TAMKANG JOURNAL OF MATHEMATICS
Volume 52, Number 2, 253-266, June 2021
DOI:10.5556/j.tkjm.52.2021.3276

A Study of New Class of Almost Contact Metric Manifolds of Kenmotsu Type

Habeeb M. Abood and Mohammed Y. Abass

Abstract. In this paper, we characterized a new class of almost contact metric manifolds and established the equivalent conditions of the characterization identity in term of Kirichenko's tensors. We demonstrated that the Kenmotsu manifold provides the mentioned class; i.e., the new class can be decomposed into a direct sum of the Kenmotsu and other classes. We proved that the manifold of dimension 3 coincided with the Kenmotsu manifold and provided an example of the new manifold of dimension 5, which is not the Kenmotsu manifold. Moreover, we established the Cartan's structure equations, the components of Riemannian curvature tensor and the Ricci tensor of the class under consideration. Further, the conditions required for this to be an Einstein manifold have been determined.

1 Introduction

In 1972, Kenmotsu [9] studied a class of almost contact metric manifolds that satisfied the identity $\nabla_X(\Phi)Y = -g(X,\Phi Y)\xi - \eta(Y)\Phi X$. In particular, he proved that it is a class of constant curvature (-1). Kenmotsu manifolds are different from the Sasakian manifolds, as discussed by Sasaki [16], on several sides; one of these sides is the Sasakian manifold of constant curvature $(+1)$, according to Tanno's classification [18]. In 1990, Chinea and Gonzalez [5] classified the almost contact metric manifolds, with the Kenmotsu manifolds falling under class C_5. In 2001, Kirichenko [10] constructed a Kenmotsu structure by taking into consideration a con-circular transformation of a cosymplectic structure. In 2002, Umnova [19] determined the Cartan's structure equations and the components of the Riemannian curvature tensor, the Ricci tensor of the Kenmotsu manifolds, and the nearly Kenmotsu manifolds in the G-structure adjoined space. Later, many researchers studied the Kenmotsu manifolds, the almost Kenmotsu manifolds, the nearly Kenmotsu manifolds and the conformal Kenmotsu manifolds. For more details, we furnish the following citations [1], [7], [8], [15], and [17].

2010 *Mathematics Subject Classification*. 53B05, 53C25, 53D10, 53D15.
Key words and phrases. Almost contact metric structure, Kenmotsu manifold, Riemannian curvature tensor, Ricci tensor, Einstein manifold.
Corresponding author: Mohammed Y. Abass.

2 Preliminaries

We denote M^{2n+1}, $X(M)$, $\Lambda(M)$ and d, the smooth manifold M of dimension $2n+1$, module of smooth vector fields of M, Grassmann algebra on M and exterior differentiation operator respectively.

Definition 1. *[3]* An almost contact metric manifold (AC-manifold) is a Riemannian manifold (M^{2n+1}, g) with the family of tensors $\{\Phi, \xi, \eta\}$, where Φ is a (1, 1)-tensor, ξ is a vector field and η is a 1-form, such that

$$\Phi(\xi) = 0; \quad \eta(\xi) = 1; \quad \eta \circ \Phi = 0; \quad \Phi^2 = -\mathrm{id} + \eta \otimes \xi;$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y); \quad \forall X, Y \in X(M).$$

Moreover, if $d\eta \neq 0$ then $(M^{2n+1}, \Phi, \xi, \eta, g)$ is called a contact metric manifold.

Now, suppose that $X^C(M)$ is the complexification of the module $X(M)$. Then $X^C(M)$ decomposes into direct sum as the following [11]:

$$X^C(M) = L^C \oplus D^0_\Phi = D^{\sqrt{-1}}_\Phi \oplus D^{-\sqrt{-1}}_\Phi \oplus D^0_\Phi;$$

where L^C is the complexification of $L = Im(\Phi)$ and $D^0_\Phi = \ker(\Phi)$ such that $\dim L = 2n$ and $\dim D^0_\Phi = 1$. Whereas each of $D^{\sqrt{-1}}_\Phi$ and $D^{-\sqrt{-1}}_\Phi$ has dimension n. Then we have projections from $X(M)$ into $D^{\sqrt{-1}}_\Phi$ and $D^{-\sqrt{-1}}_\Phi$ respectively, which are defined by

$$\Pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi); \quad \overline{\Pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi).$$

There are another projections from L into $D^{\sqrt{-1}}_\Phi$ and $D^{-\sqrt{-1}}_\Phi$ respectively, are given by

$$\sigma = \frac{1}{2}(\mathrm{id} - \sqrt{-1}\Phi); \quad \overline{\sigma} = \frac{1}{2}(\mathrm{id} + \sqrt{-1}\Phi).$$

Since the almost contact metric manifold has the structure group $U(n) \times \{e\}$, then for any orthonormal basis $\{\xi, e_1, ..., e_n, e_{\widehat{1}}, ..., e_{\widehat{n}}\}$ of $X(M)$, we can define an A-frame as

$$(p; \xi_p, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\widehat{1}}, ..., \varepsilon_{\widehat{n}}),$$

where $p \in M$, $\varepsilon_a = \sqrt{2}\sigma(e_a)$, $\varepsilon_{\widehat{a}} = \sqrt{2}\overline{\sigma}(e_a)$, $a = 1, ..., n$, and $\widehat{a} = a + n$.

Definition 2. *[11]* The set of all A-frames of AC-manifold M is called a G-structure adjoined space.

In the G-structure adjoined space, the metric g and the tensor Φ take the following formulas [19]:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & O & I_n \\ 0 & I_n & O \end{pmatrix}; \qquad (\Phi^i_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & O \\ 0 & O & -\sqrt{-1}I_n \end{pmatrix} \tag{2.1}$$

where $i, j = 0, 1, ..., 2n$ and I_n is $n \times n$ identity matrix.

