**PAPER • OPEN ACCESS**

# First Integral Method for Constructing New Exact Solutions of The important Nonlinear Evolution Equations in Physics

View the article online for updates and enhancements.

**IOP ebooks**™

Bringing together innovative digital publishing with leading authors from the global scientific community.

Start exploring the collection–download the first chapter of every title for free.

# First Integral Method for Constructing New Exact Solutions of The important Nonlinear Evolution Equations in Physics

**Faeza L. Hasan**

Mathematics Department, Education for pour science Faculty, Basrah University, Basrah, Iraq

 fahas90@yahoo.co.uk

**Abstract**

In this paper, some new exact solutions of the important nonlinear partial differential equations in physics as Gardener's equation and Sharma-Tasso-Over equation are formally derived by utilizing the first integral method, where it is equipment us with many exact solutions by using the travelling wave transform, then deduce a system of ordinary differential equations which is solved by depending on theorem in commutative algebra and with helping the mathematical software like Maple and Wolfram Mathematica.

**Keywords**

First integral method; Gardener's equation and Sharma-Tasso-Over equation, a theory of commutative algebra.

## 1. Introduction

In the recent years, A matter of getting the exact solution of nonlinear partial differential equations (NLPDEs) has aroused the interest of many scientists, due to the appearance of these equations in many scientific fields such as engineering complex physics phenomena, mechanics, chemistry and biology etc, also as a result of the development in the field of computer software like Maple or Mathematica, which enables us to perform the complicated and tedious algebraic calculations easily and high efficiency, moreover, by the exact solutions, we can easily verify the accuracy and validity of the numerical solutions and also analyses the stability of these solutions, so many efficient analytical methods have emerged to find the exact solutions of nonlinear evolution equations had proposed such as tanh method was applied by Khater et al. [9], tanh-sech method Malfliet [11], extended tanh method by El-Wakil and Abdou [3], sine-cosine method Wazwaz [18, 19], F-expansion method Sheng [15], the extended mapping method Peng and Krishnan [20], the exp(-G(ξ)) Method Fengyan [7] etc.

Feng [6], proposed a new powerful method, which called the first integral method for solving Burgers-KdV equation. This method depends on the concept of the theory of commutative algebra Ding and Li [2]. The magnificence of this method embodies in the enjoyment of the following advantages, firstly, it avoids a great deal of tiresome and complicated calculations and the second point it supplies different more exact and explicit travelling solitary solutions, moreover, it has proved ease and applicability for different types of differential equations, so it is considered an easier and quicker method than other traditional techniques. Recently this useful method was applied to solve fractional equation Eslami1 et al. [5] also used widely by many researchers [1, 10, 12, 13, 14, 16, 17]

In the present work, we would like to extend the application of the first integral method to solve important equations are Gardener's equation and Sharma-Tasso-Over equation. The Structure of this article can be arranged as follows: Section 2, gives a short introduction to the first integral method. Applying the first integral method and some new exact solutions are obtained for nonlinear partial differential equation (PDE) as Gardner's equation and the Sharma- Tasso- Olver equation In section 3. Finally, in section 4, the conclusion of this research is summarized.

## 2. The basic idea of the first integral method(FIM)

We recap the main points of FIM by considering a general nonlinear PDE in the form

$$P\left(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xy}, u_{xxx}, \dots \right) = 0, \tag{2.1}$$

where $u(x, y, t)$ is the solution of the equation (2.1). By using the wave transformations

$$u(x, y, t) = f(\xi), \qquad \xi = x + hy + kt, \tag{2.2}$$

where $h$ and $k$ are constant. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(\cdot) = \alpha \frac{d}{d\xi}(\cdot); \ \frac{\partial}{\partial x}(\cdot) = \frac{d}{d\xi}(\cdot); \ \frac{\partial}{\partial y}(\cdot) = \beta \frac{d}{d\xi}(\cdot); \ \frac{\partial^2}{\partial x^2}(\cdot) = \frac{d^2}{d\xi^2}(\cdot), \dots \tag{2.3}$$

we use (2.3) to change the partial differential equation( PDE) (2.1) to ordinary differential equation (ODE):

$$G(f, f', f'', f''', \dots) = 0. \tag{2.4}$$

Now, we introduce new independent variables $X(\xi) = f(\xi)$, $Y(\xi) = f_\xi(\xi)$ which change to a system of ODEs

$$\begin{cases} X' = Y, \\ Y' = F(X(\xi), \ Y(\xi)) \end{cases} \tag{2.5}$$

The fundamental idea of our approach is to find two first integrals of the system (2.5) at the same conditions by depending on the theorem commutative algebra of differential equations [2], then the solutions of (2.5) can be obtained directly. Since, it isn't easy to deduce this even for a single first integral, because for the presence of a plane autonomous system, there is an important theorem of commutative algebra that tells us how to can find the first integral which called the division theorem. So, it is worth noting here to recall the division theorem for two variables in the complex domain $\mathbb{C}$ [X, Y]

