hat{a}0b0,0} = R_{\hat{a}0b0,h} = R_{\hat{a}0b0,\hat{h}} = 0$;
3. $R_{a0bc,0} = R_{a0bc,h} = 0; \quad R_{a0bc,\hat{h}} = 2A^h_{abc}$;
4. $R_{\hat{a}0bc,0} = 0; \quad R_{\hat{a}0bc,h} = -2A^a_{hbc}; \quad R_{\hat{a}0bc,\hat{h}} = -2B^{ah}{}_{[bc]}$;
5. $R_{a0\hat{b}c,0} = 0; \quad R_{a0\hat{b}c,h} = -2A^b_{cah}; \quad R_{a0\hat{b}c,\hat{h}} = -A^{hb}_{ac} + B^{hd}{}_{c}\ B_{ad}{}^{b}$;

6. $R_{abcd,0} = R_{abcd,h} = 0; \quad R_{abcd,\hat{h}} = 4\{B_{f[a}{}^{h}\ A^{f}_{b]cd} + B_{f[c}{}^{h}\ A^{f}_{d]ab}\};$

7. $R_{\hat{a}bcd,0} = -4A^{a}_{bcd}; \quad R_{\hat{a}bcd,h} = 2A^{a}_{bcdh};$

8. $R_{\hat{a}bcd,\hat{h}} = 2\{A^{ah}_{bcd} + B^{af}{}_{[cd]}\ B_{fb}{}^{h} + A^{af}_{b[c}\ B_{|f|d]}^{h} + B_{f[c}{}^{h}\ B^{a\tilde{f}}{}_{d]}\ B_{b\tilde{f}}{}^{f}\};$

9. $R_{\hat{a}bc\hat{d},0} = -2\{A^{ad}_{bc} - B^{af}{}_{c}\ B_{bf}{}^{d}\};$

10. $R_{\hat{a}bc\hat{d},h} = 2A^{a}_{bcf}\ B^{fd}{}_{h} - 2A^{d}_{cfb}\ B^{fa}{}_{h} - B^{af}{}_{c}\ B_{bfh}{}^{d} - B_{bf}{}^{d}\ B^{af}{}_{ch} + \tilde{A}^{ad}_{bch};$

11. $R_{\hat{a}bc\hat{d},\hat{h}} = \tilde{A}^{adh}_{bc} - B_{bf}{}^{d}\ B^{afh}{}_{c} - B^{af}{}_{c}\ B_{bf}{}^{dh} + 2A^{daf}_{c}\ B_{fb}{}^{h} - 2A^{afd}_{b}\ B_{fc}{}^{h};$

12. $R_{\hat{a}\hat{b}cd,0} = -4B^{ab}{}_{[cd]}; \quad R_{\hat{a}\hat{b}cd,h} = 2B^{ab}{}_{[cd]h} + 4B^{f[b}{}_{h}\ A^{a]}_{fcd};$

13. $R_{\hat{a}\hat{b}cd,\hat{h}} = 2B^{abh}{}_{[cd]} + 4B_{f[d}{}^{h}\ A^{fab}_{c]}.$

Proof: The results follow from equation (3.3.12) by taking

$$\begin{aligned}(i,j,k,l) =&(a,0,b,0), (\hat{a},0,b,0), (a,0,b,c), (\hat{a},0,b,c), (a,0,\hat{b},c), (a,b,c,d),\\ &(\hat{a},b,c,d), (\hat{a},b,c,\hat{d}), (\hat{a},\hat{b},c,d);\\ t =&0, h, \hat{h},\end{aligned}$$

and regarding Theorems 2.2.1 and 2.3.1. For instance, if $(i,j,k,l) = (a,0,b,0)$, then the equation (3.3.12) given by

$$dR_{a0b0} - R_{t0b0}\ \theta^{t}_{a} - R_{atb0}\ \theta^{t}_{0} - R_{a0t0}\ \theta^{t}_{b} - R_{a0bt}\ \theta^{t}_{0} = R_{a0b0,t}\ \omega^{t}.$$

The above equation can be simplified by using the Theorems 2.2.1 and 2.3.1, as the following:

$$\begin{aligned}R_{a0b0,t}\ \omega^{t} &= -R_{\hat{h}0b0}\ \theta^{\hat{h}}_{a} - R_{a0\hat{h}0}\ \theta^{\hat{h}}_{b};\\ &= \delta^{h}_{b}\ \theta^{\hat{h}}_{a} + \delta^{h}_{a}\ \theta^{\hat{h}}_{b};\\ &= \theta^{\hat{b}}_{a} + \theta^{\hat{a}}_{b} = 0.\end{aligned}$$

So, we have $R_{a0b0,h}\ \omega^{h} + R_{a0b0,\hat{h}}\ \omega_{h} + R_{a0b0,0}\ \omega = 0$, and then

$$R_{a0b0,h} = R_{a0b0,\hat{h}} = R_{a0b0,0} = 0.$$

We use the same technique for the other cases and for some cases we must use the equations (3.3.9), (3.3.10), or (3.3.11). For example, if $(i,j,k,l) = (\hat{a}, b, c, d)$, then the equation (3.3.12) given by

$$dR_{\hat{a}bcd} - R_{tbcd}\ \theta^t_{\hat{a}} - R_{\hat{a}tcd}\ \theta^t_b - R_{\hat{a}btd}\ \theta^t_c - R_{\hat{a}bct}\ \theta^t_d = R_{\hat{a}bcd,t}\ \omega^t.$$

According to the Theorem 2.3.1, we get

$$\begin{aligned} R_{\hat{a}bcd,t}\ \omega^t &= 2dA^a_{bcd} - R_{\hat{h}bcd}\ \theta^{\hat{h}}_{\hat{a}} - R_{\hat{a}hcd}\ \theta^h_b - R_{\hat{a}\hat{h}cd}\ \theta^{\hat{h}}_b - R_{\hat{a}bhd}\ \theta^h_c \\ &\quad - R_{\hat{a}b\hat{h}d}\ \theta^{\hat{h}}_c - R_{\hat{a}bch}\ \theta^h_d - R_{\hat{a}bc\hat{h}}\ \theta^{\hat{h}}_d; \\ &= 2\Delta A^a_{bcd} - R_{\hat{a}\hat{h}cd}\ \theta^{\hat{h}}_b - R_{\hat{a}b\hat{h}d}\ \theta^{\hat{h}}_c - R_{\hat{a}bc\hat{h}}\ \theta^{\hat{h}}_d; \\ &= 2\Delta A^a_{bcd} - R_{\hat{a}\hat{f}cd}\ \theta^{\hat{f}}_b + R_{\hat{a}bd\hat{f}}\ \theta^{\hat{f}}_c - R_{\hat{a}bc\hat{f}}\ \theta^{\hat{f}}_d; \end{aligned}$$

where $f = 1, 2, ..., n$. If we return to the Theorem 2.2.1, we have $\theta^{\hat{f}}_b = -B_{fb}{}^h\ \omega_h$. So regarding the Theorem 2.3.1, the equation (3.3.9) and the previous results, we obtain the following:

$$\begin{aligned} R_{\hat{a}bcd,0} &= -4A^a_{bcd}; \\ R_{\hat{a}bcd,h} &= 2A^a_{bcdh}; \\ R_{\hat{a}bcd,\hat{h}} &= 2\{A^{ah}_{bcd} + B^{af}{}_{[cd]}\ B_{fb}{}^h + A^{af}_{b[c}\ B_{|f|d]}{}^h + B_{f[c}{}^h\ B^{a\tilde{f}}{}_{d]}\ B_{b\tilde{f}}{}^f\ \}. \end{aligned}$$

The proof of the remaining items becomes obvious, therefore, we omit it. □

Theorem 3.3.3 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ is locally symmetric if and only if the following conditions hold:*

$$A^a_{bcd} = 0; \quad B^{ab}{}_{[cd]} = 0; \quad A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d.$$

Proof: Suppose that M^{2n+1} is locally symmetric, then $\nabla_U(R)(Z,W)Y = 0$, (see the Definition 1.4.10) and thus we have

$$g(\nabla_U(R)(Z,W)Y, X) = 0; \quad \forall\ X, Y, Z, W, U \in X(M).$$

Therefore, the components $R_{ijkl,t}$ are identically zero for all $i, j, k, l, t = 0, 1, ..., 2n$. Regarding the Theorem 3.3.2, we have $A^a_{bcd} = 0$; $B^{ab}{}_{[cd]} = 0$; and $A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d$. Conversely, if $A^a_{bcd} = 0$; $B^{ab}{}_{[cd]} = 0$; $A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d$, then $\Delta A^a_{bcd} = 0$;

$\Delta B^{ab}{}_{[cd]} = 0$; and according to the Lemma 1.2.1; item (3), as well as the Theorem 2.2.3; items (2) and (3), yield the following equation:

$$\Delta A^{ad}_{bc} = \{B^{af}{}_{c}\ B_{bfh}{}^{d} + B_{bf}{}^{d}\ B^{af}{}_{ch}\}\omega^{h} + \{B_{bf}{}^{d}\ B^{afh}{}_{c} + B^{af}{}_{c}\ B_{bf}{}^{dh}\}\omega_{h} - 2A^{ad}_{bc}\ \omega.$$

So, regarding the equations (3.3.9), (3.3.10), and (3.3.11) and the Theorem 3.3.2, we get $R_{ijkl,t} = 0$. Therefore, M^{2n+1} is locally symmetric. □

