Fixed point theorems of maps satisfying generalized condition (B) in ordered partial metric spaces

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Abstract

The main purpose of this paper is to introduce and prove some fixed point (F.P) theorems by using a generalized condition (B) in ordered partial metric space. Our results improve and unify a multitude of (F.P) theorems and generalize some recent results in ordered partial metric space.

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1. Introduction

Matthews in 1992 [1] introduced the concept of partial metric space for short (P.M.S) as a generalization of metric space in which each object does not necessarily have a zero distance from itself. This concept provides to study denotational semantics of data network also play a vital role in construct models of the theory of computation. Thereafter, several authors proved some (F.P) theorems using these concepts see for instance ([2]-[16]). In 2004, Ran and Reuring [17] investigated the existence of (F.P) in partially ordered metric space, recently, Altun and Erduran [18] introduced the concept of partially ordered complete partial metric spaces and established new conditions. The reader is referred to the work of ([19]- [22]), and references therein. In this paper, we prove (F.P) theorem for generalized condition (B) in the setting of ordered P.M.S by using F -non decreasing map. Our results generalized and extend some of the known results.

2. Preliminaries:

We recall some definitions and notions of Partial metric space and partially ordered set.

Definition (2.1)[1,2]

Let M be nonempty set , then a partial metric on M is function $p: M^2 \rightarrow \Re^+$ (where $\Re^+$ is the set of all nonnegative real number), such that satisfying the following axioms;

(pm_1) $m = n \Leftrightarrow p(m,m) = p(n,n) = p(m,n)$, (separation axiom)

(pm_2) $0 \leq p(m,m) \leq p(m,n)$, (non-negatively and small self – distance)

(pm_3) $p(m,n) = p(n,m)$, (symmetry)

(pm_4) $p(m,n) \leq p(m,r) + p(r,n) - p(r,r)$, (triangular inequality)

for all $m,n,r \in M$ .Then (M, p) is said to be a P.M.S.

It is clear if $p(m,n) = 0$ then from (pm_1) and (pm_2) it follow that $m = n$ But the converse is not hold in general [1].

For each partial metric p on the set M , the function $d_p : M^2 \rightarrow \Re^+$ is defined by $d_p(m,n) = 2p(m,n) - p(m,m) - p(n,n)$.

Where d_p is a metric on M . In fact d_p is the Euclidean metric on M .

Each partial metric p on M generates a T_0-Topology $\tau(p)$ on M whose base is the family of open $p-$ball $\{B_p(m;\varepsilon), m \in M, \varepsilon > 0\}$, where

$B_p(m,\varepsilon) = \{n \in M : p(m,n) < p(m,m) + \varepsilon\}$, for all $m \in M$ and $\varepsilon > 0$.

Example (2.2)[2,5]

(1) The pair $(\Re^+, p_i)$, $i = 1,2$ where

$$p_1(m,n) = Max\{m,n\} \ \forall m,n \in \Re^+$$

$p_2(m,n) = d(m,n) + \alpha \ \forall m,n \in \Re^+$ and α is real number.

(2) Let $p : M \times M \to \Re^+$, $M \subset \Re^+$

$$p(m,n) = \min\{m,n\} \ \forall m,n \in M \subset \Re^+$$

Since (pm_2) is fail if $m > n$. Thus, (M, p) is not P.M.S.

Definition (2.3) [2,21]

1- A sequence $\{q_n\}$ in a P.M.S (M, p) is said to be convergent to a point $m \in M$ if and only if $\lim_{n\to\infty}(q_n, m) = p(m,m)$

2- A sequence $\{q_n\}$ in a P.M.S (M, p) is said to be Cauchy if and only if $\lim_{m,n\to\infty} p(q_m, q_n)$ be exists (and is finite).

Moreover, if $\lim_{n,m\to\infty} p(q_n, q_m) = 0$ then a sequence $\{q_n\}$ in P.M.S (M, p) is said to be 0-Cauchy sequence.

3- A P.M.S (M, p) is said to be complete if every Cauchy sequence $\{q_n\}$ in M converges, (with respect to the topology $\tau(p)$), to a point $m \in M$ such that $\lim_{n,m\to\infty} p(q_n, q_m) = p(m,m)$.

Moreover, a P.M.S (M,p) is called 0-complete if every 0-Cauchy sequence $\{q_n\}$ in M converges, in $\tau(p)$, to a point $m\in M$ such that $p(m,m)=0$.