In [11], Kirichenko defined tensors which are called first, second, ..., and sixth structure tensors, represented by the following formulas respectively:

$$\begin{aligned}
B(X,Y) &= -\frac{1}{8}\{\Phi \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi^2 X) + \Phi \circ \nabla_{\Phi Y}(\Phi)(\Phi X) + \Phi^2 \circ \nabla_{\Phi Y}(\Phi)(\Phi^2 X) \\
&\quad - \Phi^2 \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi X)\}; \\
C(X,Y) &= -\frac{1}{8}\{-\Phi \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi^2 X) + \Phi \circ \nabla_{\Phi Y}(\Phi)(\Phi X) + \Phi^2 \circ \nabla_{\Phi Y}(\Phi)(\Phi^2 X) \\
&\quad + \Phi^2 \circ \nabla_{\Phi^2 Y}(\Phi)(\Phi X)\}; \\
D(X) &= \frac{1}{4}\{2\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi - 2\Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi - \Phi \circ \nabla_{\xi}(\Phi)(\Phi^2 X) + \Phi^2 \circ \nabla_{\xi}(\Phi)(\Phi X)\}; \\
E(X) &= -\frac{1}{2}\{\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi + \Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi\}; \\
F(X) &= \frac{1}{2}\{\Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi - \Phi^2 \circ \nabla_{\Phi X}(\Phi)\xi\}; \\
G &= \Phi \circ \nabla_{\xi}(\Phi)\xi.
\end{aligned}$$

The nonzero components of Kirichenko's tensors in the G-structure adjoined space have the following formulas respectively:

1. $B^{ab}{}_c = -\frac{1}{2}\sqrt{-1}\Phi^a_{\widehat{b},c}; \quad B_{ab}{}^c = \frac{1}{2}\sqrt{-1}\Phi^{\widehat{a}}_{b,\widehat{c}};$
2. $C^{abc} = \frac{1}{2}\sqrt{-1}\Phi^a_{\widehat{b},\widehat{c}}; \quad C_{abc} = -\frac{1}{2}\sqrt{-1}\Phi^{\widehat{a}}_{b,c};$
3. $D^{ab} = \sqrt{-1}(\Phi^a_{0,\widehat{b}} - \frac{1}{2}\Phi^a_{\widehat{b},0}); \quad D_{ab} = -\sqrt{-1}(\Phi^{\widehat{a}}_{0,b} - \frac{1}{2}\Phi^{\widehat{a}}_{b,0});$
4. $E^a_b = \sqrt{-1}\Phi^a_{0,b}; \quad E_a{}^b = -\sqrt{-1}\Phi^{\widehat{a}}_{0,\widehat{b}};$
5. $F^{ab} = \sqrt{-1}\Phi^0_{\widehat{a},\widehat{b}}; \quad F_{ab} = -\sqrt{-1}\Phi^0_{a,b};$
6. $G^a = -\sqrt{-1}\Phi^0_{\widehat{a},0}; \quad G_a = \sqrt{-1}\Phi^0_{a,0};$

where $a, b, c = 1, ..., n$ and $\widehat{a} = a + n$.

If we suppose that ∇ is the Riemannian connection and θ its connection form, from [12], we have the following:

$$\begin{array}{rclrclrcl} \theta^a_{\hat{b}} & = & \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},k}\omega^k; & \theta^{\hat{a}}_b & = & -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,k}\omega^k; & \Phi^a_{b,k} & = & 0; \\ \theta^0_{\hat{a}} & = & \sqrt{-1}\Phi^0_{\hat{a},k}\omega^k; & \theta^0_a & = & -\sqrt{-1}\Phi^0_{a,k}\omega^k; & \Phi^{\hat{a}}_{\hat{b},k} & = & 0; \\ \theta^{\hat{a}}_0 & = & -\sqrt{-1}\Phi^{\hat{a}}_{0,k}\omega^k; & \theta^a_0 & = & \sqrt{-1}\Phi^a_{0,k}\omega^k; & \Phi^0_{0,k} & = & 0. \end{array} \tag{2.2}$$

Moreover, $\theta^i_j + \theta^{\hat{j}}_{\hat{i}} = 0$; $\theta^0_0 = 0$; $\Phi^i_{j,k} = -\Phi^{\hat{j}}_{\hat{i},k}$, where $i, j, k = 0, a, \hat{a}$ and $\hat{\hat{i}} = i$.

3 New Class of Almost Contact Metric Manifold

In this section we introduce a new class of almost contact metric manifold (M, Φ, ξ, η, g), characterized by the following identity:

$$\nabla_X(\Phi)Y - \nabla_{\Phi X}(\Phi)\Phi Y = -\eta(Y)\Phi X; \quad \forall\, X, Y \in X(M) \tag{3.1}$$

We note that the Kenmotsu manifold satisfies the equation (3.1), which means that the Kenmotsu manifold is a special class of our class.

Now, we characterize the identity (3.1) in the G-structure adjoined space as follows:

$$(\Phi^i_{j,k}X^kY^j - \Phi^i_{j,k}(\Phi X)^k(\Phi Y)^j)\varepsilon_i = -\eta_j\Phi^i_kX^kY^j\varepsilon_i; \quad i, j, k = 0, 1, ..., n, \hat{1}, ..., \hat{n}.$$

Since $(\Phi X)^0 = 0; (\Phi X)^a = \sqrt{-1}X^a; (\Phi X)^{\hat{a}} = -\sqrt{-1}X^{\hat{a}}$ and regarding the matrix of Φ in equation (2.1), we get the following assertion:

Theorem 3.1. *In G-structure adjoined space, the equation (3.1) equivalent to the following conditions:*

$$\Phi^i_{j,0} = \Phi^{\hat{b}}_{0,a} = \Phi^i_{a,b} = 0; \quad \Phi^b_{0,a} = -\sqrt{-1}\delta^b_a.$$

Now, from the components of Kirichenko's tensors in section two, we have the following theorem:

Theorem 3.2. *Suppose that $(M^{2n+1}, \Phi, \xi, \eta, g)$ is AC-manifold. Then the following conditions are equivalent:*

1. $\nabla_X(\Phi)Y - \nabla_{\Phi X}(\Phi)\Phi Y = -\eta(Y)\Phi X; \quad X, Y \in X(M)$;
2. $C = D = F = G = 0;\ E = id$;
3. *In the G-structure adjoined space, we have*

$$\begin{array}{rcl} B^{ab}{}_c & = & -B^{ba}{}_c; \quad B_{ab}{}^c = -B_{ba}{}^c; \\ C^{abc} & = & D^{ab} = F^{ab} = G^a = 0; \quad E^a_b = E_b{}^a = \delta^a_b; \\ C_{abc} & = & D_{ab} = F_{ab} = G_a = 0. \end{array}$$

Corollary 3.3. *With the assumption of the Theorem 3.2, equation (2.2) given by*

$$\begin{aligned} \theta^a_{\widehat{b}} &= -B^{ab}{}_c\,\omega^c; & \theta^{\widehat{a}}_b &= \overline{\theta^a_{\widehat{b}}}; & \theta^0_0 &= 0;\\ \theta^0_{\widehat{a}} &= -\omega^a; & \theta^0_a &= \overline{\theta^0_{\widehat{a}}}; & \theta^i_j + \theta^{\widehat{j}}_{\widehat{i}} &= 0. \end{aligned}$$

where $\overline{B^{ab}{}_c} = B_{ab}{}^c$; $\overline{\omega^a} = \omega_a$; $\overline{\omega_a} = \omega^a$.