Theorem 3.3.4 *The locally symmetric manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ of Kenmotsu type is an Einstein manifold with $\alpha = -2n$ if and only if M satisfies the following condition:*

$$B_{cab}{}^{c} = B_{ca}{}^{h}\ B_{hb}{}^{c}.$$

Proof: Suppose that M^{2n+1} is an Einstein manifold with $\alpha = -2n$, then from the Definitions 1.4.4 and 1.3.6, we have

$$r_{00} = -2n; \quad r_{a0} = r_{ab} = 0; \quad r_{\hat{a}b} = -2n\delta^{a}_{b}.$$

Since M^{2n+1} is a locally symmetric manifold of Kenmotsu type, then regarding Theorems 2.3.2, 3.3.3 and the above relations achieve the condition.
Conversely, if the condition is valid, then the conditions of the Theorem 3.3.3 with Theorem 2.3.2, lead to the result. □

Corollary 3.3.1 *The locally symmetric manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ of Kenmotsu type is an Einstein manifold with $\alpha = -2n$ if and only if, M^{2n+1} has $\Phi-$invariant Ricci tensor.*

Proof: The assertion of this corollary follows from Definition 1.4.4, Lemma 1.4.2 and Theorem 3.3.4. □

Now, suppose that $(M^{2n+1}, \xi, \eta, \Phi, g)$ is a generalized $\Phi-$recurrent manifold, then regarding Definition 1.4.11, we get

$$\Phi^2(\nabla_U(R)(Z, W)Y) = \rho(U)R(Z, W)Y + \lambda(U)\{g(Y, W)Z - g(Y, Z)W\},$$

for all $U, W, Y, Z \in X(M)$. So, for all $X \in X(M)$, we have

$$\begin{aligned} g(\Phi^2(\nabla_U(R)(Z, W)Y), X) &= g(\nabla_U(R)(Z, W)Y, \Phi^2(X)); \\ &= -g(\nabla_U(R)(Z, W)Y, X) + \eta(X)g(\nabla_U(R)(Z, W)Y, \xi). \end{aligned}$$

Then the generalized $\Phi-$recurrent $ACR-$manifold has curvature components which are given by

$$-R_{ijkl,t} + \eta_i \; R_{0jkl,t} = \rho_t \; R_{ijkl} + \lambda_t \{ g_{ik} \; g_{jl} - g_{il} \; g_{jk} \}. \tag{3.3.13}$$

So, if M^{2n+1} is the manifold of Kenmotsu type, then regarding Theorem 2.3.1 and Definition 1.3.6, equation (3.3.13) looks like the following:

1. $R_{a0b0,t} = 0;$
2. $R_{\hat{a}0b0,t} = \rho_t \; \delta^a_b - \lambda_t \; \delta^a_b;$
3. $R_{a0bc,t} = 0;$
4. $R_{\hat{a}0bc,t} = 0;$
5. $R_{a0\hat{b}c,t} = 0;$
6. $R_{abcd,t} = 0;$
7. $R_{\hat{a}bcd,t} = -2\rho_t \; A^a_{bcd};$
8. $R_{\hat{a}bc\hat{d},t} = \rho_t(-A^{ad}_{bc} + B^{ah}{}_c \; B_{bh}{}^d + \delta^a_c \; \delta^d_b) - \lambda_t \; \delta^a_c \; \delta^d_b;$
9. $R_{\hat{a}\hat{b}cd,t} = 2\rho_t(-B^{ab}{}_{[cd]} + \delta^a_{[c} \; \delta^b_{d]}) - 2\lambda_t \; \delta^a_{[c} \; \delta^b_{d]}.$

Now, if we use Theorem 3.3.2, then item 2 above gives $\rho_t = \lambda_t$, and this implies that the 1-forms ρ and λ must be equal. Moreover, if we combine the above items again with Theorem 3.3.2, then we deduce the following theorem:

Theorem 3.3.5 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ is a generalized $\Phi-$recurrent if and only if, M satisfies the following conditions:*

$$\rho = \lambda; \quad A^a_{bcd} = 0; \quad B^{ab}{}_{[cd]} = 0; \quad A^{ad}_{bc} = B^{ah}{}_c \; B_{bh}{}^d.$$

Corollary 3.3.2 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ is locally symmetric if and only if, M^{2n+1} is a generalized $\Phi-$recurrent with $\rho = \lambda$.*

Proof: The result follows from Theorems 3.3.3 and 3.3.5. □

Theorem 3.3.6 *The manifold of Kenmotsu type* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *satisfies the following relations:*

1. $g(\nabla_\xi(R)(Z,W)Y,X) = -2g(R(Z,W)Y + g(Y,W)Z - g(Y,Z)W,X)$;

2. $g(\nabla_U(R)(Z,W)\xi,X) = -g(R(Z,W)U + g(U,W)Z - g(U,Z)W,X)$;

3. $g(\nabla_U(R)(Z,\xi)Y,X) = -g(R(Z,U)Y + g(Y,U)Z - g(Y,Z)U,X)$.

Proof: Since the components of $g(\nabla_\xi(R)(Z,W)Y,X)$, $g(\nabla_U(R)(Z,W)\xi,X)$ and $g(\nabla_U(R)(Z,\xi)Y,X)$ are $R_{ijkl,0}$, $R_{i0kl,t}$ and $R_{ijk0,t}$ respectively. Then the claim of the present theorem achieving from the Theorems 2.3.1, 3.3.2 and the Definition 1.3.6. $\square$

Chapter Four

The Generalized Curvature Tensor on the Manifold of Kenmotsu Type and the Hypersurfaces of the Hermitian Manifold

Chapter 4

The Generalized Curvature Tensor on the Manifold of Kenmotsu Type and the Hypersurfaces of the Hermitian Manifold

This chapter divides into two parts, the first one focusses on the generalized curvature tensor for the manifold of Kenmotsu type. Whereas, the second part discusses the manifold of Kenmotsu type as a hypersurface of the Hermitian manifold.

4.1 The Geometry of the Generalized Curvature Tensor on the Manifold of Kenmotsu Type

In this section, we investigate the geometric properties, especially the flatness property of the generalized curvature tensor on the manifold of Kenmotsu type.

Remark 4.1.1 On the $AG-$structure space, the generalized curvature tensor $\widetilde{B}$ which mentioned in Definition 1.4.9, has the following components form:

$$\widetilde{B}_{ijkl} = a_0 R_{ijkl} + a_1\{g_{ik}\, r_{jl} - g_{il}\, r_{jk} + r_{ik}\, g_{jl} - r_{il}\, g_{jk}\} + 2a_2 s\{g_{ik}\, g_{jl} - g_{il}\, g_{jk}\}. \quad (4.1.1)$$

Theorem 4.1.1 *On AG−structure space, the components of the generalized curvature tensor $\widetilde{B}$ for the manifold of Kenmotsu type are given by*

1. $\widetilde{B}_{a0b0} = a_1\ r_{ab}$;
2. $\widetilde{B}_{\hat{a}0b0} = -(a_0 + 2na_1 - 2a_2 s)\delta^a_b + a_1\ r_{\hat{a}b}$;
3. $\widetilde{B}_{\hat{a}bcd} = 2a_0\ A^a_{bcd} + a_1\{\delta^a_c\ r_{bd} - \delta^a_d\ r_{bc}\}$;
4. $\widetilde{B}_{\hat{a}bc\hat{d}} = a_0(A^{ad}_{bc} - B^{ah}{}_c\ B_{bh}{}^d) + a_1\{\delta^a_c\ Q^d_b + \delta^d_b\ Q^a_c\} + (2a_2 s - a_0)\delta^a_c\ \delta^d_b$;
5. $\widetilde{B}_{\hat{a}\hat{b}cd} = 2a_0\ B^{ab}{}_{[cd]} + 4a_1\ \delta^{[a}_{[c}\ Q^{b]}_{d]} + 2(2a_2 s - a_0)\ \delta^{[a}_{[c}\ \delta^{b]}_{d]}$;

and the remaining components are identical to zero or given by the same properties of R or the conjugate to the above components.