*2.1. Division Theorem* [2]: Suppose that $p(X, Y)$ and $Q(X, Y)$ are polynomials of two variables X and Y in $\mathbb{c}[X, Y]$ and $p(X, Y)$ is irreducible in $\mathbb{c}[X, Y]$. If $Q(X, Y)$ vanishes at all zero points of $p(X, Y)$, then there exists a polynomial $G(X, Y)$ in $\mathbb{C}$ [X, Y] such that

$$Q(X, Y) = P(X, Y)G(X, Y).$$

# 3. Application

In this section, we will clarify the applicability of the first integral method for solving two-dimensional nonlinear partial equations.

## 3.1. Example

The Gardner equation is an integrable nonlinear partial differential equation was introduced by the scientist Clifford Gardner [8] in 1968 to generalize and modified KdV equation. This equation relates the seismic compressional wave (P-wave) velocity to the bulk density of the lithology in which the wave travels. Also, there are many applications of this equation in plasma physics, hydrodynamics, and quantum field theory

$$u_t = 6(u + \beta^2 u^2)u_x + u_{xxx}, \tag{3.6}$$

where $\beta$ is a constant, for finding exact solutions using (2.2) and (2.3), such that $\xi = x + kt$, equation (3.6) becomes

$$kf' = 6(f + \beta^2 f^2)f' + f''', \tag{3.7}$$

by integrating equation (3.7), we get

$$f'' = \alpha + kf - 3f^2 - 2\beta^2 f^3, \tag{3.8}$$

where $\alpha$ is an arbitrary integration constant. Put $X(\xi) = f(\xi)$ and $Y(\xi) = f_\xi(\xi)$, then equation (3.8) is equivalent to

$$\begin{cases} X' = Y, \\ Y' = \alpha + kX - 3X^2 - 2\beta^2 X^3. \end{cases} \tag{3.9}$$

Now, according to the concept of our approach, we suppose the $X = X(\xi)$ and $Y = Y(\xi)$ are the nontrivial solutions to (3.9), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in $\mathbb{C}[X, Y]$, such that

$$q\big(X(\xi), Y(\xi)\big) = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{3.10}$$

noting that $a_i(X)$ $(i = 0,1,2,,\mathrm{m})$ are polynomials of X and $a_m(X) \neq 0$. Equation (3.10) is called the first integral to equation (3.9). According to the Division Theorem, there exists a polynomial $g(x) + h(X)Y$ in $\mathbb{C}[X,Y]$, such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{dX}{d\xi} + \frac{\partial q}{\partial Y}\frac{dY}{d\xi} = (g(x) + h(X)Y)\left(\sum_{i=0}^{m} a_i(X)Y^i\right), \tag{3.11}$$

in our study we will take discuss two cases are clarify as follows:

**Case 1**

By taking $m = 1$ in (3.11). Now the $\frac{dq}{d\xi}$ is a polynomial in X and Y, and $q(X,Y) = 0$ implies $\frac{dq}{d\xi} = 0$,

$$\sum_{i=0}^{1} a_i'(X)Y^{i+1} + \sum_{i=0}^{1} i a_i(X)Y^{i-1}(\alpha + kX - 3X^2 - 2\beta^2 X^3)$$
$$= (g(x) + h(X)Y)\left(\sum_{i=0}^{1} a_i(X)Y^i\right), \tag{3.12}$$

on both sides of (3.12) by equating the coefficients of $Y^i$ when $i = 2, 1, 0$, we will get

$$a_1'(X) = a_1(X)h(X), \tag{3.13a}$$
$$a_0'(X) = a_1(X)g(X) + a_0(X)h(X), \tag{3.13b}$$
$$a_1(X)(\alpha + kX - 3X^2 - 2\beta^2 X^3) = a_0(X)g(X), \tag{3.13c}$$

note that $a_1(X)$ is a polynomial of X, then by (3.13a), the $a_1(X)$ can be concluded as a constant and $h(X) = 0$. For simplicity, we suppose that $a_1(X) = 1$, and balancing the degrees of $a_0(X), a_1(X)$, and $g(X)$, we can deduce that degree of $g(X)$ equal one only, after that we suppose $g(X) = AX + B$, we find $a_0(X)$ from (3.13b)

$$a_0(X) = \frac{AX^2}{2} + BX + C \tag{3.14}$$

here $A, B,$ and $C$ are constants. By substituting $a_1(X), a0(X) and g(X)$ in (3.13c) and all the coefficients of power X equaling to zero. We will get a system of nonlinear algebraic equations

$$\frac{A^2}{2} = -2\beta^2 \;,\; \frac{3}{2}AB = -3\;,\; B^2 + AC = k\;, CB = \alpha\;, \tag{3.15}$$