Proof. From the equation (2.2), the components of Kirichenko's tensors and the Theorem 3.2, we have

$$\begin{aligned} \theta^a_{\widehat{b}} &= \frac{\sqrt{-1}}{2}\Phi^a_{\widehat{b},k}\,\omega^k;\\ &= \frac{\sqrt{-1}}{2}\Phi^a_{\widehat{b},0}\,\omega^0 + \frac{\sqrt{-1}}{2}\Phi^a_{\widehat{b},c}\,\omega^c + \frac{\sqrt{-1}}{2}\Phi^a_{\widehat{b},\widehat{c}}\,\omega^{\widehat{c}};\\ &= \frac{\sqrt{-1}}{2}\Phi^a_{\widehat{b},c}\,\omega^c;\\ &= -B^{ab}{}_c\,\omega^c. \end{aligned}$$

and similarly for the remaining components. □

Corollary 3.4. *An AC-manifold M of dimension 3, which satisfies the equation (3.1) is Kenmotsu manifold.*

Proof. If we suppose that M satisfies the equation (3.1) and its dimension is $2n+1 = 3$. Then $n = 1$ and $a, b, c = 1$. Moreover, the components of the first structure tensor B are $B^{ab}{}_c = B^{11}{}_1$ and $B_{ab}{}^c = B_{11}{}^1$. But from the Theorem 3.2; item 3, we have $B^{ab}{}_c = -B^{ba}{}_c$ and $B_{ab}{}^c = -B_{ba}{}^c$ and this implies that $B^{11}{}_1 = B_{11}{}^1 = 0$. Since the Kenmotsu manifold satisfies the Theorem 3.2; item 3 with $B^{ab}{}_c = B_{ab}{}^c = 0$ according to [11], we derive that M is Kenmotsu manifold. □

Now, we construct an example of AC-manifold of dimension 5, which satisfies (3.1), but is not Kenmotsu manifold, due to the following:

Example 1. Suppose that (M, ξ, η, Φ, g) is an AC-manifold of dimension 5, such that

$$M = \{(x, y, z, u, v) \in \mathbb{R}^5 : xzv \neq 0\};$$

and $\{e_0 = \xi, e_1, e_2, e_3, e_4\}$ is a basis of $X(M)$, given by

$$e_0 = \frac{\partial}{\partial v}; \quad e_1 = \exp(-v)\frac{\partial}{\partial x}; \quad e_2 = \exp(-(v+x+z))\frac{\partial}{\partial y}; \quad e_3 = \exp(-v)\frac{\partial}{\partial z};$$
$$e_4 = \exp(-(v+x+z))\frac{\partial}{\partial u}.$$

Then we have the following Lie brackets:

$$[e_1, e_0] = e_1; \quad [e_4, e_1] = \exp(-v)e_4; \quad [e_3, e_0] = e_3; \quad [e_4, e_0] = e_4; \quad [e_2, e_1] = \exp(-v)e_2;$$
$$[e_1, e_3] = 0; \quad [e_2, e_0] = e_2; \quad [e_2, e_3] = \exp(-v)e_2; \quad [e_2, e_4] = 0; \quad [e_4, e_3] = \exp(-v)e_4.$$

Moreover, if we have the following:

$$\Phi(e_0) = 0; \quad \Phi(e_1) = e_3; \quad \Phi(e_2) = e_4; \quad \Phi(e_3) = -e_1; \quad \Phi(e_4) = -e_2;$$
$$\eta(e_0) = 1; \quad \eta(e_1) = \eta(e_2) = \eta(e_3) = \eta(e_4) = 0;$$
$$g(e_i, e_j) = \begin{cases} 1, & i = j; \\ 0, & i \neq j; \end{cases}$$

where $i, j = 0, 1, 2, 3, 4$. Then from the Koszul's formula that given by [6] as follows:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z]) \\ &\quad + g(Z, [X, Y]); \quad \forall\, X, Y, Z \in X(M). \end{aligned}$$

We deduce the following values of the Riemannian connection ∇ of the metric g:

$$\begin{array}{llllllll}
\nabla_{e_0} e_0 &=& 0; & \nabla_{e_0} e_1 &=& 0; & \nabla_{e_0} e_2 &=& 0; \\
\nabla_{e_1} e_0 &=& e_1; & \nabla_{e_1} e_1 &=& -e_0; & \nabla_{e_1} e_2 &=& 0; \\
\nabla_{e_2} e_0 &=& e_2; & \nabla_{e_2} e_1 &=& \exp(-v)e_2; & \nabla_{e_2} e_2 &=& -\exp(-v)(e_1 + e_3) - e_0; \\
\nabla_{e_3} e_0 &=& e_3; & \nabla_{e_3} e_1 &=& 0; & \nabla_{e_3} e_2 &=& 0; \\
\nabla_{e_4} e_0 &=& e_4; & \nabla_{e_4} e_1 &=& \exp(-v)e_4; & \nabla_{e_4} e_2 &=& 0;
\end{array}$$

$$\begin{array}{llllll}
\nabla_{e_0} e_3 &=& 0; & \nabla_{e_0} e_4 &=& 0; \\
\nabla_{e_1} e_3 &=& 0; & \nabla_{e_1} e_4 &=& 0; \\
\nabla_{e_2} e_3 &=& \exp(-v)e_2; & \nabla_{e_2} e_4 &=& 0; \\
\nabla_{e_3} e_3 &=& -e_0; & \nabla_{e_3} e_4 &=& 0; \\
\nabla_{e_4} e_3 &=& \exp(-v)e_4; & \nabla_{e_4} e_4 &=& -\exp(-v)(e_1 + e_3) - e_0.
\end{array}$$