Lemma (2.4)

Let (M,p) be a P.M.S. Then

1- A sequence $\{q_n\}$ is Cauchy in a P.M.S if and only if $\{q_n\}$ is a Cauchy in a metric space (M,d_p),

2- A P.M.S (M,p) is complete if, and only if a metric space (M,d_p) is complete. In addition, $\lim_{n\to\infty} d_p(q_n,m)=0 \Leftrightarrow p(m,m)=\lim_{n\to\infty} p(q_n,m)=\lim_{n,m\to\infty} p(q_n,q_m)$

Definition (2.5) [21]

A mapping $T:M\to M$ is said to be continuous at $m_0\in M \Leftrightarrow \forall\varepsilon>0\ \exists\delta>0$ such that $T(B_p(m_0,\delta))\subseteq B_p(T(m_0),\varepsilon)$.

Lemma (2.6) [7,20]

Let (M,p) be a complete P.M.S. Then

(i) If $p(m,n)=0\Rightarrow m=n$

(ii) If $m\neq n\Rightarrow p(m,n)>0$

(iii) If $\{q_n\}\subset M, q_n\to m$ as $n\to\infty$ and $p(m,m)=0$ then $\lim_{n\to\infty} p(q_n,r)=p(m,r)$.

Definition (2.7) [17]

Suppose $(M,\preceq)$ be partially ordered set, then we have

(1) $\forall m,n\in M$ are said to be comparable if either $m\preceq n$ or $n\preceq m$ holds.

(2) A subset N of M is said to be "well ordered " if each two elements of N are comparable.

(3) A map $T:M \to M$ is called non-decreasing (non-increasing) if $m \preceq n$ then $Tm \preceq Tn$ $(Tm \succeq Tn)$

Definition (2.8) [5]

$(M, p, \preceq)$ is said to be ordered P.M.S if

(1) (M, p) is a P.M.S

(2) $((M, \preceq)$ be partially ordered set.

Definition (2.9) Let $(M, p, \preceq)$ be ordered P.M.S. Then T is said to be almost generalized contractive condition if there exist $\alpha \in [0,1)$ and $L \geq 0 \ni$ for all $m, n \in M$, where

$$p(Tm,Tn) \leq \alpha M_p(m,n) + LN_p(m,n)$$
$$M_p(m,n) = Max\{p(m,n), p(m,Tm), p(n,Tn), ½[p(m,Tn) + p(n,Tm)]\}$$
$$N_p(m,n) = \min\{p(m,Tm), p(n,Tn), p(m,Tn), p(n,Tm)]\}$$

3. Main Result

Theorem (3.1) Let $(M, p, \preceq)$ be a 0-complete ordered P.M.S. Let $F:M \to M$ be non-decreasing map such that for all comparable $m, n \in M$, k is some positive integer, $\alpha \in [0,1)$ and $L \geq 0 \ni$ with $\alpha + 2L < 1$,

$$p(F^k m, F^k n) \leq \alpha M_p(m,n) + LN_p(m,n) \qquad (3.1)$$

Where

$$M_p(m,n) = Max\{p(m,n), p(m,F^k m), p(n,F^k n), ½[p(m,F^k n) + p(n,F^k m)]\}$$
$$N_p(m,n) = \min\{p(m,F^k m), p(n,F^k n), p(m,F^k n), p(n,F^k m)]\}$$

If F is continuous and there exists $q_0 \in M$ with $q_0 \preceq F^k q_0$. Then F has a unique (F.P) r and $p(Fr, Fr) = 0 = p(r,r)$.

Proof: Let $q_0 \in M$ be an arbitrary point.

Define a sequence $\{q_n\}$ in M such that $q_n = F^k q_{n-1}$ for all $n \in N$

If $p(q_n,q_{n+1})=0$ for some $n\geq 0$, then $F^k q_n = q_{n+1} = q_n$ .that is q_n a fixed point of F^k and since $F\ q_n = F(F^k q_n) = F^k(F\ q_n)$.Hence Fq_n is a fixed point of F^k and by uniqueness of fixed point we get $Fq_n = q_n$.

Now, assume that $p(q_n,q_{n+1})>0$, for all $n\in N$.