solving the system of equations in equation (3.15), we have

$$k = \frac{-1 + 2\alpha\beta^4}{\beta^2}\;,\quad C = -i\alpha\beta\;,\quad B = \frac{i}{\beta}\;,\quad A = 2i\beta\;, \tag{3.16a}$$

$$k = \frac{-1 + 2\alpha\beta^4}{\beta^2}\;,\quad C = i\alpha\beta\;,\quad B = -\frac{i}{\beta}\;,\quad A = -2i\beta\;, \tag{3.16b}$$

using (3.16a) and (3.16b) in (3.10), we obtain

$$Y = \frac{i(\alpha\beta^2 - X(1 + \beta^2 X))}{\beta}, \tag{3.17a}$$

$$Y = \frac{i(-\alpha\beta^2 + X(1 + \beta^2 X))}{\beta}, \tag{3.17b}$$

respectively. Combining equations (3.17a, b) with (3.9), we get the exact solutions of equation (3.8) as follows:

$$f(\xi) = \frac{-1 + \sqrt{-1 - 4\alpha\beta^4} \ \text{Tan}\left(\frac{\sqrt{-1 - 4\alpha\beta^4} \ (-i\xi + \beta C_1)}{2\beta}\right)}{2\beta^2}, \tag{3.18a}$$

and

$$f(\xi) = \frac{-1 + \sqrt{-1 - 4\alpha\beta^4} \ \text{Tan}\left(\frac{\sqrt{-1 - 4\alpha\beta^4} \ (i\xi + \beta C_1)}{2\beta}\right)}{2\beta^2}, \tag{3.18b}$$

where $C_1$ is a constant. Then the exact solution of equation (3.6) with variables $x$ and $t$ become

$$u(x,t) = \frac{-1 + \sqrt{-1 - 4\alpha\beta^4} \ \text{Tan}\left(\frac{\sqrt{-1 - 4\alpha\beta^4} \ (-i(x + \frac{-1 + 2\alpha\beta^4}{\beta^2}t + \beta C_1)}{2\beta}\right)}{2\beta^2}, \tag{3.19a}$$

$$u(x,t) = -\frac{-1 + \sqrt{-1 - 4\alpha\beta^4} \ \text{Tan}\left(\frac{\sqrt{-1 - 4\alpha\beta^4} \ (i(x + \frac{-1 + 2\alpha\beta^4}{\beta^2}t + \beta C_1)}{2\beta}\right)}{2\beta^2}, \tag{3.19b}$$

**Case 2**

We assume that $m = 2$ in (10), and $q(X, Y) = 0$ this implies $\frac{dq}{d\xi} = 0$,

$$\sum_{i=0}^{2} a_i'(X)Y^{i+1} + \sum_{i=0}^{2} ia_i(X)Y^{i-1}(\alpha + kX - 3X^2 - 2\beta^2 X^3)$$
$$= (g(x) + h(X)Y)\left(\sum_{i=0}^{2} a_i(X)Y^i\right), \quad (3.20)$$

on both sides of (3.20) by equating the coefficients of $Y^i$ when $i = 2, 1, 0$, we will get

$$a_2'(X) = a_2(X)h(X), \tag{3.21a}$$
$$a_1'(X) = a_2(X)g(X) + a_1(X)h(X), \tag{3.21b}$$
$$a_0'(X) + 2a_2(X)(\alpha + kX - 3X^2 - 2\beta^2 X^3) = a_1(X)g(X) + a_0(X)h(X), \tag{3.21c}$$
$$a_1(X)(\alpha + kX - 3X^2 - 2\beta^2 X^3) = a_0(X)g(X), \tag{3.21d}$$

note that $a_2(X)$ is a polynomial of X, then by (3.21a), the $a_2(X)$ can be concluded as a constant and $h(X) = 0$. For simplicity, we suppose that $a_2(X) = 1$, and balancing the degrees of $a_0(X), a_1(X)$, and $g(X)$, we can deduce that the degree of $g(X)$ equal one only, after that we suppose $g(X) = AX + B$, and $A \neq 0$, then we evaluate $a_1(X)$, and $a_0(X)$ from (3.21b, c)

$$a_1(X) = \left(\frac{A}{2}X^2 + BX + C\right), \tag{3.22a}$$

$$a_0(X) = d + (BC - 2\alpha)X + (\frac{B^2}{2} + \frac{1}{2}AC - k)X^2 + (2 + \frac{1}{2}AB)X^3 + (\frac{A^2}{8} + \beta^2)X^4, \tag{3.22b}$$

where $A, B, C$ and $d$ are constants. By substituting $a_2(X), a_1(X), a_0(X)$ $and$ $g(X)$ in (3. 21d) and all the coefficients of power $X$ equaling to zero. We will get a system of nonlinear algebraic equations