Then from the above discussion, we deduce that M satisfies the identity (3.1), but M is not Kenmotsu manifold. For instance, if $X = e_4$ and $Y = e_1$ then

$$\nabla_X(\Phi)Y = \exp(-v)(e_2 + e_4) \neq 0 = -g(X, \Phi Y)\xi - \eta(Y)\Phi X.$$

4 Cartan's Structure Equations

In this section, we establish the structure equation of the new class of AC-manifold in section 3 Boothby [4] stated the following assertion:

Lemma 4.1. *Suppose that (M^n, g) is Riemannian manifold, ω^i the dual of the orthonormal frame and θ^i_j the connection form of Levi-Civita connection of the metric g, where $i, j = 1, 2, ..., n$. Then we have*

1. $d\omega^i = -\theta^i_j \wedge \omega^j$;
2. $d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \frac{1}{2} R^i_{jkl}\, \omega^k \wedge \omega^l$,

where R^i_{jkl} are the components of the Riemannian curvature tensor of type (3, 1), (see [13]).

Now, if we have a class of AC-manifold satisfies the identity (3.1), then the first family of structure equations of this class is given in the theorem below.

Theorem 4.1. *Suppose that M^{2n+1} is AC-manifold satisfies the equation (3.1), then we have*

1. $d\omega = 0$;
2. $d\omega^a = -\theta^a_b \wedge \omega^b + B^{ab}{}_c\, \omega^c \wedge \omega_b - \omega^a \wedge \omega$;
3. $d\omega_a = \theta^b_a \wedge \omega_b + B_{ab}{}^c\, \omega_c \wedge \omega^b - \omega_a \wedge \omega$.

Proof. We put $i = 0$, taking into account the Lemma 4.1; item 1, and the Corollary 3.3 we get

$$d\omega^0 = d\omega = -\theta^0_j \wedge \omega^j = -\theta^0_0 \wedge \omega^0 - \theta^0_a \wedge \omega^a - \theta^0_{\widehat{a}} \wedge \omega^{\widehat{a}} = 0.$$

Once again, regarding the Lemma 4.1; item 1, and the Corollary 3.3, if we put $i = a = 1, ..., n$, then we have

$$\begin{aligned} d\omega^a &= -\theta^a_j \wedge \omega^j = -\theta^a_0 \wedge \omega - \theta^a_b \wedge \omega^b - \theta^a_{\widehat{b}} \wedge \omega^{\widehat{b}}; \\ &= -\omega^a \wedge \omega - \theta^a_b \wedge \omega^b + B^{ab}{}_c\, \omega^c \wedge \omega_b. \end{aligned}$$

On the other hand, if we put $i = \widehat{a} = n + 1, ..., 2n$, then the result follows from the conjugate of the case when $i = a$. □

Now, we can find the second family of the structure equations by using the Theorem 4.1 and the following lemma:

Lemma 4.2. [14] *Suppose that M is a smooth manifold, then there exist a unique operator $d : \Lambda(M) \to \Lambda(M)$, satisfies the following properties:*

1. *d is linear over R.*
2. $d(\Lambda^k(M)) \subset \Lambda^{k+1}(M)$.

3. $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$, *where* $\omega_1 \in \Lambda^k(M)$; $\omega_2 \in \Lambda^l(M)$.

4. $d \circ d = 0$.

5. *For* $f \in C^\infty(M)$ *and* $X \in X(M)$, *then* $df(X) = X(f)$.

Theorem 4.2. *The second family of the structure equations of the class of AC-manifold* M^{2n+1} *which satisfies the identity (3.1), is given by*

1. $d\theta^a_b = -\theta^a_c \wedge \theta^c_b + A^{ad}_{bc}\,\omega^c \wedge \omega_d + A^a_{bcd}\,\omega^c \wedge \omega^d + A^{acd}_b\,\omega_c \wedge \omega_d;$

2. $dB^{ab}{}_c = B^{ab}{}_d\,\theta^d_c - B^{db}{}_c\,\theta^a_d - B^{ad}{}_c\,\theta^b_d + B^{ab}{}_{cd}\,\omega^d + B^{abd}_c\,\omega_d - B^{ab}{}_c\,\omega;$

3. $dB_{ab}{}^c = -B_{ab}{}^d\,\theta^c_d + B_{db}{}^c\,\theta^d_a + B_{ad}{}^c\,\theta^d_b + B_{ab}{}^{cd}\,\omega_d + B_{abd}^c\,\omega^d - B_{ab}{}^c\,\omega,$

$$\textit{where}\quad A^{ad}_{[bc]} - B^{ad}{}_{[cb]} - B^{ah}{}_{[b}\,B_{|h|c]}{}^d = 0;\quad A^{[bc]}_{ad} + B_{ad}{}^{[cb]} + B_{ah}{}^{[b}\,B^{|h|c]}{}_d = 0;$$
$$A^{acd}_b - B^{a[cd]}{}_b + B^{a[c}{}_h B^{|h|d]}{}_b = 0;\quad A^b_{acd} + B_{a[cd]}{}^b - B_{a[c}{}^h B_{|h|d]}{}^b = 0;\quad A^a_{[bcd]} = A^{[bcd]}_a = 0.$$

Proof. By applying the exterior differentiation operator d in the Theorem 4.1; item 2 and using the Lemma 4.2, we get

$$\begin{aligned} 0 = &-d\theta^a_b \wedge \omega^b + \theta^a_b \wedge (-\theta^b_c \wedge \omega^c + B^{bc}{}_d\,\omega^d \wedge \omega_c - \omega^b \wedge \omega) \\ &+ dB^{ab}{}_c \wedge \omega^c \wedge \omega_b + B^{ab}{}_c(-\theta^c_d \wedge \omega^d + B^{cd}{}_h\,\omega^h \wedge \omega_d - \omega^c \wedge \omega) \wedge \omega_b \\ &- B^{ab}{}_c\,\omega^c \wedge (\theta^d_b \wedge \omega_d + B_{bd}{}^h\,\omega_h \wedge \omega^d - \omega_b \wedge \omega) \\ &- (-\theta^a_b \wedge \omega^b + B^{ab}{}_c\,\omega^c \wedge \omega_b - \omega^a \wedge \omega) \wedge \omega; \end{aligned}$$

$$\begin{aligned} 0 = &-(d\theta^a_b + \theta^a_c \wedge \theta^c_b) \wedge \omega^b + (dB^{ab}{}_c + B^{db}{}_c\,\theta^a_d + B^{ad}{}_c \wedge \theta^b_d - B^{ab}{}_d\,\theta^d_c) \wedge \omega^c \wedge \omega_b \\ &- B^{a[c}{}_h\,B^{|h|d]}{}_b\,\omega^b \wedge \omega_c \wedge \omega_d - B^{ab}{}_c\,\omega^c \wedge \omega \wedge \omega_b + B^{ah}{}_{[b}\,B_{|h|c]}{}^d\,\omega^b \wedge \omega^c \wedge \omega_d. \end{aligned} \tag{4.1}$$