Since F is non-decreasing, we have

$q_0 \preceq F^k q_0 = q_1 \preceq F^k q_1 = q_2 \preceq .... \preceq q_n \preceq F^k q_n = q_{n+1} \preceq F^k q_{+1} = q_{n+2},...$By using condition (3.1), we get

$$p(q_n,q_{n+1}) = p(F^k q_{n-1}, F^k q_n) \leq \alpha M_p(q_{n-1},q_n) + LN_p(q_{n-1},q_n)$$

$$M_p(q_{n-1},q_n) = Max\{p(q_{n-1},q_n), p(q_{n-1},F^k q_{n-1}), p(q_n,F^k q_n),$$

$$\tfrac{1}{2}[p(q_{n-1},F^k q_n) + p(q_n,F^k q_{n-1})]\}$$

$$M_p(q_{n-1},q_n) = Max\{p(q_{n-1},q_n), p(q_{n-1},q_n), p(q_n,q_{n+1}),$$

$$\tfrac{1}{2}[p(q_{n-1},q_{n+1}) + p(q_n,q_n)]\}$$

$M_p(q_{n-1},q_n) = Max\{p(q_{n-1},q_n), p(q_n,q_{n+1}), \tfrac{1}{2}[p(q_{n-1},q_{n+1}) + p(q_n,q_n)]\}$ By using (pm_4)

$$\tfrac{1}{2}[p(q_{n-1},q_{n+1}) + p(q_n,q_n)] \leq \tfrac{1}{2}[p(q_{n-1},q_n) + p(q_n,q_{n+1})]$$
$$\leq Max\{p(q_{n-1},q_n), p(q_n,q_{n+1})\}$$

Hence

$$M_p(q_{n-1},q_n) = Max\{p(q_{n-1},q_n), p(q_n,q_{n+1})\}$$

$$N_p(q_{n-1},q_n) = \min\{p(q_{n-1},q_n), p(q_n,q_{n+1}), p(q_{n-1},q_{n+1}), p(q_n,q_n)\}$$

$$N_p(q_{n-1},q_n) = \min\{p(q_{n-1},q_{n+1}), p(q_n,q_n)\}$$

We obtain that

$p(q_n,q_{n+1}) \leq \alpha Max\{p(q_{n-1},q_n), p(q_n,q_{n+1})\} + L\min\{p(q_{n-1},q_{n+1}), p(q_n,q_n)\}$ Now, we have four cases:

(1) If $Max\{p(q_{n-1},q_n), p(q_n,q_{n+1})\} = p(q_{n-1},q_n)$ and

$$\min\{p(q_{n-1},q_{n+1}), p(q_n,q_n)\} = p(q_{n-1},q_{n+1})$$

Then

$$p(q_n,q_{n+1}) \le \alpha p(q_{n-1},q_n) + Lp(q_{n-1},q_{n+1})$$

$$p(q_n,q_{n+1}) \le \alpha p(q_{n-1},q_n) + L[p(q_{n-1},q_n) + p(q_n,q_{n+1}) - p(q_n,q_n)]$$

$$\le \alpha p(q_{n-1},q_n) + Lp(q_{n-1},q_n) + Lp(q_n,q_{n+1})$$

$$(1-L)p(q_n,q_{n+1}) \le (\alpha + L)p(q_{n-1},q_n)$$

$$\le (\alpha + L)/(1-L)p(q_{n-1},q_n)$$

$$\le \beta_1 p(q_{n-1},q_n)$$

Where $\beta_1 = (\alpha + L)/(1-L) < 1$.

(2) If $Max\{p(q_{n-1},q_n), p(q_n,q_{n+1})\} = p(q_{n-1},q_n)$ and

$\min\{p(q_{n-1},q_{n+1}), p(q_n,q_n)\} = p(q_n,q_n)$ Then

$$p(q_n,q_{n+1}) \le \alpha p(q_{n-1},q_n) + Lp(q_n,q_n)$$

$$\le \alpha p(q_{n-1},q_n) + L[p(q_n,q_{n+1}) + p(q_{n+1},q_n) - p(q_{n+1},q_{n+1})]$$

$$\le \alpha p(q_{n-1},q_n) + Lp(q_n,q_{n+1}) + Lp(q_{n+1},q_n)$$

$$(1-2L)p(q_n,q_{n+1}) \le \alpha p(q_{n-1},q_n)$$

$$p(q_n,q_{n+1}) \le \alpha/(1-2L)p(q_{n-1},q_n)$$

$$p(q_n,q_{n+1}) \le \beta_2 p(q_{n-1},q_n)$$

Where $\beta_2 = \alpha/(1-2L) < 1$.