$$A = -4\mathrm{i}\beta, \quad B = -\frac{2\mathrm{i}}{\beta}, \quad C = \frac{\mathrm{i} + \mathrm{i}k\beta^2}{\beta^3}, \quad D = -\frac{0.25 + 0.5k\beta^2 + 0.25k^2\beta^4}{\beta^6}, \quad \alpha = \frac{0.5 + 0.5k\beta^2}{\beta^4},$$

$$(3.23a)$$

$$A = 4\mathrm{i}\beta, \quad B = \frac{2\mathrm{i}}{\beta}, \quad C = -\frac{\mathrm{i} + \mathrm{i}k\beta^2}{\beta^3}, \quad D = -\frac{0.25 + 0.5k\beta^2 + 0.25k^2\beta^4}{\beta^6}, \quad \alpha = \frac{0.5 + 0.5k\beta^2}{\beta^4},$$

$$(3.23b)$$

using (3.23a, b) in (3.10), we obtain

$$Y = -\frac{\frac{1}{2}(x^2\beta^4 + 2x\beta^2 - k\beta^2 - 1)}{\beta^3}, \tag{3.24a}$$

$$Y = -\frac{\frac{1}{2}(x^2\beta^4 + 2x\beta^2 - k\beta^2 - 1)}{\beta^3}, \tag{3.24b}$$

respectively. Combining equations (3.24a, b) with (3.9), we will obtain the new exact solutions of equation (3.8) as:

$$f(\xi) = -\frac{1}{4}\left[\tanh\left(\frac{\sqrt{\beta^4 + 8k\beta^2 + 8}\;(C_1 + \xi)}{2\beta^3}\right)\sqrt{\beta^4 + 8k\beta^2 + 8} + \beta^2\right], \tag{3.25a}$$

$$f(\xi) = \frac{1}{4}\left[\tanh\left(\frac{\sqrt{\beta^4 + 8k\beta^2 + 8}\;(C_1 + \xi)}{2\beta^3}\right)\sqrt{\beta^4 + 8k\beta^2 + 8} + \beta^2\right], \tag{3.25b}$$

where $C_1$ is a constant. Then the new exact solutions of equation (3.6) as follows:

$$u(x,t) = -\frac{1}{4}\left[\tanh\left(\frac{\sqrt{\beta^4 + 8k\beta^2 + 8}\;(C_1 + x + kt)}{2\beta^3}\right)\sqrt{\beta^4 + 8k\beta^2 + 8} + \beta^2\right], \tag{3.26a}$$

$$u(x,t) = \frac{1}{4}\left[\tanh\left(\frac{\sqrt{\beta^4 + 8k\beta^2 + 8}\;(C_1 + x + kt)}{2\beta^3}\right)\sqrt{\beta^4 + 8k\beta^2 + 8} + \beta^2\right]. \tag{3.26b}$$

*3.2. Example*

Assuming n=2 in Burgers hierarchy equation we obtain the Sharma- Tasso- Olver equation, this equation attracted the attention of many researchers from them Erbas and Yusufoglu [4].

$$u_t + \alpha(u^3)_x + \frac{3}{2}\alpha(u^2)_x + \alpha u_{xxx} = 0, \tag{3.27}$$

where a constant for finding exact solutions using (2.2) and (2.3), Eq. (3.27) becomes

$$k\,f' + \alpha(f^3)' + \frac{3}{2}\alpha(f^2)' + \alpha f''' = 0, \tag{3.28}$$

by integrating equation (3.28) and rearrangement we get

$$f'' = \frac{C_0}{\alpha} - \frac{k}{\alpha}f - f^3 - 3ff', \tag{3.29}$$

here $C_0$ is a constant. Let $X(\xi) = f(\xi)$, $Y(\xi) = f'(\xi)$, then equation (3.29) is equivalent to

$$\begin{cases} X' = Y, \\[2mm] Y'(\xi) = \dfrac{C_0}{\alpha} - \dfrac{k}{\alpha}X - X^3 - 3XY. \end{cases} \tag{3.30}$$

Now, according to the concept of our approach, we suppose the $X = X(\xi)$ and $Y = Y(\xi)$ are the nontrivial solutions to (3.30), and $q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in $\mathbb{C}[X, Y]$, such that

$$q\big(X(\xi), Y(\xi)\big) = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{3.31}$$

where at $a_i(X)$ when $i = 0,1,2,,m$ are polynomials of X and $a_m(X) \neq 0$. Equation (3.31) represents the first integral of equation (3.30). By utilizing the Division Theorem, there exists a polynomial $g(x) + h(X)Y$ in $\mathbb{C}[X,Y]$, such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{dX}{d\xi} + \frac{\partial q}{\partial Y}\frac{dY}{d\xi} = (g(x) + h(X)Y)\left(\sum_{i=0}^{m} a_i(X)Y^i\right), \tag{3.32}$$

in our study will take discuss two cases are clarify as follows:

**Case 1**

Suppose $m = 1$ in equation (3.31). Find that $\frac{dq}{d\xi}$ is a polynomial of X and Y, when $q(X,Y) = 0$ then $\frac{dq}{d\xi} = 0$,

$$\sum_{i=0}^{1} a_i'(X)Y^{i+1} + \sum_{i=0}^{1} ia_i(X)Y^{i-1}\left(\frac{C_0}{\alpha} - \frac{k}{\alpha}X - X^3 - 3XY\right)$$
$$= (g(x) + h(X)Y)\left(\sum_{i=0}^{1} a_i(X)Y^i\right). \tag{3.33}$$

On equating the coefficient of $Y^i$ ($i = 2,1,0$) on both sides of (3.33), we obtain

$$a_1'(X) = a_1(X)h(X), \tag{3.34a}$$
$$a_0'(X) + a_1(X)(-3X) = a_1(X)g(X) + a_0(X)h(X), \tag{3.34b}$$
$$a_1(X)\left(\frac{C_0}{\alpha} - \frac{k}{\alpha}X - X^3\right) = a_0(X)g(X), \tag{3.34c}$$

note that $a_1(X)$ is a polynomial of X, then by (3.34a), the $a_1(X)$ can be concluded as a constant and $h(X) = 0$. For simplicity, we suppose that $a_1(X) = 1$, and balancing the degrees of $a_0(X), a_1(X)$, and $g(X)$, we can deduce that degree of $g(X)$ equal one only, after that we suppose $g(X) = AX + B$, we find $a_0(X)$ from (3.34b)

$$a_0(X) = \frac{(A+3)X^2}{2} + BX + C \tag{3.35}$$

here $A, B$, and $C$ are constants. By substituting $a_1(X), a0(X) and g(X)$ in (3.34c) and all the coefficients of power $X$ equaling to zero. We will get a system of nonlinear algebraic equations

$$BC = \frac{C_0}{\alpha}, \quad B^2 + AC = \frac{-k}{\alpha}, \quad \frac{3B}{2} + \frac{3}{2}AB = 0, \quad \frac{3A}{2} + \frac{A^2}{2} = -1, \tag{3.36}$$

solving the last algebraic equations in (3.36), we have

$$A = -1, \quad C = \frac{k + B^2\alpha}{\alpha}, \quad C_0 = B(k + B^2\alpha), \tag{3.37a}$$

$$A = -2, \quad B = 0, \quad C = \frac{k}{2\alpha}, \quad C_0 = 0, \tag{3.37b}$$

using (3.37a) and (3.37b) in (3.10), we obtain

$$Y = -\frac{k + B^2\alpha + \alpha X(B + X)}{\alpha}, \tag{3.38a}$$

$$Y = -\frac{k + \alpha X^2}{\alpha}, \tag{3.38b}$$

respectively. Combining equations (3.38a, b) with (3.9), the exact solutions of equation (3.8) has become:

$$f(\xi) = \frac{1}{2}\left[-B - \frac{Tan\left(\frac{\sqrt{3B^2\alpha^2 + 4\alpha k}\ \ (C_1 + \xi)}{2\alpha}\right)\sqrt{3B^2\alpha + 4k}}{\sqrt{\alpha}}\right], \qquad (3.39a)$$

$$f(\xi) = -\frac{\sqrt{k}\ Tan\left(\frac{\sqrt{k}\ (C_1 + \xi)}{2\sqrt{\alpha}}\right)}{\sqrt{\alpha}} \qquad (3.39b)$$

where $C_1$ is a constant. Then the new exact solution of equation (3.27) as follows:

$$u(x,t) = -\frac{1}{2}\left[B + \frac{tan\left(\frac{\sqrt{3B^2\alpha^2 + 4\alpha k}\ \ (C_1 + x + kt)}{2\alpha}\right)\sqrt{3B^2\alpha + 4k}}{\sqrt{\alpha}}\right], \qquad (3.40a)$$

$$u(x,t) = -\frac{\sqrt{k}\ Tan\left(\frac{\sqrt{k}\ (C_1 + x + kt)}{2\sqrt{\alpha}}\right)}{\sqrt{\alpha}}. \qquad (3.40b)$$