Suppose that

$$\begin{aligned} d\theta^a_b + \theta^a_c \wedge \theta^c_b = &A^{adh}_{bcf}\,\theta^c_d \wedge \theta^f_h + A^{ad}_{bch}\,\theta^c_d \wedge \omega^h + A^{adh}_{bc}\,\theta^c_d \wedge \omega_h + A^{ad}_{bc0}\,\theta^c_d \wedge \omega + A^a_{bcd}\,\omega^c \wedge \omega^d \\ &+ A^{ad}_{bc}\,\omega^c \wedge \omega_d + A^a_{bc0}\,\omega^c \wedge \omega + A^{acd}_b\,\omega_c \wedge \omega_d + A^{ac0}_b\,\omega_c \wedge \omega; \end{aligned}$$

$$dB^{ab}{}_c + B^{db}{}_c\,\theta^a_d + B^{ad}{}_c\,\theta^b_d - B^{ab}{}_d\,\theta^d_c = B^{abh}{}_{cd}\,\theta^d_h + B^{ab}{}_{cd}\,\omega^d + B^{abd}_c\,\omega_d + B^{ab0}_c\,\omega.$$

Then the equation (4.1) becomes

$$\begin{aligned} &- A^{adh}_{bcf}\,\theta^c_d \wedge \theta^f_h \wedge \omega^b - A^{ad}_{[b|c|h]}\,\theta^c_d \wedge \omega^h \wedge \omega^b - A^{adh}_{bc}\,\theta^c_d \wedge \omega_h \wedge \omega^b - A^{ad}_{bc0}\,\theta^c_d \wedge \omega \wedge \omega^b \\ &- A^a_{[bcd]}\,\omega^c \wedge \omega^d \wedge \omega^b - A^{ad}_{[bc]}\,\omega^c \wedge \omega_d \wedge \omega^b - A^a_{[bc]0}\,\omega^c \wedge \omega \wedge \omega^b - A^{acd}_b\,\omega_c \wedge \omega_d \wedge \omega^b \\ &- A^{ac0}_b\,\omega_c \wedge \omega \wedge \omega^b + B^{abh}{}_{cd}\,\theta^d_h \wedge \omega^c \wedge \omega_b + B^{ab}{}_{[cd]}\,\omega^d \wedge \omega^c \wedge \omega_b + B^{a[bd]}_c\,\omega_d \wedge \omega^c \wedge \omega_b \end{aligned}$$

$$+ B^{ab0}{}_{c}\, \omega \wedge \omega^{c} \wedge \omega_{b} - B^{a[c}{}_{h}\, B^{|h|d]}{}_{b}\, \omega^{b} \wedge \omega_{c} \wedge \omega_{d} - B^{ab}{}_{c}\, \omega^{c} \wedge \omega \wedge \omega_{b}$$
$$+ B^{ah}{}_{[b}\, B_{|h|c]}{}^{d}\, \omega^{b} \wedge \omega^{c} \wedge \omega_{d} = 0.$$

Then we get

$$\begin{gathered} A^{adh}_{bcf} = A^{ad}_{[b|c|h]} = A^{ad}_{bc0} = A^{a}_{[bcd]} = A^{a}_{[bc]0} = 0; \qquad A^{ad}_{[bc]} - B^{ad}{}_{[cb]} - B^{ah}{}_{[b}\, B_{|h|c]}{}^{d} = 0; \\ A^{acd}_{b} - B^{a[cd]}{}_{b} + B^{a[c}{}_{h} B^{|h|d]}{}_{b} = 0; \quad A^{ac0}_{b} - B^{ac0}{}_{b} - B^{ac}{}_{b} = 0; \quad A^{adh}_{bc} + B^{ahd}{}_{bc} = 0. \end{gathered} \tag{4.2}$$

Now, applying the same argument above for the Theorem 4.1; item 3, and using the truth $\theta^{b}_{a} = -\overline{\theta^{a}_{b}}$ and $B_{ab}{}^{c} = \overline{B^{ab}{}_{c}}$, then we get

$$\begin{gathered} A_{ad}^{[b|c|h]} = A_{a}^{[bcd]} = A_{a}^{[bc]0} = 0; \qquad A_{ad}^{[bc]} + B_{ad}{}^{[cb]} + B_{ah}{}^{[b}\, B^{|h|c]}{}_{d} = 0; \\ A^{b}_{acd} + B_{a[cd]}{}^{b} - B_{a[c}{}^{h} B_{|h|d]}{}^{b} = 0; \quad A^{b}_{ac0} + B_{ac0}{}^{b} + B_{ac}{}^{b} = 0; \quad A^{bc}_{adh} - B_{ahd}{}^{bc} = 0. \end{gathered} \tag{4.3}$$

Regarding the Theorem 3.2; item 3, we have $B^{[ab]}{}_{c} = B^{ab}{}_{c}$ and $B_{[ab]}{}^{c} = B_{ab}{}^{c}$, so all the components of their derivatives have the same property. Then from this fact and the equations (4.2) and (4.3) we deduce the required results. □