(3) If $Max\{p(q_{n-1},q_n), p(q_n,q_{n+1})\} = p(q_n,q_{n+1})$ and

$min\{ p(q_{n-1},q_{n+1}), p(q_n,q_n)\} = p(q_{n-1},q_{n+1})$

Then

$$p(q_n,q_{n+1}) \le \alpha p(q_n,q_{n+1}) + Lp(q_{n-1},q_{n+1})$$

$$\le \alpha p(q_n,q_{n+1}) + L[p(q_{n-1},q_n) + p(q_n,q_{n+1}) - p(q_n,q_n)]$$

$$\le \alpha p(q_n,q_{n+1}) + Lp(q_{n-1},q_n) + Lp(q_n,q_{n+1})$$

$$(1-\alpha-L)p(q_n,q_{n+1}) \le Lp(q_{n-1},q_n)$$

$$p(q_n,q_{n+1}) \le L/(1-\alpha-L)p(q_{n-1},q_n)$$

$$p(q_n,q_{n+1}) \le \beta_3 p(q_{n-1},q_n)$$

Where $\beta_3 = L/(1-\alpha-L) < 1$.

(4) If $Max\{p(q_{n-1},q_n),p(q_n,q_{n+1})\}=p(q_n,q_{n+1})$ and

$\min\{p(q_{n-1},q_{n+1}),p(q_n,q_n)\}=p(q_n,q_n)$ Then

$$
\begin{aligned}
p(q_n,q_{n+1}) &\le \alpha p(q_n,q_{n+1})+Lp(q_n,q_n)\\
&\le \alpha p(q_n,q_{n+1})+Lp(q_{n-1},q_{n+1})\\
&\le \alpha p(q_n,q_{n+1})+L[p(q_{n-1},q_n)+p(q_n,q_{n+1})-p(q_n,q_n)]
\end{aligned}
$$

$$p(q_n,q_{n+1})\le \alpha p(q_n,q_{n+1})+Lp(q_{n-1},q_n)+Lp(q_n,q_{n+1})$$

$$(1-\alpha-L)p(q_n,a_{n+1})\le Lp(q_{n-1},q_n)$$

$$p(q_n,q_{n+1})\le L/(1-\alpha-L)\,p(q_{n-1},q_n)$$

$$p(q_n,q_{n+1})\le \beta_4 p(q_{n-1},q_n)$$

Where $\beta_4=L/(1-\alpha-L)<1.$

Therefore choose $\beta=Max\{\beta_1,\beta_2,\beta_3,\beta_4\}$

$0<\beta<1$, for every $n\in N$, we have

$p(q_n,q_{n+1})\le \beta p(q_{n-1},q_n)$, and

$p(q_n,q_{n+1})\le \beta^n p(q_0,q_1)$.

Next to claim that $\{q_n\}$ is 0-Cauchy sequence, let $m,n\in N$

$$
\begin{aligned}
p(q_n,q_{n+m}) \le\ & [p(q_n,q_{n+1})+p(q_{n+1},q_{n+2})+\ldots\ldots+p(q_{n+m-1},q_{n+m})]-\\
& [p(q_{n+1},q_{n+1})+p(q_{n+2},q_{n+2})+\ldots+p(q_{n+m-1},q_{n+m-1})]
\end{aligned}
$$

$$
\begin{aligned}
p(q_n,q_{n+m}) &\le p(q_n,q_{n+1})+p(q_{n+1},q_{n+2})+\ldots\ldots+p(q_{n+m-1},q_{n+m})\\
&\le \beta^n p(q_0,q_1)+\beta^{n+1}p(q_0,q_1)+\ldots\ldots+\beta^{n+m-1}p(q_0,q_1)\\
&\le \beta^n[1+\beta+\beta^2+\ldots+\beta^{m-1}]p(q_0,q_1)\\
&\le \beta^n/(1-\beta)\,p(q_0,q_1)
\end{aligned}
$$

Since $\beta\in(0,1)$, we get $\beta^n/(1-\beta)\to 0$ as $n\to\infty$.

Hence $\lim_{n,m\to\infty} p(q_n,q_{n+m})=0$

Therefore $\{q_n\}$ is a 0-Cauchy sequence in (M,p).