Proof: Since $r(X,Y) = g(X,QY)$, then $r_{ij} = g_{ik}Q^k_j$. Consquently, regarding the Definition 1.3.6, we have

$$r_{\hat{a}b} = g_{\hat{a}k}Q^k_b = g_{\hat{a}0}Q^0_b + g_{\hat{a}c}Q^c_b + g_{\hat{a}\hat{c}}Q^{\hat{c}}_b = Q^a_b.$$

Since $\widetilde{B}$ defined on the manifold of Kenmotsu type, then the substitutions of the values of $R_{ijkl} = R^{\hat{i}}_{jkl}$ and g_{ij} from Theorem 2.3.1 and Definition 1.3.6, respectively in the equation (4.1.1), we get the desired. □

Theorem 4.1.2 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ has flat generalized curvature tensor if and only if, M is an $\eta-$Einstein manifold with $\alpha = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s)$, $A^a_{bcd} = 0$, $\beta = -(2n+\alpha)$, $A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d + \frac{a_1}{a_0}\beta\ \delta^a_c\delta^d_b$ and $B^{ab}{}_{[cd]} = \frac{a_1}{a_0}\beta\ \delta^a_{[c}\delta^b_{d]}$, provided that $a_0, a_1 \neq 0$.*

Proof: Suppose that M^{2n+1} has a flat generalized curvature tensor with $a_0 \neq 0$ and $a_1 \neq 0$, then $\widetilde{B}_{ijkl} = 0$ and according to the Theorem 4.1.1, we have

$$r_{ab} = 0; \quad r_{\hat{a}b} = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s)\delta^a_b; \quad A^a_{bcd} = 0.$$

Then taking into account Definition 1.4.4 and the above value of $r_{\hat{a}b}$, we get $\alpha = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s)$. Since M is the manifold of Kenmotsu type, then from Theorem 2.3.2, we have $r_{00} = -2n = \alpha + \beta$ and this gives β. Again, Theorem 4.1.1; item

4 gives $A^{ad}_{bc} = B^{ah}{}_{c}\ B_{bh}{}^{d} + \frac{a_1}{a_0}\beta\ \delta^a_c\ \delta^d_b$. Moreover, Theorem 4.1.1; item 5 gives $B^{ab}{}_{[cd]} = \frac{a_1}{a_0}\beta\ \delta^a_{[c}\ \delta^b_{d]}$. The converse is also true. □

Now, we introduce the notion of generalized $\Phi-$holomorphic sectional ($G\Phi HS-$) curvature tensor which is embodied in the following definition:

Definition 4.1.1 *A $G\Phi HS-$curvature tensor S of any $(M^{2n+1}, \xi, \eta, \Phi, g)$ manifold is defined by*

$$S(X) = \frac{\widetilde{B}(\Phi X, X, X, \Phi X)}{(g(X,X))^2}; \quad \forall\ X \in \ker(\eta); \quad X \neq 0.$$

Moreover, M is called of pointwise constant $G\Phi HS-$curvature if $S(X) = \gamma$ and γ does not depend on X.

Clearly that, $G\Phi HS-$curvature tensor is $\Phi HS-$curvature tensor if and only if, $a_0 = 1$, and $a_1 = a_2 = 0$. Therefore, we can drive the necessary and sufficient condition for $ACR-$manifold to have pointwise constant $G\Phi HS-$curvature on $AG-$structure space.

Theorem 4.1.3 *$(M^{2n+1}, \xi, \eta, \Phi, g)$ has pointwise constant $G\Phi HS-$curvature if and only if, on $AG-$structure space, the generalized curvature tensor $\widetilde{B}$ of M satisfies the equality below.*

$$\widetilde{B}^{(a\ d)}_{(bc)} = \frac{\gamma}{2}\widetilde{\delta}^{ad}_{bc}.$$

Proof: Since the tensor $\widetilde{B}$ has the same properties of Riemannian curvature tensor R, then we can follow the same proof in [71] or equivalently in [111]. □

Theorem 4.1.4 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ has pointwise constant $G\Phi HS-$curvature if and only if, on $AG-$structure space, M satisfies the following equality:*

$$A^{ad}_{bc} = B_{bc}{}^{[ad]} - B_{hb}{}^{a}\ B^{dh}{}_{c} - \frac{2a_1}{a_0}\delta^{(a}_{(b}Q^{d)}_{c)} + \frac{\gamma - 2a_2 s + a_0}{2a_0}\widetilde{\delta}^{ad}_{bc}.$$

Proof: Suppose that M is the manifold of Kenmotsu type and has pointwise constant $G\Phi HS-$curvature. Regarding the Theorem 4.1.3 and Theorem 4.1.1; item 4, we get

$$A^{(ad)}_{(bc)} = B^{(a|h|}{}_{(b}\ B_{c)h}{}^{d)} - \frac{2a_1}{a_0}\delta^{(a}_{(b}Q^{d)}_{c)} + \frac{\gamma - 2a_2 s + a_0}{2a_0}\widetilde{\delta}^{ad}_{bc}.$$

The above equation can be rewritten as follows:

$$A^{(ad)}_{(bc)} = -B_{h(b}{}^{(a}\ B^{d)h}{}_{c)} - \frac{2a_1}{a_0}\delta^{(a}_{(b}Q^{d)}_{c)} + \frac{\gamma - 2a_2 s + a_0}{2a_0}\widetilde{\delta}^{ad}_{bc}.$$

Since $A^{ad}_{bc} = A^{[ad]}_{[bc]} + A^{[ad]}_{(bc)} + A^{(ad)}_{[bc]} + A^{(ad)}_{(bc)}$, then taking into account the Theorem 2.2.3 with the technique of the Theorem 3.1.1 and the above result, we attain the requirement. □

Recently, Yildiz and De [118] introduced the notions of Φ-projectively semisymmetric and Φ-Weyl semisymmetric. Regarding these ideas, we can introduce the following definition:

Definition 4.1.2 *An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is called a Φ-generalized semi (Φ$GS-$) symmetric if $\widetilde{B}(Z, W) \cdot \Phi = 0$, for all $Z, W \in X(M)$, or equivalently*

$$\widetilde{B}(X, \Phi Y, Z, W) + \widetilde{B}(\Phi X, Y, Z, W) = 0; \quad \forall\ X, Y, Z, W \in X(M).$$

Lemma 4.1.1 On $AG-$structure space, the $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is $\Phi GS-$symmetric if and only if,

$$\widetilde{B}_{a0b0} = \widetilde{B}_{\hat{a}0b0} = \widetilde{B}_{a0bc} = \widetilde{B}_{\widehat{a}0bc} = \widetilde{B}_{a0\widehat{b}c} = \widetilde{B}_{abcd} = \widetilde{B}_{\hat{a}\hat{b}cd} = 0.$$

Proof: According to the Definition 4.1.2, we have M is $\Phi GS-$symmetric if and only if,

$$\widetilde{B}(X, \Phi Y, Z, W) + \widetilde{B}(\Phi X, Y, Z, W) = 0; \quad \forall\ X, Y, Z, W \in X(M).$$

On the $AG-$structure space, the above identity equivalent to the following:

$$\widetilde{B}_{iqkl}\ \Phi^q_j + \widetilde{B}_{tjkl}\ \Phi^t_i = 0; \quad q, t = 0, 1, ..., 2n.$$

If we take

$$(i, j, k, l) = (a, 0, b, 0), (\hat{a}, 0, b, 0), (a, 0, b, c), (\hat{a}, 0, b, c), (a, 0, \hat{b}, c), (a, b, c, d), (\hat{a}, \hat{b}, c, d),$$

and using the Definition 1.3.6, we obtain the result. □

It is not hard to conclude the following:

Corollary 4.1.1 *The $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ of flat generalized curvature tensor is usually $\Phi GS-$symmetric.*

Corollary 4.1.2 *The manifold of Kenmotsu type* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *has flat generalized curvature tensor if and only if, M is $\Phi GS-$symmetric with* $A^a_{bcd} = 0$ *and* $A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d + \frac{a_1}{a_0}\mu\ \delta^a_c\delta^d_b$*, where* $\mu = -\frac{1}{a_1}(a_0 + 4na_1 - 2a_2 s)$*, provided that* $a_0, a_1 \neq 0$.

Proof: Suppose that M is the manifold of Kenmotsu type and it has flat generalized curvature tensor, then from Corollary 4.1.1, we see that M is $\Phi GS-$symmetric and regarding Theorem 4.1.1, we get the other conditions.
Conversely, If M is $\Phi GS-$symmetric with the above conditions then according to Lemma 4.1.1 and Theorem 4.1.1, we have M has flat generalized curvature tensor. □

Theorem 4.1.5 *The manifold of Kenmotsu type* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *posses $\Phi GS-$symmetric if and only if, M is an $\eta-$Einstein manifold with* $\alpha = \frac{1}{a_1}(a_0+2na_1-2a_2 s)$*,* $\beta = -(2n+\alpha)$ *and* $B^{ab}{}_{[cd]} = \frac{a_1}{a_0}\beta\ \delta^a_{[c}\delta^b_{d]}$*, provided that* $a_0, a_1 \neq 0$.

Proof: Suppose that M is $\Phi GS-$symmetric manifold of Kenmotsu type, then from Lemma 4.1.1 and Theorem 4.1.1, we have

$$r_{ab} = 0; \quad r_{\hat{a}b} = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s)\delta^a_b; \quad B^{ab}{}_{[cd]} = -\frac{1}{a_0}(a_0 + 4na_1 - 2a_2 s)\delta^a_{[c}\delta^b_{d]}.$$

Regarding Definition 1.4.4 and Theorem 2.3.2, we attain the values of α and β.
The converse is verified directly from Theorem 4.1.1 and Lemma 4.1.1. □

Corollary 4.1.3 *The manifold of Kenmotsu type* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *posses $\Phi GS-$symmetric and $G\Phi HS-$curvature if and only if, M is $\eta-$Einstein manifold with* $\alpha = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s)$*,* $\beta = -(2n+\alpha)$*,* $B^{ab}{}_{[cd]} = \frac{a_1}{a_0}\beta\delta^a_{[c}\delta^b_{d]}$*, and*

$$A^{ad}_{bc} = \frac{\gamma}{2a_0}\widetilde{\delta}^{ad}_{bc} - B_{hb}{}^a\ B^{dh}{}_c + \frac{a_1}{a_0}\beta\delta^a_b\delta^d_c,$$

provided that $a_0, a_1 \neq 0$.