5 Riemannian Curvature Tensor and Ricci Tensor

In this section we calculate the components of the Riemannian curvature tensor and the Ricci tensor of the almost contact metric manifold, which was characterized in the previous sections. The method of calculation depends on the Lemma 4.1; item 2, Theorem 4.1, Lemma 4.2, and Theorem 4.2, due to the following:

$$\begin{aligned} d\theta^{i}_{j} + \theta^{i}_{k} \wedge \theta^{k}_{j} &= \frac{1}{2} R^{i}_{jkl}\, \omega^{k} \wedge \omega^{l}; \\ d\theta^{i}_{j} + \theta^{i}_{0} \wedge \theta^{0}_{j} + \theta^{i}_{c} \wedge \theta^{c}_{j} + \theta^{i}_{\widehat{c}} \wedge \theta^{\widehat{c}}_{j} &= R^{i}_{jc0}\, \omega^{c} \wedge \omega + R^{i}_{j\widehat{c}0}\, \omega_{c} \wedge \omega + \frac{1}{2} R^{i}_{jcd}\, \omega^{c} \wedge \omega^{d} \\ &\quad + R^{i}_{jc\widehat{d}}\, \omega^{c} \wedge \omega_{d} + \frac{1}{2} R^{i}_{j\widehat{c}\widehat{d}}\, \omega_{c} \wedge \omega_{d}. \end{aligned} \tag{5.1}$$

So, there are several cases regarding the values of $i, j = 0, a, \widehat{a}$. These cases are designing as the following:

Case (1). If $i = j = 0$, then the equation (5.1) and the Corollary 3.3 give

$$R^{0}_{0c0} = R^{0}_{0\widehat{c}0} = R^{0}_{0cd} = R^{0}_{0c\widehat{d}} = R^{0}_{0\widehat{c}\widehat{d}} = 0.$$

Case (2). If $i = a, \quad j = 0$, then according to the equation (5.1) and the Corollary 3.3, we obtain

$$R^{a}_{0c0} = -\delta^{a}_{c}; \quad R^{a}_{0\widehat{c}0} = R^{a}_{0cd} = R^{a}_{0c\widehat{d}} = R^{a}_{0\widehat{c}\widehat{d}} = 0.$$

Case (3). If $i = a, \quad j = b$, then taking into account the equation (5.1), the Corollary 3.3 and the Theorem 4.2, we have

$$(A^{ad}_{bc} - B^{ah}{}_{c}\, B_{bh}{}^{d} - \delta^a_c\, \delta^d_b)\omega^c \wedge \omega_d + A^a_{bcd}\, \omega^c \wedge \omega^d + A^{acd}_b\, \omega_c \wedge \omega_d = R^a_{bc0}\, \omega^c \wedge \omega$$
$$+ R^a_{b\widehat{c}0}\, \omega_c \wedge \omega + \frac{1}{2} R^a_{bcd}\, \omega^c \wedge \omega^d + R^a_{bc\widehat{d}}\, \omega^c \wedge \omega_d + \frac{1}{2} R^a_{b\widehat{c}\widehat{d}}\, \omega_c \wedge \omega_d.$$

Thus we conclude that

$$R^a_{bc0} = R^a_{b\widehat{c}0} = 0; \quad R^a_{bcd} = 2A^a_{bcd}; \quad R^a_{bc\widehat{d}} = A^{ad}_{bc} - B^{ah}{}_{c}\, B_{bh}{}^{d} - \delta^a_c\, \delta^d_b; \quad R^a_{b\widehat{c}\widehat{d}} = 2A^{acd}_b.$$

Case (4). If $i = a, \quad j = \widehat{b}$, then from the equation (5.1), Corollary 3.3, Theorem 4.1, Lemma 4.2 and Theorem 4.2, we have

$$(B^{ab}{}_{[cd]} - \delta^a_{[c}\, \delta^b_{d]})\omega^c \wedge \omega^d + (B^{abd}{}_{c} - B^{ab}{}_{h}\, B^{hd}{}_{c})\omega^c \wedge \omega_d = R^a_{\widehat{b}c0}\, \omega^c \wedge \omega + R^a_{\widehat{b}\widehat{c}0}\, \omega_c \wedge \omega$$
$$+ \frac{1}{2} R^a_{\widehat{b}cd}\, \omega^c \wedge \omega^d + R^a_{\widehat{b}c\widehat{d}}\, \omega^c \wedge \omega_d + \frac{1}{2} R^a_{\widehat{b}\widehat{c}\widehat{d}}\, \omega_c \wedge \omega_d;$$

So, we get

$$R^a_{\widehat{b}c0} = R^a_{\widehat{b}\widehat{c}0} = R^a_{\widehat{b}\widehat{c}\widehat{d}} = 0; \quad R^a_{\widehat{b}cd} = 2(B^{ab}{}_{[cd]} - \delta^a_{[c}\, \delta^b_{d]}); \quad R^a_{\widehat{b}c\widehat{d}} = B^{abd}{}_{c} - B^{ab}{}_{h}\, B^{hd}{}_{c}.$$

From the above cases we have the following theorem:

Theorem 5.1. *In the G-structure adjoined space, the components of the Riemannian curvature tensor of the AC-manifold M^{2n+1} which satisfy the identity (3.1), are given by*

1. $R^a_{0c0} = -\delta^a_c$;
2. $R^a_{bcd} = 2A^a_{bcd}$;
3. $R^a_{bc\widehat{d}} = A^{ad}_{bc} - B^{ah}{}_{c}\, B_{bh}{}^{d} - \delta^a_c\, \delta^d_b$;
4. $R^a_{\widehat{b}cd} = 2(B^{ab}{}_{[cd]} - \delta^a_{[c}\, \delta^b_{d]})$;
5. $R^a_{\widehat{b}c\widehat{d}} = B^{abd}{}_{c} - B^{ab}{}_{h}\, B^{hd}{}_{c}$.

and the other components are identical to zero or given by the symmetric property or conjugate to the above components.