Since (M,p) is 0-complete,it follows there exist $q \in M$, such that $q_n \to q$ in (M,p) and $p(q,q)=0$.

Moreover, $\lim_{n\to\infty} p(q_n,q)=p(q,q)=0$

Next, we shall prove that $Fq=q$

$$p(q,F^k q)\le p(q,q_{n+1})+p(q_{n+1},F^k q)-p(q_{n+1},q_{n+1})$$
$$p(q,F^k q)\le p(q,q_{n+1})+p(q_{n+1},F^k q)$$

By using the continuity of F and condition (3.1) we get

$$p(q,F^k q)\le \lim_{n\to\infty} p(q,q_{n+1})+\lim_{n\to\infty} p(q_{n+1},F^k q)$$
$$p(q,F^k q)\le p(q,q)+p(F^k q,F^k q)=p(F^k q,F^k q)$$

Thus $p(q,F^k q)\le p(F^k q,F^k q)$

But by using (pm_2) we have $p(F^k q,F^k q)\le p(q,F^k q)$

Hence

$$p(q,F^k q)=p(F^k q,F^k q)$$

suppose $p(q,F^k q)>0$. Since $q \preccurlyeq q$

$$p(q,F^k q)=p(F^k q,F^k q)\le \alpha M_p(q,q)+LN_p(q,q)$$

$$p(q,F^k q)\le \alpha Max\{p(q,q),p(q,F^k q),p(q,F^k q), \tfrac{1}{2}[p(q,F^k q)+p(q,F^k q)]\}+Lp(q,F^k q)$$

$$\le \alpha p(q,F^k q)+Lp(q,F^k q)$$
$$\le (\alpha+L)p(q,F^k q)$$
$$\le (\alpha+2L)p(q,F^k q)$$
$$< p(q,F^k q)$$

a contradiction. Thus, we have

$$p(q,F^k q)=0.$$

Thus, we get $F^k q=q$. That is q a (F.P) of F^k

Since k is a positive integer, F is continuous.

Therefore $Fq = q$. We claim the uniqueness of (F.P) of F ,

Assume $F^k u = u$, $F^k v = v$ and $u \neq v$. Then by condition (3.1)

$$p(u,v) = p(F^k u, F^k v) \leq \alpha M_p(u,v) + LN_p(u,v)$$

$$p(u,v) \leq \alpha Max\{p(u,v), p(u,F^k u), p(v,F^k v), \tfrac{1}{2}[p(u,F^k v) + p(v,F^k u)]\}$$

$$+ L\min\{p(u,F^k u), p(v,F^k v), p(u,F^k v), p(v,F^k u)\}$$

$$p(u,v) \leq \alpha Max\{p(u,v), p(u,u), p(v,v), \frac{1}{2}[p(u,v) + p(v,u)]\}$$

$$+ L\min\{p(u,u), p(v,v), p(u,v), p(v,u)\}$$

Then

$$p(u,v) \leq \alpha p(u,v) + L.0$$

$$p(u,v) < p(u,v)$$

Since $0 < \alpha < 1$

Hence $u = v$. So F^k has a unique (F.P) this implies that F has a unique (F.P).

Example (3.2)

Let $M = [0,1] \cup \{2\}$ be a set endowed with the P.M.S, $p(m,n) = \max\{m,n\}$ for all $m,n \in M$,we define M with the usual order $\leq$ and $F : M \to M$

Such that $F(m) = 0$ if $m \in [0,1]$, $F(m) = \frac{1}{3}$ if $m = 2$ .Clearly $F[0,1] = 0$ and $F(2) = \frac{1}{3}$ .We get $F^2(m) = F^3(m) = ... = F^k(m) = 0$ for all $m \in M$ Let $m = 0$ and $n = 1$ applying condition (3.1) ,we have ,

$$p(F^k 0, F^k 1) \le \alpha M_p(0,1) + LN_p(0,1),$$

$$M_p(0,1) = \alpha Max\{p(0,1), p(0,F^k 0), p(1,F^k 1),$$

$$\tfrac{1}{2}[p(0,F^k 1) + p(1,F^k 0)]\},$$

$$= \alpha p(0,1) = \alpha$$

$$N_p(0,1) = LMin\{p(0,F^k 0), p(1,F^k 1), p(0,F^k 1), p(1,F^k 0)\},$$

$$= L.0 = 0$$

$$p(F^k 0, F^k 1) = P(0,0) \le \alpha + 0 = \alpha \in [0,1]$$

So, Theorem (3.1) is verified and F^k has a fixed point 0.Consequently F has a fixed point 0.