**Case 2**

We assume that $m = 2$ in (10), and $q(X, Y) = 0$ this implies $\frac{dq}{d\xi} = 0$,

$$\sum_{i=0}^{2} a_i'(X)Y^{i+1} + \sum_{i=0}^{2} ia_i(X)Y^{i-1}\left(\frac{C_0}{\alpha} - \frac{k}{\alpha}X - X^3 - 3XY\right)$$
$$= (g(x) + h(X)Y)\left(\sum_{i=0}^{2} a_i(X)Y^i\right), \qquad (3.41)$$

on both sides of (3.41) by equating the coefficients of $Y^i$ when $i = 0, 1, 2$, we will get

$$a_2'(X) = a_2(X)h(X), \qquad (3.42a)$$
$$a_1'(X) + 2a_2(X)(-3X) = a_2(X)g(X) + a_1(X)h(X), \qquad (3.42b)$$
$$a_0'(X) + a_1(X)(-3X) + 2a_2(X)\left(\frac{C_0}{\alpha} - \frac{k}{\alpha}X - X^3\right) = a_1(X)g(X) + a_0(X)h(X), \qquad (3.42c)$$
$$a_1(X)\left(\frac{C_0}{\alpha} - \frac{k}{\alpha}X - X^3\right) = a_0(X)g(X), \qquad (3.42d)$$

note that $a_2(X)$ is a polynomial of X, then by (3.42a), the $a_2(X)$ can be concluded as a constant and $h(X) = 0$. For simplicity, we suppose that $a_2(X) = 1$, and balancing the degrees of $a_0(X), a_1(X)$, and $g(X)$, we can deduce that the degree of $g(X)$ equal one only, after that we suppose $g(X) = AX + B$, and $A \neq 0$, then we evaluate $a_1(X)$, and $a_0(X)$ from (3.42b, c)

$$a_1(X) = \left(\frac{(A + 6)}{2}X^2 + BX + C\right), \qquad (3.43a)$$
$$a_0(X) = d + BCX - \frac{2C_0}{\alpha}X + \frac{B^2}{2}X^2 + \frac{3C}{2}X^2 + \frac{AC}{2}X^2 + \frac{K}{\alpha}X^2 + 2BX^3 + \frac{AB}{2}X^3 + \frac{11}{4}X^4 + \frac{9A}{8}X^4$$
$$+ \frac{A^2}{8}X^4, \qquad (3.43b)$$

where $A, B, C$ and $d$ are constants. By substituting $a_2(X), a_1(X), a_0(X)$ and $g(X)$ in (3. 42d) and all the coefficients of power $X$ equaling to zero. We will get a system of nonlinear algebraic equations

$$A = -2, \quad B = 0, \quad d = \frac{Ck}{2\alpha}, \quad C_0 = 0 \qquad (3.44a)$$

$$A = -3, \quad B = 0, \quad C = \frac{3k}{2\alpha}, \quad d = \frac{k^2}{2\alpha^2}, \quad C_0 = 0 \qquad (3.44b)$$

$$A = -3, \qquad B = -\frac{i\sqrt{k}}{\sqrt{\alpha}}, \qquad C = \frac{k}{2\alpha}, \qquad d = 0, \qquad C_0 = 0, \qquad (3.44c)$$

$$A = -3, \qquad B = \frac{i\sqrt{k}}{\sqrt{\alpha}}, \qquad C = \frac{k}{2\alpha}, \qquad d = 0, \quad C_0 = 0, \qquad (3.44d)$$

using (3.44 a, b, c and d) in (3.10), we obtain

$$Y = -\frac{C\alpha + 2X^2\alpha + \sqrt{(C + 2X^2)\alpha(-2k + C\alpha)}}{2\alpha} \text{ ,and } \; Y = \frac{-C\alpha - 2X^2\alpha + \sqrt{(C + 2X^2)\alpha(-2k + C\alpha)}}{2\alpha},$$
$$(3.45a)$$

$$Y = -\frac{k + X^2\alpha}{2\alpha} \quad and \quad Y = -\frac{k + X^2\alpha}{\alpha}, \qquad (3.45b)$$

$$Y = X\left(-X + \frac{i\sqrt{k}}{\sqrt{\alpha}}\right) \quad and \quad Y = X\left(-X - \frac{i\sqrt{k}}{\sqrt{\alpha}}\right), \qquad (3.45c)$$

respectively. Combining equations (3.45a,b,c) with (3.28), we have other new exact solutions of equation (3.27) as:

$$f_1(\xi) = \left(\frac{\text{ArcTan}\left(\frac{\sqrt{\alpha}\xi}{\sqrt{k}}\right) - \text{ArcTan}\left(\frac{\sqrt{-2k + C\alpha}\xi}{\sqrt{k}\sqrt{C + 2\xi^2}}\right)}{2\sqrt{\alpha k}}\right)^{-1} - \frac{\xi}{2\alpha} + C_1$$

and

$$f_2(\xi) = \left(\frac{\text{ArcTan}\left(\frac{\sqrt{\alpha}\xi}{\sqrt{k}}\right) + \text{ArcTan}\left(\frac{\sqrt{-2k + C\alpha}\xi}{\sqrt{k}\sqrt{C + 2\xi^2}}\right)}{2\sqrt{\alpha k}}\right)^{-1} - \frac{\xi}{2\alpha} + C_1 \qquad (3.46a)$$