Proof: Suppose that M is the manifold of Kenmotsu type, then the necessary and sufficient conditions for the present corollary are satisfied from the Theorems 4.1.4 and 4.1.5. □

Now, we introduce a generalization of the notion of $ACR-$manifold of constant curvature used by Abood and Al-Hussaini [2]. We shall show this idea in the following definition:

Definition 4.1.3 *An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is said to have constant generalized curvature κ if the following identity holds:*

$$\widetilde{B}(X, Y, Z, W) = \kappa\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\}; \quad \forall\ X, Y, Z, W \in X(M).$$

On the $AG-$structure space, Definition 4.1.3 equivalent to the identity below.

$$\widetilde{B}_{ijkl} = \kappa\{g_{ik}\ g_{jl} - g_{il}\ g_{jk}\}. \qquad (4.1.2)$$

Directly, regarding Definitions 4.1.3, 1.4.8 and 1.4.9, we have the following result:

Theorem 4.1.6 *Suppose that $(M^{2n+1}, \xi, \eta, \Phi, g)$ is an $ACR-$manifold of constant generalized curvature $\kappa = 2a_2 s$. Then M has flat conharmonic curvature tensor if and only if, $a_0 = 1$ and $a_1 = -\frac{1}{2n-1}$.*

Theorem 4.1.7 *An $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ has constant generalized curvature κ if and only if, on the $AG-$structure space, $\widetilde{B}$ has the following components:*

1. $\widetilde{B}_{\hat{a}0b0} = \kappa\ \delta^a_b$;
2. $\widetilde{B}_{\hat{a}bc\hat{d}} = \kappa\ \delta^a_c \delta^d_b$;
3. $\widetilde{B}_{\hat{a}\hat{b}cd} = 2\kappa\ \delta^a_{[c}\delta^b_{d]}$;

and the remaining components are identical to zero or establishing from the above components by the same properties of R or by taking the conjugate operation.

Proof: The result follows from equation (4.1.2) by taking

$$(i, j, k, l) = (\hat{a}, 0, b, 0), (\hat{a}, b, c, \hat{d}), (\hat{a}, \hat{b}, c, d);$$

and regarding Definition 1.3.6. □

Theorem 4.1.8 *The $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is $\Phi GS-$symmetric if and only if, M has constant generalized curvature $\kappa = 0$.*

Proof: The claim of this theorem is achieving from Lemma 4.1.1 and Theorem 4.1.7. □

Theorem 4.1.9 *If an $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ has constant generalized curvature κ, then M has pointwise constant $G\Phi HS-$curvature equal to $\gamma = \kappa$.*

Proof: The allegation of the present theorem occurs from the Theorems 4.1.3 and 4.1.7. □

Theorem 4.1.10 *The manifold of Kenmotsu type $(M^{2n+1}, \xi, \eta, \Phi, g)$ has constant generalized curvature κ if and only if, M is an $\eta-$Einstein manifold with $\alpha = \frac{1}{a_1}(a_0 + 2na_1 - 2a_2 s + \kappa)$, $A^a_{bcd} = 0$, $\beta = -(2n + \alpha)$, $A^{ad}_{bc} = B^{ah}{}_c\, B_{bh}{}^d + \frac{a_1}{a_0}\beta\ \delta^a_c \delta^d_b$ and $B^{ab}{}_{[cd]} = \frac{a_1}{a_0}\beta\ \delta^a_{[c}\delta^b_{d]}$, provided that $a_0, a_1 \neq 0$.*

Proof: The assertion of this theorem can be happen, if we are combining the results of Theorems 4.1.1 and 4.1.7. □

Now, we try to find the geometric properties of $ACR-$manifold if the generalized curvature tensor, the concircular curvature tensor and the projective curvature tensor are related.

Suppose that $(M^{2n+1}, \xi, \eta, \Phi, g)$ is an $ACR-$manifold satisfies the following condition:

$$\widetilde{B}(X, Y, Z, W) = \frac{a_0}{3}\{P(X, Y, Z, W) - P(Y, X, Z, W) + \widetilde{C}(X, Y, Z, W)\}. \quad (4.1.3)$$

Regarding equations (1.4.1), (1.4.2) and (4.1.1), equation (4.1.3) can be written on the $AG-$structure space as follows:

$$\begin{aligned} 0 &= (a_1 + \frac{a_0}{6n})\{g_{ik}\ r_{jl} - g_{il}\ r_{jk} + r_{ik}\ g_{jl} - r_{il}\ g_{jk}\} \\ &+ (2a_2 + \frac{a_0}{6n(2n+1)})s\{g_{ik}\ g_{jl} - g_{il}\ g_{jk}\}. \end{aligned} \quad (4.1.4)$$

The contracting of the equation (4.1.4), that is multiplies it by g^{ik}, we can deduce that

$$r_{jl} = -\frac{(\alpha + 2n\beta)s}{(2n-1)\alpha} g_{jl}, \quad (4.1.5)$$

where $\alpha = a_1 + \frac{a_0}{6n}$ and $\beta = 2a_2 + \frac{a_0}{6n(2n+1)}$. Moreover, the contracting of the equation (4.1.5) gives $a_0 + 4na_1 + 4n(2n+1)a_2 = 0$. Then we can state the following theorem:

Theorem 4.1.11 *Any $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ which satisfies the identity (4.1.3) is an Einstein manifold with $a_0 + 4na_1 + 4n(2n+1)a_2 = 0$, provided that $\alpha \neq 0$. Moreover, if M is the manifold of Kenmotsu type then $s = \frac{2n(2n-1)\alpha}{\alpha+2n\beta}$, provided that $\alpha + 2n\beta \neq 0$.*

Proof: The first part of this theorem is obvious from the above discussion. Now, if M is the manifold of Kenmotsu type then from Theorem 2.3.2, we have $r_{00} = -2n$. Then the result is achieved from Definition 1.3.6 and equation (4.1.5). $\square$

4.2 The Manifold of Kenmotsu Type as Hypersurface for the Hermitian Manifold

This section shall study the manifold of Kenmotsu type as a hypersurface of Hemitian manifold.

Remark 4.2.1 [95] Suppose that $(M^{2n-1}, \xi, \eta, \Phi, g)$ is an $ACR-$manifold, then there exists an almost complex structure J on $M \times \mathbb{R}$ defined by $J(X, f\frac{d}{dt}) = (\Phi X - f\xi, \eta(X)\frac{d}{dt})$, where $X \in X(M)$, $t \in \mathbb{R}$ and f is a smooth function on $\mathbb{R}$. The Riemannian metric h on $M \times \mathbb{R}$ is defined by

$$h((X, f_1\frac{d}{dt}), (Y, f_2\frac{d}{dt})) = g(X, Y) + f_1\ f_2; \quad \forall\ X, Y \in X(M); \quad f_1, f_2 \in C^{\infty}(\mathbb{R}).$$

The structure on $M \times \mathbb{R}$ is Hermitian if and only if the structure on M is normal.

Remark 4.2.2 Since the manifold of Kenmotsu type is normal because it belongs to the class $C_3 \oplus C_4 \oplus C_5$, where C_5 is taken here to be Kenmotsu manifold mentioned in Theorem 1.4.3 (see [34] for more details about the classes C_3 and C_4). Then the structure on the product of the manifold of Kenmotsu type and the real line is Hermitian structure (i.e. $W_3 \oplus W_4$) according to Remark 4.2.1.

Now, we discuss the opposite problem, that is, if (N^{2n}, J, h) is Hermitian manifold, then can we find a hypersurface of N which is the manifold of Kenmotsu type? For this reason, we suppose that $\alpha, \beta, \gamma = 1, 2, ..., n-1$ and $\sigma_{ij} = \sigma_{ji}$;

$i, j = 1, 2, ..., 2n-1$ are the components of the second quadratic form. From Banaru [10], we see that the Hermitian manifold N satisfies $C^{abc} = C_{abc} = 0$, where $a, b, c = 1, 2, ..., n$, then Theorem 1.5.1 reduces to the following form:

Theorem 4.2.1 *The $ACR-$manifold on a hypersurface of Hermitian manifold has the following first family of Cartan's structure equations:*

$$\begin{aligned} d\omega^\alpha &= \omega^\alpha_\beta \wedge \omega^\beta + C^{\alpha\beta}_\gamma \ \omega^\gamma \wedge \omega_\beta + (\sqrt{2}C^{\alpha n}_\beta + \sqrt{-1}\sigma^\alpha_\beta)\omega^\beta \wedge \omega \\ &+ (\sqrt{-1}\sigma^{\alpha\beta} - \frac{1}{\sqrt{2}}C^{\alpha\beta}_n)\omega_\beta \wedge \omega; \\ d\omega_\alpha &= -\omega^\beta_\alpha \wedge \omega_\beta + C^\gamma_{\alpha\beta} \ \omega_\gamma \wedge \omega^\beta + (\sqrt{2}C^\beta_{\alpha n} - \sqrt{-1}\sigma^\beta_\alpha)\omega_\beta \wedge \omega \\ &- (\sqrt{-1}\sigma_{\alpha\beta} + \frac{1}{\sqrt{2}}C^n_{\alpha\beta})\omega^\beta \wedge \omega; \\ d\omega &= (\sqrt{2}C^{n\alpha}_\beta - \sqrt{2}C^\alpha_{n\beta} - 2\sqrt{-1}\sigma^\alpha_\beta)\omega^\beta \wedge \omega_\alpha + (C^n_{n\beta} + \sqrt{-1}\sigma_{n\beta})\omega \wedge \omega^\beta \\ &+ (C^{n\beta}_n - \sqrt{-1}\sigma^\beta_n)\omega \wedge \omega_\beta, \end{aligned}$$

where ω^α_β play the same role of θ^α_β.