Theorem (3.3) let $(M, p, \preceq)$ be a 0-complete ordered P.M.S. Let $F: M \to M$ be non-decreasing mapping such that;

$$p(F^k m, F^k n) \le \alpha Max\{p(m,n), p(m,F^k m), p(n,F^k n),$$

$$\tfrac{1}{2}[p(m,F^k n) + p(n,F^k m)]\} \qquad (3.2)$$

for all comparable $m, n \in M$, k is some positive integer and $\alpha \in (0,1)$,Suppose that F is continuous and let there exists $q_0 \in M$ with $q_0 \preceq F^k q_0$. Then F has a unique (F.P) q and

$$p(Fq, Fq) = 0 = p(q,q).$$

Proof: It follows from Theorem (3.1) with $L = 0$.

Theorem (3.4) let $(M, p, \preceq)$ be a 0-complete ordered (P.M.S) . Let $F: M \to M$ be non-decreasing continuous mapping such that

$$p(F^k m, F^k n) \le \alpha p(m, F^k m) + \beta p(n, F^k n) + \gamma p(m,n) \qquad (3.3)$$

for every comparable $m, n \in M$, k is some positive integer, $\alpha, \beta, \gamma > 0$ and, $\alpha + \beta + \gamma < 1$.suppose there exists $q_0 \in M$ with $q_0 \preceq F^k q_0$. Then F has a unique (F.P) q and $p(Fq, Fq) = 0 = p(q,q)$.

Proof: Let $q_0 \in M$ be an arbitrary point.

Define a sequence $\{q_n\}$ in M such that $q_n = F^k q_{n-1}$ for all $n \in N$.

If $p(q_n, q_{n+1}) = 0$ for some $n \in N$ then $F^k q_n = q_{n+1} = q_n$, q_n is a (F.P) of F^k

So that q_n is a (F.P) of F .

Assume that $p(q_n, q_{n+1}) > 0$ for all $n \ge 0$.

Since F is non decreasing map, we have

$$q_0 \preceq F^k q_0 = q_1 \preceq F^k q_1 = q_2 \preceq .... \preceq q_n \preceq F^k q_n = q_{n+1} \preceq F^k q_{+1} = q_{n+2}, ...$$

Applying condition (3.3)

$$p(q_n, q_{n+1}) = p(F^k q_{n-1}, F^k q_n) \le \alpha p(q_{n-1}, F^k q_{n-1}) + \beta p(q_n, F^k q_n) +$$
$$\gamma p(q_{n-1}, q_n).$$

$$p(q_n, q_{n+1}) \le \alpha p(q_{n-1}, q_n) + \beta p(q_n, q_{n+1}) + \gamma p(q_{n-1}, q_n)$$
$$(1-\beta) p(q_n, q_{n+1}) \le (\alpha + \gamma) p(q_{n-1}, q_n)$$
$$p(q_n, q_{n+1}) \le {(\alpha + \gamma)}/{(1-\beta)}\, p(q_{n-1}, q_n)$$

$$p(q_n, q_{n+1}) \le \delta p(q_{n-1}, q_n)$$

Where $\delta = {(\alpha + \gamma)}/{(1-\beta)}$ $0 < \delta < 1.$

And by induction we get, $p(q_n, q_{n+1}) \le \delta^n p(q_0, q_1)$

Now proceeding as in theorem (3.1) we can prove that $\{q_n\}$ is a 0-Cauchy sequence. Since M is 0-complete metric space then every 0-cauchy sequence in M converges (with respect to τ_p) to a point $q \in M$, such that

$$\lim_{n,m \to \infty} p(q_n, q_m) = p(q, q) = 0.$$

Moreover, by lemma (2-6) we have

$$\lim_{n \to \infty} p(q_n, q) = p(q, q) = 0$$

Now we shall prove that $Fq = q$

Letting $n \to \infty$, and using the continuity of F , we get

$$p(q,F^k q) \le p(q,q_{n+1}) + p(q_{n+1},F^k q) - p(q_{n+1},q_{n+1})$$
$$\le p(q,q_{n+1}) + p(q_{n+1},F^k q)$$
$$\le \lim_{n\to\infty} p(q,q_{n+1}) + \lim_{n\to\infty} p(F^k q_n,F^k q)$$
$$p(q,F^k q) \le p(q,q) + p(F^k q,F^k q) = p(F^k q,F^k q)$$