$$f_1(\xi) = -\sqrt{\frac{k}{\alpha}}\,\text{Tan}\left(\frac{\sqrt{k}\,(\xi - 2\alpha\,C_1)}{2\sqrt{\alpha}}\right), \text{and} \qquad f_2(\xi) = \sqrt{\frac{k}{\alpha}}\,\text{Tan}\left(\frac{\sqrt{k}\,(-\xi + \alpha\,C_1)}{\sqrt{\alpha}}\right) \qquad (3.46b)$$

$$f_1(\xi) = \frac{i\sqrt{k}}{\sqrt{\alpha} + i\text{Cos}\left(\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) + \text{Sin}\left(\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right)},$$

and

$$f_2(\xi) = \frac{i\sqrt{k}}{\sqrt{\alpha} - i\text{Cos}\left(\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) - \text{Sin}\left(\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right)} \qquad (3.46c)$$

$$f_1(\xi) = \left(\frac{1}{1 + \alpha\text{Cos}\left(2\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) - i\alpha\text{Sin}\left(2\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right)}\right)$$

$$\cdot \sqrt{k}\left(\text{Cos}\left(\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right)\right.$$

$$\left. - i\left(\sqrt{\alpha}\text{Cos}\left(2\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) + \text{Sin}\left(\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) - i\sqrt{\alpha}\text{Sin}\left(2\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right)\right)\right)$$

and

$$f_1(\xi) = \left( \frac{1}{1 + \alpha \cos\left(2\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) - i\alpha \sin\left(2\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right)} \right)$$

$$\cdot -\sqrt{k}\left( \cos\left(\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) \right.$$

$$\left. + i\left( \sqrt{\alpha}\cos\left(2\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) - \sin\left(\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) + i\sqrt{\alpha}\sin\left(2\sqrt{k}\left(\frac{\xi}{\sqrt{\alpha}} - C_1\right)\right) \right) \right)$$

$$(3.46d)$$

where $C_1$ is a constant. Then the exact solution with variables $x\ and\ t$ of equation (3.27) become as:

$$u_1(x,t) = \left( \frac{\text{ArcTan}\left(\frac{\sqrt{\alpha}(x+kt)}{\sqrt{k}}\right) - \text{ArcTan}\left(\frac{\sqrt{-2k+C\alpha}(x+kt)}{\sqrt{k}\sqrt{C+2(x+kt)^2}}\right)}{2\sqrt{\alpha k}} \right)^{-1} - \frac{(x+kt)}{2\alpha} + C_1,$$

and

$$u_2(x,t) = \left( \frac{\text{ArcTan}\left(\frac{\sqrt{\alpha}(x+kt)}{\sqrt{k}}\right) + \text{ArcTan}\left(\frac{\sqrt{-2k+C\alpha}(x+kt)}{\sqrt{k}\sqrt{C+2(x+kt)^2}}\right)}{2\sqrt{\alpha k}} \right)^{-1} - \frac{(x+kt)}{2\alpha} + C_1 \quad (3.47a)$$

$$u_1(x,t) = -\sqrt{\frac{k}{\alpha}}\,\text{Tan}\left(\frac{\sqrt{k}\left((x+kt) - 2\alpha\,C_1\right)}{2\sqrt{\alpha}}\right), \quad u_2(x,t) = \sqrt{\frac{k}{\alpha}}\,\text{Tan}\left(\frac{\sqrt{k}\left(-(x+kt) + \alpha\,C_1\right)}{\sqrt{\alpha}}\right) \quad (3.47b)$$

$$u_1(x,t) = \frac{i\sqrt{k}}{\sqrt{\alpha} + i\cos\left(\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) + \sin\left(\sqrt{k}\left(\frac{\xi(x+kt)}{\sqrt{\alpha}} - C_1\right)\right)},$$

and

$$u_2(x,t) = \frac{i\sqrt{k}}{\sqrt{\alpha} - i\cos\left(\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) - \sin\left(\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right)}, \quad (3.47c)$$

$$u_1(x,t) = \left( \frac{\sqrt{k}}{1 + \alpha\cos\left(2\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) - i\alpha\sin\left(2\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right)} \right)$$

$$\cdot \left( \cos\left(\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) \right.$$

$$- i\left( \sqrt{\alpha}\cos\left(2\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) + \sin\left(\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) \right.$$

$$\left. \left. - i\sqrt{\alpha}\sin\left(2\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) \right) \right)$$

$$u_2(x,t) = \left( \frac{-\sqrt{k}}{1 + \alpha \cos\left(2\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) - i\alpha \sin\left(2\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right)} \right.$$

$$\cdot \left( \cos\left(\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) \right.$$

$$+ i\left( \sqrt{\alpha}\cos\left(2\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) - \sin\left(\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right) \right.$$

$$\left.\left.\left. + i\sqrt{\alpha}\sin\left(2\sqrt{k}\left(\frac{(x+kt)}{\sqrt{\alpha}} - C_1\right)\right)\right)\right)\right). \qquad (3.47d)$$

All these solutions for m=1, 2 for two nonlinear evolution equations in physics are new exact solutions.