Regarding Theorem 2.2.2, we note that the manifold $(M^{2n-1}, \xi, \eta, \Phi, g)$ of Kenmotsu type satisfies the following theorem on a certain basis of $X(M)$:

Theorem 4.2.2 *The manifold of Kenmotsu type has the following first group of Cartan's structure equations:*

$$\begin{aligned} d\omega^\alpha &= \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma \ \omega^\gamma \wedge \omega_\beta - \omega^\alpha \wedge \omega; \\ d\omega_\alpha &= -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma \ \omega_\gamma \wedge \omega^\beta - \omega_\alpha \wedge \omega; \\ d\omega &= 0, \end{aligned}$$

where $\omega^\alpha_\beta = -\theta^\alpha_\beta$.

Now, if the manifold of Kenmotsu type $(M^{2n-1}, \xi, \eta, \Phi, g)$ is a hypersurface of the Hermitian manifold (N^{2n}, J, h), then the Cartan's structure equations mentioned in Theorems 4.2.1 and 4.2.2 must be equal. Then we get

$$\begin{aligned} &C^{\alpha\beta}_\gamma = B^{\alpha\beta}{}_\gamma; \quad \sqrt{2}C^{\alpha n}_\beta + \sqrt{-1}\sigma^\alpha_\beta = -\delta^\alpha_\beta; \quad \sqrt{-1}\sigma^{\alpha\beta} - \frac{1}{\sqrt{2}}C^{\alpha\beta}_n = 0; \\ &C^\gamma_{\alpha\beta} = B_{\alpha\beta}{}^\gamma; \quad \sqrt{2}C^\beta_{\alpha n} - \sqrt{-1}\sigma^\beta_\alpha = -\delta^\beta_\alpha; \quad \sqrt{-1}\sigma_{\alpha\beta} + \frac{1}{\sqrt{2}}C^n_{\alpha\beta} = 0; \qquad (4.2.6) \\ &\sqrt{2}C^{n\alpha}_\beta - \sqrt{2}C^\alpha_{n\beta} - 2\sqrt{-1}\sigma^\alpha_\beta = 0; \quad C^n_{n\beta} + \sqrt{-1}\sigma_{n\beta} = 0; \quad C^{n\beta}_n - \sqrt{-1}\sigma^\beta_n = 0. \end{aligned}$$

Since $\sigma_{[\alpha\beta]} = 0$ and $C^{\gamma}_{[\alpha\beta]} = C^{\gamma}_{\alpha\beta}$, then equation (4.2.6) gives the following relations:

$$\sigma_{\alpha\beta} = 0; \quad \sigma_{n\beta} = 0; \quad \sigma^{\alpha}_{\beta} = \sqrt{-1}(\sqrt{2}C^{\alpha n}_{\beta} + \delta^{\alpha}_{\beta}). \tag{4.2.7}$$

Thus from the above discussion, we can establish the theorem below.

Theorem 4.2.3 *If the Hermitian manifold has the manifold of Kenmotsu type as a hypersurface, then the second quadratic form σ has components agree with the equation* (4.2.7).

On the other hand, we can establish a relation between the components of Riemannian curvature tensors of the AH-manifold and its hypersurfaces. For this purpose, we suppose that $\mathcal{R}^i_{jkl}$ are the components of Riemannian curvature tensor of AH-manifold (N^{2n}, J, h) and $\widetilde{\mathcal{R}}^i_{jkl}$ are the components of Riemannian curvature tensor of its hypersurface $(M^{2n-1}, \xi, \eta, \Phi, g)$. Then from the second group of Cartan's structure equations, we have

$$d\omega^i_j = \omega^i_k \wedge \omega^k_j + \frac{1}{2}\mathcal{R}^i_{jkl}\ \omega^k \wedge \omega^l;$$
$$d\theta^i_j = \theta^i_k \wedge \theta^k_j + \frac{1}{2}\widetilde{\mathcal{R}}^i_{jkl}\ \theta^k \wedge \theta^l,$$

where ω^i_j and θ^i_j are Riemannian connection forms of N and M respectively. Whereas, ω^k and θ^k are the dual A-frames on $AG-$structure spaces of N and M respectively. Moreover, from [13], we have

$$\theta^i = C^i_j\ \omega^j; \quad \omega^i = \widetilde{C}^i_j\ \theta^j; \quad \theta^i_j = C^i_k\ \omega^k_r\ \widetilde{C}^r_j; \quad \omega^i_j = \widetilde{C}^i_k\ \theta^k_r\ C^r_j,$$

where $C = (C^i_j)$ and $C^{-1} = (\widetilde{C}^i_j)$ were defined in [13]. Then the substitution of the above relations in the second group of Cartan's structure equations, we conclude the following theorem:

Theorem 4.2.4 *If $\mathcal{R}^i_{jkl}$ and $\widetilde{\mathcal{R}}^q_{rst}$ are the components of Riemannian curvature tensor of AH-manifold (N^{2n}, J, g) and its hypersurface $(M^{2n-1}, \Phi, \xi, \eta, g)$ respectively, then they are related as follows:*

$$\mathcal{R}^i_{jkl} = \widetilde{C}^i_q\ \widetilde{\mathcal{R}}^q_{rst}\ C^r_j\ C^s_k\ C^t_l.$$

Chapter Five

The Geometry of ACR −Manifolds of Class C_{12}

Chapter 5

The Geometry of $ACR-$Manifolds of Class C_{12}

This chapter is devoted to investigating the structure equations of the class C_{12} and the curvature components of the aforementioned class on the $AG-$structure space.

5.1 The Structure Equations of the Class C_{12}

In this section, we determine the Cartan's structure equations for $ACR-$manifolds of class C_{12} on the $AG-$structure space using the same techniques of chapter 2.

Regarding Chinea and Gonzalez [34], we note that $(M^{2n+1}, \xi, \eta, \Phi, g)$ belongs to the class C_{12} if it satisfies the following identity:

$$\nabla_X(\Omega)(Y, Z) = \eta(X)\{\eta(Z)\nabla_\xi(\eta)\Phi Y - \eta(Y)\nabla_\xi(\eta)\Phi Z\},$$

for all $X, Y, Z \in X(M)$, where $\Omega(X, Y) = g(X, \Phi Y)$.
Regarding the citation [35], we have

$$\nabla_X(\Omega)(Y, Z) = -g(\nabla_X(\Phi)Y, Z);$$
$$\nabla_X(\eta)Y = -g(\nabla_X(\Phi)\xi, \Phi Y),$$

for all $X, Y, Z \in X(M)$. Then C_{12} identity can be rewritten in the following form:

$$\nabla_X(\Phi)Y = -\eta(X)\{\eta(Y)\Phi(\nabla_\xi\xi) + g(\nabla_\xi\xi, \Phi Y)\xi\}. \tag{5.1.1}$$

If we replace X in the equation (5.1.1) by ΦX or $\Phi^2 X$, we get

$$\nabla_{\Phi X}(\Phi)Y = \nabla_{\Phi^2 X}(\Phi)Y = 0. \tag{5.1.2}$$

Moreover, if we put $Y = \xi$ in the equation (5.1.1), yield

$$\nabla_X \xi = \eta(X)\nabla_\xi \xi. \tag{5.1.3}$$

Theorem 5.1.1 *The $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ belongs to the class C_{12} if and only if the Kirichenko's tensors which are mentioned in chapter 1, attain that*

$$B = C = D = E = F = 0; \quad G = \nabla_\xi \xi.$$

Proof: Regarding the equation (5.1.2), we have $B = C = E = F = 0$. While according to the citation [100], we have

$$\Phi \circ \nabla_X(\Phi)\xi = \nabla_X \xi; \quad \forall\ X \in X(M).$$

So, we get $G = \nabla_\xi \xi$. Since the equation (5.1.1) has the following form on the $AG-$structure space:

$$\begin{aligned}
\Phi^i_{j,k}\ Y^j\ X^k\ \varepsilon_i &= -\eta_k\ X^k\{\eta_j\ Y^j\ \Phi^i_l\ G^l\ \varepsilon_i + \Omega_{lj}\ G^l\ Y^j\ \xi\};\\
\Phi^i_{j,k}\ Y^j\ X^k\ \varepsilon_i &= -\eta_k\ X^k\{\eta_j\ Y^j\ \Phi^i_l\ G^l\ \varepsilon_i + \Omega_{lj}\ G^l\ Y^j\ \delta^i_0\ \varepsilon_i\};\\
\Phi^i_{j,k} &= -\eta_k\{\eta_j\ \Phi^i_l\ G^l + \Omega_{lj}\ G^l\ \delta^i_0\}.
\end{aligned}$$

Then the last equation gives $\Phi^a_{0,\hat{b}}$, $\Phi^a_{\hat{b},0}$ and their conjugate are zero. These imply that $B^{ab} = B_{ab} = 0$, then $D = 0$. □

Regarding Theorem 5.1.1, we conclude that the components of Kirichenko's tensors on the class C_{12} are zero except the components of the tensor G. So, according to Theorems 1.4.5 and 5.1.1, we have that $ACR-$manifold of class C_{12} on $AG-$structure space achieve the following first collection of Cartan's structure equations:

$$\begin{aligned}
d\omega^a &= -\theta^a_b \wedge \omega^b;\\
d\omega_a &= \theta^b_a \wedge \omega_b;\\
d\omega &= C_b\ \omega \wedge \omega^b + C^b\ \omega \wedge \omega_b.
\end{aligned} \tag{5.1.4}$$