Now, we discuss some properties of the Riemannian curvature tensor, which has components as mentioned in Theorem 5.1.

$$R(X, Y)\xi = R^i_{0jk}\, X^j Y^k \varepsilon_i$$

$$
\begin{aligned}
&= R^i_{00b}\, X^0 Y^b \varepsilon_i + R^i_{00\widehat{b}}\, X^0 Y^{\widehat{b}} \varepsilon_i + R^i_{0b0}\, X^b Y^0 \varepsilon_i + R^i_{0\widehat{b}0}\, X^{\widehat{b}} Y^0 \varepsilon_i \\
&= \delta^i_b\, X^0 Y^b \varepsilon_i + \delta^i_{\widehat{b}}\, X^0 Y^{\widehat{b}} \varepsilon_i - \delta^i_b\, X^b Y^0 \varepsilon_i - \delta^i_{\widehat{b}}\, X^{\widehat{b}} Y^0 \varepsilon_i \\
&= \eta(X)Y - \eta(Y)X.
\end{aligned}
$$

$$
\begin{aligned}
R(X,\xi)Y &= R^i_{jk0}\, X^k Y^j \varepsilon_i \\
&= R^i_{0b0}\, X^b Y^0 \varepsilon_i + R^i_{0\widehat{b}0}\, X_b Y^0 \varepsilon_i + R^i_{b\widehat{c}0}\, X_c Y^b \varepsilon_i + R^i_{\widehat{b}c0}\, X^c Y_b \varepsilon_i \\
&= -\delta^i_b\, X^b Y^0 \varepsilon_i - \delta^i_{\widehat{b}}\, X_b Y^0 \varepsilon_i + \delta^i_0\, \delta^c_b\, X_c Y^b \varepsilon_i + \delta^i_0\, \delta^b_c\, X^c Y_b \varepsilon_i \\
&= g(X,Y)\xi - \eta(Y)X.
\end{aligned}
$$

Suppose that r is the Ricci tensor of type (2, 0), then $r(X,Y) = g(R(Z,Y)X,Z)$. So, we have $r(X,\xi) = g(R(Y,\xi)X,Y) = -2n\eta(X)$.

Thereafter, we compute the components of the Ricci tensor in the G-structure adjoined space of the class, which satisfies (3.1) due to the following:

$$
\begin{aligned}
r_{ij} &= -R^k_{ijk}; \quad i,j,k = 0,1,...,2n, \\
&= -R^0_{ij0} - R^c_{ijc} - R^{\widehat{c}}_{ij\widehat{c}}.
\end{aligned}
$$

Then for $a,b,c = 1,...,n,$ *and* $\widehat{a} = a+n,$ *we have*

$$
\begin{aligned}
r_{ab} &= -R^0_{ab0} - R^c_{abc} - R^{\widehat{c}}_{ab\widehat{c}} \\
&= -2A^c_{abc} + B_{cab}{}^c - B_{ca}{}^h\, B_{hb}{}^c. \\
r_{\widehat{a}b} &= -R^0_{\widehat{a}b0} - R^c_{\widehat{a}bc} - R^{\widehat{c}}_{\widehat{a}b\widehat{c}} \\
&= -\delta^a_b - 2(B^{ca}{}_{[bc]} - \delta^c_{[b}\, \delta^a_{c]}) + A^{ac}_{cb} - B^{ah}{}_b\, B_{ch}{}^c - \delta^a_b\, \delta^c_c \\
&= -2(n\delta^a_b + B^{ca}{}_{[bc]}) + A^{ac}_{cb} - B^{ah}{}_b\, B_{ch}{}^c.
\end{aligned}
$$

From the above discussion we have the following theorem:

Theorem 5.2. *The components of the Ricci tensor in the G-structure adjoined space of the AC-manifold M^{2n+1} which satisfy the identity (3.1), are given by*

1. $r_{00} = -2n$;
2. $r_{a0} = 0$;
3. $r_{ab} = -2A^c_{abc} + B_{cab}{}^c - B_{ca}{}^h\, B_{hb}{}^c$;
4. $r_{\widehat{a}b} = -2(n\delta^a_b + B^{ca}{}_{[bc]}) + A^{ac}_{cb} - B^{ah}{}_b\, B_{ch}{}^c$.

and the remaining components are identical to zero or given by the symmetric property or conjugate to the above components.

Definition 3. *[9]* An AC-manifold (M, Φ, ξ, η, g) is called η-Einstein manifold if, it's Ricci tensor r satisfies the following equation:

$$r(X, Y) = \lambda\, g(X, Y) + \mu\, \eta(X)\, \eta(Y); \quad \forall X, Y \in X(M) \tag{5.2}$$

where $\lambda, \mu \in C^{\infty}(M)$. In particular, if $\mu = 0$ then M is called Einstein manifold.

The equation (5.2) equivalent to $r_{ij} = \lambda\, g_{ij} + \mu\, \eta_i\, \eta_j$, where $\eta_j = g_{0j}$ and $i, j = 0, 1, ..., 2n$. In the rest of this paper, we suppose that $\widetilde{M}$ is the AC-manifold of dimension $2n+1$, that satisfies the identity (3.1).

Theorem 5.3. *An AC-manifold $\widetilde{M}$ is an Einstein manifold if and only if, the following conditions hold true:*

$$\lambda = -2n; \quad A^c_{abc} = 0; \quad B_{cab}{}^c = B_{ca}{}^h\, B_{hb}{}^c; \quad B^{ca}{}_{[bc]} = 0; \quad A^{ac}_{cb} = B^{ah}{}_b\, B_{ch}{}^c.$$

Proof. Regarding the Theorem 5.2, and the equation (2.1) we have $r_{00} = -2ng_{00}$, then $\lambda = -2n$. Moreover, we must have $r_{ab} = 0$ and $r_{\hat{a}b} = -2n\delta^a_b$. This equivalent to the following equations:

$$-2A^c_{abc} + B_{cab}{}^c - B_{ca}{}^h\, B_{hb}{}^c = 0; \quad -2B^{ca}{}_{[bc]} + A^{ac}_{cb} - B^{ah}{}_b\, B_{ch}{}^c = 0.$$

From the fact that $B_{ab}{}^c = -B_{ba}{}^c$ and $B^{ab}{}_c = -B^{ba}{}_c$, we get

$$-2A^c_{abc} - B_{acb}{}^c + B_{ac}{}^h\, B_{hb}{}^c = 0; \quad 2B^{ac}{}_{[bc]} + A^{ac}_{cb} + B^{ah}{}_b\, B_{hc}{}^c = 0.$$

Since $R^a_{bcd} = 2A^a_{bcd} = -R^a_{bdc} = -2A^a_{bdc}$, then by taking the anisymmetric of the indexes b and c of the above equations, we obtain

$$3A^c_{acb} - (A^c_{acb} + B_{a[cb]}{}^c - B_{a[c}{}^h\, B_{|h|b]}{}^c) = 0; \quad 3B^{ac}{}_{[bc]} + (A^{ac}_{[cb]} - B^{ac}{}_{[bc]} - B^{ah}{}_{[c}\, B_{|h|b]}{}^c) = 0.$$