## 4. Conclusions

In this study, we have illustrated a first integral method and applied it for exploring some new exact solutions of important two-dimensional problems in physical phenomena with the help of symbolic computation software Maple. The periodic wave solutions and the solitary wave solutions have originated from the exact solutions. This method was proved the applicability and effectiveness for solving the nonlinear evolution equations.

## References

[1]  Davood, R. et al.,  2011 The First Integral Method for Solving Maccari's System, App. Math., 2, 258-263.
[2]  Ding, T. and Li, C., 1996 Ordinary differential equations, Peking: Peking university press.
[3]  El-Wakil, S. and Abdou, M., 2007 New Exact Travelling Wave Solutions Using Modified Extended Tanh-Function Method, Chao. Solit. Frac., 31, No. 4, 840-852.
[4]  Erbas B., Yusufoglu E., 2008 Exp-Function for constricting Exact solution of Sharma-Tasso- Olver Equation, https://doi.org/10.1016/j.chaos.2008.09.003.
[5]  Eslami1, M. et. al., 2014  Application of first integral method to fractional partial differential equations, Indian J Phys., 88(2), 177–184, DOI 10.1007/s12648-013-0401-6.
[6]  Feng, Z., 2002  On explicit exact solutions to the compound Burgers-KdV Equation, Physics Letters A 293(1-2):57-66, DOI:10.1016/S0375-9601(01)00825-8.
[7]   Fengyan Z,  Dongxue Li and Junmei, W., 2019  The Exact   Solutions of the Sharma-Tasso-Olver Equation Using exp(-G(ξ)) Method and Ansatz Method., Adv. in Appl. Math.,  8(2), 219-226.
[8]   Gardner, G.H.F.; Gardner L. and Gregory A., 1974 Formation Velocity and Density the Diagnostic Basics for Stratigraphic Traps. Geophysics 39, 770-780.
[9]   Khater, A., Malfli, W., Callebaut, D. and Kamel, E., 2002 The Tanh Method, a simple transformation and Exact Analytical Solutions for Nonlinear Reaction-Diffusion equations, Chao. Solit. Frac., 14, No. 3, 513-522.
[10] Li, H. and Guo, Y., 2006 New Exact Solutions to The Fitzhugh-Nagumo Equation, App. Math. and Compu., 180, No. 2, 524-528.
[11] Malfliet, w., 1992 Solitary Wave Solutions of Nonlinear Wave Equations, Am. J. Phys. 60, 7, 650-654, DOI: 10.1119/1.17120.
[12] Mousa, L. Biazar, J. and Zainab, A., 2018 The First Integral Method for Solving Some Conformable Fractional Differential Equations, Opt Quant Electron, 50-55, DOI: 10.1007/s11082-017-1307.
[13] Ndogmo, C. J., 2018 First Integrals and Hamiltonians of Some Classes of ODEs of Maximal Symmetry, J. Hindawi of Appl. Math., Article ID 7302081, 1-9.
[14] Shanna,  P. and Kushel, O., 2010  The First Integral Method for Huxley Equation, Int. J. of Noni. Sci., 10, No. l, 46-52.

[15] Sheng, Z., 2006 The Periodic Wave Solutions for The (2+1)-Dimensional Konopelchenko Dubrovsky Equations, Chao. Solit. Frac., 30, 1213-1220.

[16] Soliman, A. and Raslan, K., 2009 First Integral Method for The Improved Modified KdV Equation, Int. J. of Nonl. Sci. 8, No.I, 11-18.

[17] Taghizadeh, N., et. al., 2010 New Exact Solutions of Some Nonlinear Partial Differential Equations by The First Integral Method, Int. J. of Applic. and Appl. Math., 05, No. 10, 1543- 1553.

[18] Wazwaz, A., 2004 The Sine-Cosine Method for Obtaining Solutions with Compact and on Compact structures, Appl. Math. Comput, 159, No. 2, 559-576.

[19] Wazwaz, A., 2006 Two Reliable Methods for Solving Variants of The KdV Equation with Compact and Noncompact Structures ", Chao. Solit. Frac., 28, No.2, 454-462.

[20] Peng, Y.and Krishnan, E., 2006 Exact Travelling Wave Solutions for a Class of Nonlinear Partial Differential Equations, Int. J. Pure & Appl. Math. Sci., 3, No .I, 11-20.