Since θ is the 1-form of the Levi-Civita (Rieman) connection for the $ACR-$ manifold of class C_{12}, then regarding Corollary 1.3.1 and the fact that all components of the tensors B, C, D, E, F are zero, we conclude that θ satisfies the following:

$$\theta^a_0 = C^a\ \omega; \quad \theta^a_{\hat{b}} = 0. \tag{5.1.5}$$

Now, if we are acting the operator d on the first part of equation (5.1.4), then we obtain

$$\triangle\Theta^a_b \wedge \omega^b = 0, \tag{5.1.6}$$

where $\triangle\Theta^a_b = d\theta^a_b + \theta^a_c \wedge \theta^c_b$. Since $\triangle\Theta^a_b$ is 2-form, then we can write

$$\begin{aligned}\triangle\Theta^a_b = {} & A^{adh}_{bcf}\ \theta^c_d \wedge \theta^f_h + A^{ad}_{bch}\ \theta^c_d \wedge \omega^h + A^{adh}_{bc}\ \theta^c_d \wedge \omega_h + A^{ad}_{bc0}\ \theta^c_d \wedge \omega + A^a_{bcd}\ \omega^c \wedge \omega^d \\ & + A^{ad}_{bc}\ \omega^c \wedge \omega_d + A^a_{bc0}\ \omega^c \wedge \omega + A^{acd}_b\ \omega_c \wedge \omega_d + A^{ac0}_b\ \omega_c \wedge \omega.\end{aligned}$$

Substitute the above equation in equation (5.1.6), we have

$$A^{adh}_{bcf} = A^{ad}_{[b|c|h]} = A^{adh}_{bc} = A^{ad}_{bc0} = A^a_{[bcd]} = A^{ad}_{[bc]} = A^a_{[bc]0} = A^{acd}_b = A^{ac0}_b = 0.$$

Now, repeating the same argument to the second part of equation (5.1.4), we get

$$A^{adh}_{bcf} = A^{ad}_{bch} = A^{[a|d|h]}_{bc} = A^{ad}_{bc0} = A^a_{bcd} = A^{[ad]}_{bc} = A^a_{bc0} = A^{[acd]}_b = A^{[ac]0}_b = 0.$$

So, we have

$$d\theta^a_b = -\theta^a_c \wedge \theta^c_b + A^{ad}_{bc}\ \omega^c \wedge \omega_d,$$

where $A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$. Moreover, the exterior differentiation of the third part of equation (5.1.4) leading to

$$dC_b \wedge \omega \wedge \omega^b + C_b\ d\omega \wedge \omega^b - C_b\ \omega \wedge d\omega^b + dC^b \wedge \omega \wedge \omega_b + C^b\ d\omega \wedge \omega_b - C^b\ \omega \wedge d\omega_b = 0.$$

The above equation implies that

$$\begin{aligned}(dC_b - C_d\ \theta^d_b) \wedge \omega \wedge \omega^b + C_{[b}\ C_{a]}\ \omega \wedge \omega^a \wedge \omega^b + (dC^b + C^d\ \theta^b_d) \wedge \omega \wedge \omega_b \\ + C^{[b}\ C^{a]}\ \omega \wedge \omega_a \wedge \omega_b = 0.\end{aligned}$$

Since $C_{[b}\ C_{a]} = \frac{1}{2}(C_b\ C_a - C_a\ C_b) = 0$ and similarly $C^{[b}\ C^{a]} = 0$, then the above equation reduces to

$$(dC_b - C_d\ \theta^d_b) \wedge \omega \wedge \omega^b + (dC^b + C^d\ \theta^b_d) \wedge \omega \wedge \omega_b = 0. \tag{5.1.7}$$

Since the forms $(dC_b - C_d\ \theta_b^d)$ and $(dC^b + C^d\ \theta_d^b)$ are 1-forms, then they can be written in the following formulae:

$$dC_b - C_d\ \theta_b^d = C_{bh}^d\ \theta_d^h + C_{bd}\ \omega^d + C_b^d\ \omega_d + C_{b0}\ \omega,$$
$$dC^b + C^d\ \theta_d^b = C_d^{bh}\ \theta_h^d + C^{bd}\ \omega_d + C_d^b\ \omega^d + C^{b0}\ \omega,$$

then the substitution of the above formulae in equation (5.1.7) gives $C_{bh}^d = C_d^{bh} = C_{[bd]} = C^{[bd]} = 0$. So, we can state the following theorem:

Theorem 5.1.2 *The second family of Cartan's structure equations of the class C_{12} on the $AG-$structure space are given by the following formulae:*

1. $d\theta_b^a = -\theta_c^a \wedge \theta_b^c + A_{bc}^{ad}\ \omega^c \wedge \omega_d$;

2. $dC_b = C_d\ \theta_b^d + C_{bd}\ \omega^d + C_b^d\ \omega_d + C_{b0}\ \omega$;

3. $dC^b = -C^d\ \theta_d^b + C^{bd}\ \omega_d + C_d^b\ \omega^d + C^{b0}\ \omega$,

where $A_{bc}^{[ad]} = A_{[bc]}^{ad} = C_{[bd]} = C^{[bd]} = 0$.

The above theorem agrees with Theorem 1.4.6, and we have $C_{bd} = \nabla_{\varepsilon_d} C_b$, $C_b^d = \nabla_{\varepsilon_{\hat{d}}} C_b$, $C_{b0} = \nabla_{\xi} C_b$ and so on.

Corollary 5.1.1 *The $ACR-$manifold of class C_{12} is cosymplectic manifold if and only if $G = 0$.*

Proof: The allegation of this corollary is verified from equation (5.1.1). □

5.2 The Curvature Tensors on the Class C_{12}

In this section, we determine the components of the Riemannian curvature tensor and Ricci tensor for the $ACR-$manifold of class C_{12}. Moreover, we investigate the (κ, μ)-nullity distribution of the class C_{12}.

We begin this section with an example on $ACR-$manifold of class C_{12} of dimension 3.

Example 5.2.1 Suppose that $(M^3, \xi, \eta, \Phi, g)$ is $ACR-$manifold of dimension three, such that

$$M = \{(x, y, z) \in \mathbb{R}^3 : y \neq 0\}$$

and suppose that $\{e_0, e_1, e_2\}$ is a Φ-basis of the Lie algebra of smooth vector fields $X(M)$, such that

$$[e_0, e_1] = -e_0, \quad [e_0, e_2] = [e_1, e_2] = 0,$$

and

$$e_0 = \xi, \quad \Phi(e_1) = e_2, \quad \Phi(e_2) = -e_1,$$

where

$$e_0 = e^y \frac{\partial}{\partial x}, \quad e_1 = \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial z}.$$

Moreover, we define the Riemannian metric g and the 1-form η as follows:

$$g(e_i, e_j) = \delta_{ij}, \quad i, j = 0, 1, 2, \quad \eta(X) = g(X, \xi), \quad X \in X(M),$$

where δ_{ij} is the Krönecker delta. Then from the following Koszul's formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) - g(X, [Y, Z]) \\ &\quad - g(Y, [X, Z]) + g(Z, [X, Y]); \quad \forall\ X, Y, Z \in X(M), \end{aligned}$$

we note that

$$\begin{aligned} &\nabla_{e_0} e_0 = e_1, \quad \nabla_{e_0} e_1 = -e_0, \quad \nabla_{e_0} e_2 = 0, \\ &\nabla_{e_1} e_0 = 0, \qquad \nabla_{e_1} e_1 = 0, \qquad \nabla_{e_1} e_2 = 0, \\ &\nabla_{e_2} e_0 = 0, \qquad \nabla_{e_2} e_1 = 0, \qquad \nabla_{e_2} e_2 = 0. \end{aligned}$$

Then $(M^3, \Phi, \xi, \eta, g)$ satisfies equation (5.1.1) and then it is 3-dimensional $ACR-$manifold of class C_{12}.

Now, we can determine the components R^i_{jkl} of Riemannian curvature tensor on $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ of class C_{12} over the $AG-$structure space by using the following equations from Theorem 1.4.1; item (2):

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \frac{1}{2} R^i_{jkl}\ \omega^k \wedge \omega^l,$$

where $i, j, k, l = 0, 1, ..., 2n$. Since M^{2n+1} satisfies equation (5.1.5) and Theorem 5.1.2, then we can conclude the following theorem:

Theorem 5.2.1 *On AG-structure space, the components of Riemannian curvature tensor R of the class C_{12} are given as the following:*

1. $R^a_{0b0} = C^a_b - C^a\ C_b$;

2. $R^a_{0\hat{b}0} = C^{ab} - C^a\ C^b$;

3. $R^a_{bc\hat{d}} = A^{ad}_{bc}$,

and the other components are zero or given by the properties of R or the conjugate to the above components (i.e. $\overline{R^i_{jkl}} = R^{\hat{i}}_{\hat{j}\hat{k}\hat{l}}$).