So $p(q,F^k q) \le p(F^k q,F^k q)$

But by (pm_2) we have $p(F^k q,F^k q) \le p(q,F^k q)$

hence, we get $p(q,F^k q) = p(F^k q,F^k q)$

Now if we suppose $p(q,F^m q) > 0$, then

$$p(q,F^k q) = p(F^k q,F^k q) \le \alpha p(q,F^k q) + \beta p(q,F^k q) + \gamma p(q,q)$$
$$\le (\alpha+\beta) p(q,F^k q)$$
$$< p(q,F^k q)$$

a contradiction. Hence, we get $p(q,F^k q) = 0$.

by using lemma (2.6) (i),we have $F^k q = q$

and since F is continuous and k is positive integer we get $Fq = q$

Now We claim the uniqueness of (F.P) of F^k .

Assume that $F^k u = u$ and $F^k w = w$, $u \ne w$, then

$$p(u,w) = p(F^k u,F^k w) \le \alpha p(u,F^k u) + \beta p(w,F^k w) + \gamma p(u,w)$$
$$p(u,w) \le \alpha p(u,u) + \beta p(w,w) + \gamma p(u,w)$$
$$p(u,w) \le \gamma p(u,w)$$
$$p(u,w) < p(u,w)$$

A contradiction. Thus $u = w$. So F^k has a unique (F.P) this implies that F has a unique (F.P).

Corollary(3.5) let $(M, p, \preceq)$ be a 0-complete ordered P.M.S. Let $F: M \to M$ be non-decreasing continuous mapping such that

$$p(F^k m,F^k n) \le \alpha p(m,n) \tag{3.4}$$

for all comparable $m,n \in M$, k is some positive integer and $\alpha \in [0,1)$. Suppose there exists $q_0 \in M$ with $q_0 \preceq F^k q_0$ Then F has a unique (F.P) q and $p(Fq,Fq)=0=p(q,q)$

Corollary (3.6) (Theorem 8 in [21]) let $(M,p,\preceq)$ be a 0-complete ordered P.M.S . Let $F:M \to M$ be non-decreasing continuous mapping such that

$$p(F^k m, F^k n) \le \alpha[p(m,F^k m)+p(n,F^k n)] \qquad (3.5)$$

for all comparable $m,n \in M$, k is some positive integer and $\alpha \in (0,\frac{1}{2})$ Suppose there exists $q_0 \in M$ with $q_0 \preceq F^k q_0$ Then F has a unique (F.P) r and $p(Fr,Fr)=p(r,r)=0$

Corollary (3.7)

Let $(M,p,\preceq)$ be a 0-complete ordered P.M.S . Let $F:M \to M$ be non- decreasing continuous mapping such that

$$p(F^k m, F^k n) \le \alpha p(m,n)+Lp(n,F^k m) \qquad (3.6)$$

for every comparable $m,n \in M$, k is some positive integer, $\alpha \in [0,1)$, $L \ge 0$ with $\alpha + 2L < 1$.Suppose there exists $q_0 \in M$ with $q_0 \preceq F^k q_0$ Then F has a unique (F.P) q and $p(Fq,Fq)=0=p(q,q)$.

Remark (3.8)

(1) In metric space (M,d) condition (3.1) with $k=1$ is called generalized condition (B) [23].

(2)In metric space (M,d) condition (3.6) with $k=1$ is called almost contraction which introduce by Berinde [24].

(3) In partial metric space (M,p) condition (3.6) with $k=1$ and L=0 is called Ciric contraction condition [11].

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حول مبرهنات النقاط الصامدة للدوال التي تحقق شرط (B) في الفضاء المتري الجزئي المرتب

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المستخلص

يتناول هذا البحث استعراض ودراسة النقاط الصامدة باستخدام شرط (B) المعمم في الفضاء الجزئي المرتب. النتائج التي حصلنا عليها هي تحسين وتوحيد العديد من النتائج في مبرهنات النقطة الصامدة وتعميم بعض النتائج الحديثة في الفضاء المتري الجزئي المرتب