Now, from the Theorem 4.2, we deduce that $A^c_{acb} = B^{ac}{}_{[bc]} = 0$ and this gives the result. □

Theorem 5.4. *An AC-manifold $\widetilde{M}$ is an η-Einstein manifold if and only if, the following conditions hold good:*

$$\lambda + \mu = -2n; \quad A^c_{abc} = 0; \quad B_{cab}{}^c = B_{ca}{}^h\, B_{hb}{}^c; \quad B^{ca}{}_{[bc]} = \frac{\mu}{3}\delta^a_b; \quad A^{ac}_{cb} = B^{ah}{}_b\, B_{ch}{}^c - \frac{\mu}{3}\delta^a_b.$$

Proof. Regarding the definition of the η-Einstein manifold, we have $r_{00} = \lambda + \mu$, then $\lambda + \mu = -2n$. Moreover, we must have $r_{ab} = 0$ and $r_{\hat{a}b} = \lambda\, g_{\hat{a}b} = (-2n - \mu)\delta^a_b$. Similar to the manner of the proof of Theorem 5.3, we obtain the assertion of this theorem. □

Remark. From the above theorems, it is clearly that λ and μ are scalars.

Now, suppose that Q is the Ricci operator of the Ricci tensor r of type $(2, 0)$, that is, $r(X, Y) = g(QX, Y)$.

Definition 4. *[2]* An AC-manifold (M, Φ, ξ, η, g) is said to be of Φ-invariant Ricci tensor if, $\Phi \circ Q = Q \circ \Phi$.

Lemma 5.1. [2] *An AC-manifold (M, Φ, ξ, η, g) has Φ-invariant Ricci tensor if and only if, $Q_0^{\hat{a}} = Q_b^{\hat{a}} = 0$, or equivalently, $r_{a0} = r_{ab} = 0$.*

Theorem 5.5. *An AC-manifold $\widetilde{M}$ is an Einstein manifold if and only if, $\widetilde{M}$ has Φ-invariant Ricci tensor and satisfies the following equations:*

$$\lambda = -2n; \quad B^{ca}{}_{[bc]} = 0; \quad A^{ac}_{cb} = B^{ah}{}_b \, B_{ch}{}^c.$$

Proof. The result follows from the Theorem 5.3 and the Lemma 5.1. □

Theorem 5.6. *An AC-manifold $\widetilde{M}$ is an η-Einstein manifold if and only if, $\widetilde{M}$ has Φ-invariant Ricci tensor and satisfies the following equations:*

$$\lambda + \mu = -2n; \quad B^{ca}{}_{[bc]} = \frac{\mu}{3}\delta^a_b; \quad A^{ac}_{cb} = B^{ah}{}_b \, B_{ch}{}^c - \frac{\mu}{3}\delta^a_b.$$

Proof. The assertion of this theorem follows from the Theorem 5.4 and the Lemma 5.1. □

References

[1] R. Abdi and E. Abedi, *On the Ricci tensor of submanifolds in conformal Kenmotsu manifolds*, Kyushu J. Math., **71**(2)(2017), 257-269.

[2] H. M. Abood and F. H. J. Al-Hussaini, *Locally conformal almost cosymplectic manifold of Φ-holomorphic sectional conharmonic curvature tensor*, Eur. J. Pure Appl. Math., **11**(3)(2018), 671-681.

[3] A. Abu-Saleem and A. R. Rustanov, *Curvature identities special generalized manifolds Kenmotsu second kind*, Malaysian J. Math. Sci., **9**(2)(2015), 187-207.

[4] W. M. Boothby, An introduction to differentiable manifolds and Riemannian geometry, Academic Press, New York, 1975.

[5] D. Chinea and C. Gonzalez, *A classification of almost contact metric manifolds*, Annali di Matematica Pura ed Applicata, **156**(1)(1990), 15-36.

[6] U. C. De and P. Majhi, *On invariant submanifolds of Kenmotsu manifolds*, J. Geom., **106**(1)(2015), 109-122.

[7] G. Dileo, *On the geometry of almost contact metric manifolds of Kenmotsu type*, Differential Geom. Appl., **29**(2011), S58-S64.

[8] I. K. Erken, P. Dacko and C. Murathan, *On the existence of proper nearly Kenmotsu manifolds*, Mediterr. J. Math., **13**(6)(2016), 4497-4507.

[9] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tôhoku Math. J., **24**(1)(1972), 93-103.

[10] V. F. Kirichenko, *On the geometry of Kenmotsu manifolds*, (in Russian), Dokl. Akad. Nauk, **380**(5)(2001), 585-587.

[11] V. F. Kirichenko, Differential-geometric structures on manifolds, (in Russian), Second edition, Pechatnyy dom, Odessa, 2013.

[12] V. F. Kirichenko and N. N. Dondukova, *Contactly geodesic transformations of almost contact metric structures*, Math. Notes, **80**(2)(2006), 204-213.

[13] J. M. Lee, Riemannian manifolds an introduction to curvature, Springer-Verlag, New York, 1997.

[14] J. M. Lee, Introduction to smooth manifolds, Second edition, Springer Science+Business Media, New York, 2013.

[15] G. Pitis, Geometry of Kenmotsu Manifolds, Editura Universitatii Transilvania, Brasov, 2007.

[16] S. Sasaki, Lecture notes on almost contact manifolds, Part II, Tohoku University, 1967.

[17] A. A. Shaikh, F. R. Al-Solamy and H. Ahmad, *Some transformations on Kenmotsu manifolds*, SUT J. Math., **49**(2)(2013), 109-119.

[18] S. Tanno, *Sasakian manifolds with constant ϕ-holomorphic sectional curvature*, Tôhoku Math. J., **21**(3)(1969), 501-507.

[19] S. V. Umnova, Geometry of Kenmotsu manifolds and their generalizations, (in Russian), Ph. D. thesis, Moscow State Pedagogical University, Moscow, 2002.

Habeeb M. Abood Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq.

E-mail: iraqsafwan2006@gmail.com

Mohammed Y. Abass Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq.

E-mail: mohammedyousif42@yahoo.com