Proof: If we take into account Theorem 1.4.1; item (2) and setting $i = a,\ j = 0$, then we arrive to the following:

$$d\theta^a_0 + \theta^a_0 \wedge \theta^0_0 + \theta^a_b \wedge \theta^b_0 + \theta^a_{\hat{b}} \wedge \theta^{\hat{b}}_0 = R^a_{0b0}\ \omega^b \wedge \omega + R^a_{0\hat{b}0}\ \omega_b \wedge \omega + \frac{1}{2} R^a_{0bd}\ \omega^b \wedge \omega^d + R^a_{0b\hat{d}}\ \omega^b \wedge \omega_d + \frac{1}{2} R^a_{0\hat{b}\hat{d}}\ \omega_b \wedge \omega_d.$$

According to equation (5.1.5) and Lemma 1.2.1; item 3, we get

$$dC^a \wedge \omega + C^a\ d\omega + C^b\ \theta^a_b \wedge \omega = R^a_{0b0}\ \omega^b \wedge \omega + R^a_{0\hat{b}0}\ \omega_b \wedge \omega + \frac{1}{2} R^a_{0bd}\ \omega^b \wedge \omega^d + R^a_{0b\hat{d}}\ \omega^b \wedge \omega_d + \frac{1}{2} R^a_{0\hat{b}\hat{d}}\ \omega_b \wedge \omega_d.$$

Then the items 1 and 2 of the present theorem are done by the substitution of equation (5.1.4) and Theorem 5.1.2 in the above equality. Therefore, to carry out item 3, we put $i = a,\ j = b$ in Theorem 1.4.1; item (2) and follows the same technique given above. $\square$

Lemma 5.2.1 In the $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ of class C_{12}, the following identity:

$$2d\eta(X, Y) = \eta(X)g(G, Y) - \eta(Y)g(G, X),$$

holds for all $X, Y \in X(M)$.

Proof: Using equation (5.1.3), Theorem 5.1.1 and the fact that

$$\eta(\nabla_X \xi) = \eta(X)\eta(\nabla_\xi \xi) = \eta(X)\eta(G) = \eta(X)\eta \circ \Phi(\nabla_\xi(\Phi)\xi) = 0.$$

Also, from the citation [35], it follows that:

$$
\begin{aligned}
2d\eta(X,Y) &= \nabla_X(\Omega)(\xi,\Phi Y) - \nabla_Y(\Omega)(\xi,\Phi X);\\
&= -g(\nabla_X(\Phi)\xi,\Phi Y) + g(\nabla_Y(\Phi)\xi,\Phi X);\\
&= g(\Phi(\nabla_X\xi),\Phi Y) - g(\Phi(\nabla_Y\xi),\Phi X);\\
&= g(\nabla_X\xi,Y) - g(\nabla_Y\xi,X);\\
&= \eta(X)g(G,Y) - \eta(Y)g(G,X).
\end{aligned}
$$

□

Theorem 5.2.2 *The $ACR-$manifold $(M^{2n+1},\xi,\eta,\Phi,g)$ of class C_{12} attains the following curvature identity:*

$$R(X,Y)\xi = 3d\eta(X,Y)G - X(\eta(Y))G + Y(\eta(X))G + \eta(Y)\nabla_X G - \eta(X)\nabla_Y G,$$

for all vector fields $X,Y \in X(M)$.

Proof: Using the equality $d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])$, equation (5.1.3) and Lemma 5.2.1, we obtain

$$
\begin{aligned}
R(X,Y)\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi;\\
&= \nabla_X(\eta(Y)G) - \nabla_Y(\eta(X)G) - \eta([X,Y])G;\\
&= (\nabla_X(\eta)Y)G + \eta(Y)\nabla_X G - (\nabla_Y(\eta)X)G - \eta(X)\nabla_Y G - \eta([X,Y])G;\\
&= 2d\eta(X,Y)G + \eta(Y)\nabla_X G - \eta(X)\nabla_Y G - \eta([X,Y])G;\\
&= 3d\eta(X,Y)G + \eta(Y)\nabla_X G - \eta(X)\nabla_Y G - X(\eta(Y))G + Y(\eta(X))G.
\end{aligned}
$$

□

Corollary 5.2.1 *On the $ACR-$manifold $(M^{2n+1},\xi,\eta,\Phi,g)$ of class C_{12}, the following curvature identities hold:*

1. $R(X,Y)\xi = 0$, *if* $X,Y \in \ker(\eta)$;
2. $R(\Phi X,\Phi Y)\xi = R(\Phi^2 X,\Phi^2 Y)\xi = R(\Phi X,\Phi^2 Y)\xi = 0; \quad \forall\ X,Y \in X(M).$

Proof: The outcomes are obvious from Lemma 5.2.1 and Theorem 5.2.2. □

Now, we are in position to calculate the components of Ricci tensor r of $ACR-$ manifold of class C_{12} on $AG-$structure space.

Theorem 5.2.3 *On $AG-$structure space, the components of Ricci tensor of $ACR-$manifold of class C_{12} are given below.*

1. $r_{00} = 2(C^a_a - C^a\ C_a)$;

2. $r_{a0} = 0$;

3. $r_{ab} = C_{ab} - C_a\ C_b$;

4. $r_{\hat{a}b} = C^a_b - C^a\ C_b + A^{ac}_{cb}$,

and the remaining components are conjugate to the above components or given by the symmetric property.

Proof: Regarding Definition 1.4.3 and Theorem 5.2.1, we have the following:

$$\begin{aligned} r_{00} &= -R^k_{00k}; \\ &= -R^0_{000} - R^a_{00a} - R^{\hat{a}}_{00\hat{a}}; \\ &= 0 + R^a_{0a0} + \overline{R^a_{0a0}}; \\ &= 2R^a_{0a0}; \\ &= 2(C^a_a - C^a\ C_a). \end{aligned}$$

So, we can follow the same above technique to proof the others items. □

Theorem 5.2.4 *On $AG-$structure space, an $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ of class C_{12} is an $\eta-$Einstein manifold if and only if, M^{2n+1} satisfies the following conditions:*

$$\alpha + \beta = 2(C^a_a - C^a\ C_a), \quad C_{ab} = C_a\ C_b, \quad \alpha\ \delta^a_b = C^a_b - C^a\ C_b + A^{ac}_{cb}.$$

Proof: According to Definition 1.4.4, we have that M^{2n+1} is an $\eta-$Einstein manifold if and only if its Ricci tensor r satisfies the following for all vector fields X, Y over M:

$$r(X, Y) = \alpha\ g(X, Y) + \beta\ \eta(X)\ \eta(Y),$$

where $\alpha, \beta \in C^\infty(M)$. On the $AG-$structure space, the above equation equivalent to the following:

$$r_{ij} = \alpha\ g_{ij} + \beta\ \eta_i\ \eta_j.$$

Making use of Definition 1.3.6, it follows that:

$$r_{00} = \alpha + \beta, \quad r_{a0} = r_{ab} = 0, \quad r_{\hat{a}b} = \alpha\ \delta^a_b.$$

Regarding Theorem 5.2.3 and the last equations, we get the requirement. □

Corollary 5.2.2 *If $(M^{2n+1}, \xi, \eta, \Phi, g)$ is an $\eta-$Einstein manifold of class C_{12} with $C^a_b = C^a\ C_b$, then $\alpha + \beta = 0$ and $\alpha = n^{-1}A^{ac}_{ca}$.*

Proof: Using Theorem 5.2.4 and contracting the following conditions:

$$C^a_b = C^a\ C_b, \quad \alpha\ \delta^a_b = C^a_b - C^a\ C_b + A^{ac}_{cb}.$$

Subsequently, we get the desired. □

Now, we discuss the nullity conditions for $ACR-$manifold of class C_{12}. From Definition 1.4.12, we have

$$R(Z,W)Y = \kappa\{g(W,Y)Z - g(Z,Y)W\} + \mu\{g(W,Y)hZ - g(Z,Y)hW\}.$$

Since $R(X,Y,Z,W) = g(R(Z,W)Y,X)$, then we get

$$\begin{aligned} R(X,Y,Z,W) &= \kappa\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\} \\ &\quad + \mu\{g(Y,W)g(X,hZ) - g(Y,Z)g(X,hW)\}. \end{aligned}$$

On $AG-$structure space, the above identity equivalent to the following:

$$R_{ijkl} = \kappa(g_{ik}\ g_{jl} - g_{il}\ g_{jk}) + \mu(g_{jl}\ g_{is}\ h^s_k - g_{jk}\ g_{is}\ h^s_l), \tag{5.2.8}$$

where $i, j, k, l, s = 0, 1, ..., 2n$. Then we have the following:

Lemma 5.2.2 If $(M^{2n+1}, \xi, \eta, \Phi, g)$ belongs to the class C_{12}, then on $AG-$ structure space, the tensor $h = \frac{1}{2}\mathfrak{L}_\xi(\Phi)$ has the following components forms:

$$h^0_a = -\frac{\sqrt{-1}}{2}C_a; \quad h^a_0 = -\sqrt{-1}C^a,$$

and the other components are identical to zero or the conjugate to the above components.

Proof: Regarding Definition 1.4.12, we have

$$h(X) = \frac{1}{2}\{\nabla_\xi(\Phi)X - \nabla_{\Phi X}\xi + \Phi(\nabla_X \xi)\}; \quad \forall\ X \in X(M).$$

So, regarding equation (5.1.3) and Theorem 5.1.1, we can rewrite the above equation as follow:

$$h(X) = \frac{1}{2}\{\nabla_\xi(\Phi)X + \eta(X)\Phi(G)\}; \quad \forall\ X \in X(M).$$

On $AG-$structure space, the above equation has the following form:

$$h^i_j = \frac{1}{2}\{\Phi^i_{j,0} - \eta_j\ \Phi^i_k\ G^k\}; \quad i, j, k = 0, a, \hat{a}.$$

Since the tensor G has the components C^a and C_a, then $G^k = 0$ at $k = 0$. So, regarding the components of G, Definition 1.3.6 and setting $(i, j) = (0, a), (a, 0)$ in the above equation, we attain the requirements. □

Theorem 5.2.5 *The $ACR-$manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ of class C_{12} has (κ, μ)-nullity distribution if and only if, the following conditions hold:*

1. $C^a_b = C^a\ C_b + \kappa\ \delta^a_b$;
2. $C^{ab} = C^a\ C^b$;
3. $A^{ad}_{bc} = \kappa\ \delta^a_c\ \delta^d_b$.

Proof: Since $R^i_{jkl} = R_{\hat{i}jkl}$, then according to equation (5.2.8) and Definition 1.3.6, we get

$$\begin{aligned} R_{\hat{a}0b0} &= \kappa(g_{\hat{a}b}\ g_{00} - g_{\hat{a}0}\ g_{0b}) + \mu(g_{00}\ g_{\hat{a}s}\ h^s_b - g_{0b}\ g_{\hat{a}s}\ h^s_0); \\ &= \kappa\ \delta^a_b + \mu\ h^a_b. \end{aligned}$$

So, regarding Theorem 5.2.1 and Lemma 5.2.2, we attain item 1. Therefore, we can follow the same argument to prove the remaining items. □

Corollary 5.2.3 *If $(M^{2n+1}, \xi, \eta, \Phi, g)$ is an $ACR-$manifold of class C_{12} with (κ, μ)-nullity distribution, then $\kappa = 0$ or $n = 1$.*

Proof: From Theorem 5.1.2, we have $A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$, then making use of Theorem 5.2.5; item 3, we get

$$\begin{aligned} 0 &= \kappa\ \delta^{[a}_{c}\ \delta^{d]}_{b}; \\ 0 &= \kappa\{\delta^{a}_{c}\ \delta^{d}_{b} - \delta^{d}_{c}\ \delta^{a}_{b}\}. \end{aligned} \tag{5.2.9}$$

Then the contracting of equation (5.2.9) with respect to the indexes (a, c), we get $(n-1)\kappa = 0$ and this implies that $\kappa = 0$ or $n = 1$.
Then we attain the claim of the corollary. □

Theorem 5.2.6 *If $(M^{2n+1}, \xi, \eta, \Phi, g)$ is an $ACR-$manifold of class C_{12} with $n > 1$ and it satisfies (κ, μ)-nullity condition, then M has flat Riemannian curvature tensor. That is*

$$R(X, Y)Z = 0; \quad \forall\ X, Y, Z \in X(M).$$

Proof: Suppose that $X, Y, Z \in X(M)$, then $R(X, Y)Z = R^{i}_{\imath\jmath\ell}\ X^{\imath}\ Y^{\jmath}\ Z^{\ell}\ \varepsilon_i$, where $i, \imath, \jmath, \ell = 0, 1, ..., 2n$. Regarding Theorems 5.2.1, 5.2.5 and Corollary 5.2.3, we conclude that $R^{i}_{\imath\jmath\ell} = 0$, and this leads to the result. □

Theorem 5.2.7 *Suppose that M is $ACR-$manifold $(M^3, \xi, \eta, \Phi, g)$ of class C_{12}. Then M satisfies (κ, μ)-nullity condition if and only if M is an Einstein manifold with $\alpha = 2\kappa$.*

Proof: According to Theorems 5.2.3 and 5.2.5, we get the desired result. □

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List of Published and Submitted Papers

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[**1**] H. M. Abood and M. Y. Abass, A study of new class of almost contact metric manifolds of Kenmotsu type, Accepted in the Tamkang Journal of Mathematics 2020.

[**2**] H. M. Abood and M. Y. Abass, On the geometry of almost contact metric manifolds of class C_{12} with nullity condition, Submitted.

[**3**] M. Y. Abass and H. M. Abood, $\Phi-$Holomorphic sectional curvature and generalized Sasakian space forms for a class of Kenmotsu type, Journal of Basrah Researches ((Sciences)) **45** (2019), no. 2, 108-117.

[**4**] M. Y. Abass and H. M. Abood, Generalized curvature tensor and the hypersurfaces of the Hermitian manifold for the class of Kenmotsu type, Submitted.

[**5**] M. Y. Abass and H. M. Abood, On generalized $\Phi-$recurrent manifolds of Kenmotsu type, Submitted.

المستخلص:

في هذه الاطروحة ميّزنا فئة جديدة من منطويات متصلة مترية تقريبية واستنتجنا الشروط المكافئة للمتطابقة المميزة بدلالة تناسر كريجنكا. أثبتنا بان منطوي كينموتسو يحقق الفئة المذكورة او بعبارة أخرى الفئة الجديدة يمكن ان تتحلل الى جمع مباشر من منطوي كينموتسو وفئات أخرى. برهنا بان المنطوي ذو بعد 3 يتطابق مع منطوي كينموتسو ووفرنا مثالاً للمنطوي الجديد ذي البعد 5 بحيث لا يكون منطوي كينموتسو. بالإضافة الى ذلك، استنتجنا معادلات كارتان التركيبية ومركبات تنسر انحناء ريمان وتنسر ريشي للفئة قيد الدراسة. أضف الى ذلك، تم تحديد الشروط المطلوبة لجعل الفئة المذكورة تكون منطوي اينشتاين. لقد اسمينا الفئة سالفة الذكر التي تم تمييزها بالفئة من نوع كينموتسو.

علاوةً على ذلك، في هذه الاطروحة استنتجنا مثالاً للفئة من نوع كينموتسو كضرب مشوه للمنطوي الهرميشي في المستقيم الحقيقي. على فضاء البنية $G-$ المترابطة، تم الحصول على الشروط المطلوبة للفئة المذكورة ليكون لها تنسر انحناء مقطعي $\Phi-$ هولومورفي ثابت نقطياً. صنّفنا فئات جديدة من منطويات اتصال متري تقريبي تبعاً لتناسر انحنائها ووجدنا علاقاتهم مع فئتنا. بالإضافة الى ذلك، استنتجنا الشروط التي تجعل فئتنا تحقق تعميم نماذج فضاء ساساكي والفئات الجديدة ومنطوي اينشتاين.

درست الاطروحة الحالية تعميم متكرر $\Phi-$ لمنطويات من نوع كينموتسو. الهدف من هذه الدراسة هو تحديد مركبات مشتقة التغاير لتنسر الانحناء الريماني. بالإضافة الى ذلك، تم استنتاج الشروط التي تجعل منطوي من نوع كينموتسو متناظراً محلياً او تعميماً متكرراً $\Phi-$. ايضاً استنتجت الاطروحة بان المنطوي من نوع كينموتسو المتناظر محلياً يكون تعميم متكرر $\Phi-$ تحت شرط مناسب والعكس صحيح. أضف الى ذلك، الدراسة استنتجت العلاقة بين منطويات اينشتاين والمنطوي من نوع كينموتسو المتناظر محلياً.

لنفس الفئة حددنا مركبات تنسر الانحناء العام واستنتجنا بان الفئة المذكورة تكون منطوي $\eta-$ اينشتاين تحت تسطح تنسر الانحناء العام؛ العكس يبقى صحيحاً تحت شروط مناسبة. بالإضافة الى ذلك، قدمنا مفهوم تعميم تنسر الانحناء المقطعي $\Phi-$ هولومورفي

ومن ثم وجدنا الشرط الضروري والكافي الذي يجعل المفهوم المذكور سابقاً ثابتاً للفئة من نوع كينموتسو. ايضاً تم تقديم مفهوم Φ – المعمم شبه المتناظر وتم استنتاج علاقته مع الفئة من نوع كينموتسو ومنطوي η– اينشتاين. أضف الى ذلك، عممنا مفهوم المنطوي ذي الانحناء الثابت حيث البنية هي اتصال تقريبي وحققنا علاقته بالأفكار المذكورة. اخيراً بينا بان الفئة من نوع كينموتسو موجودة كسطح فوقي للمنطوي الهرميشي وتم اشتقاق العلاقة بين المركبات لتناسر الانحناء الريمانية للمنطوي الهرميشي التقريبي والسطوح الفوقية له.

هذه الاطروحة ناقشت ايضاً هندسة منطوي الاتصال المتري التقريبي من الفئة C_{12}. وبشكل خاص، تم تحديد المعادلات التركيبية ومركبات تنسري الانحناء والريشي على فضاء البنية – G المترابطة. ايضاً الاطروحة تدرس بعض متطابقات الانحناء لهذه الفئة. بالإضافة الى ذلك، هذه الاطروحة ناقشت توزيع العدم – (κ,μ) للفئة C_{12} واستنتجت الشروط الضرورية والكافية للفئة المذكورة لكي تمتلك توزيع العدم – (κ,μ) ولكي تحقق معيار η – اينشتاين. اخيراً تم بناء مثال للمنطوي من الفئة C_{12} ذي البعد 3.

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هندسة بعض تناسر انحناء لمنطوي متصل متري تقريبي

أطروحة مقدمة

الى مجلس كلية التربية للعلوم الصرفة – جامعة البصرة

وهي جزء من متطلبات نيل درجة دكتوراه فلسفة

في علوم الرياضيات

من قبل

محمد يوسف عباس

بإشراف

أ. د. حبيب مطشر عبود

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