

Applications of the Operator ${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right)$

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Abstract

In this paper, we construct the q -exponential operator ${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right)$. We use the operator ${}_3\phi_2$ to obtain an extension of Euler identities, Ramanujan's sum, q -Chu- Vanermonde summation formula and we give some other identities. Also we use the operator ${}_3\phi_2$ to get an extension of the Ramanujan's identity, the Askey beta integral, Ramanujan's beta integral and we give some other integrals formulas.

1- Introduction

In this paper we will use the standard notations for basic hypergeometric series given in [5], we assume that $|q| < 1$.

Definition 1.1. [5]. Let a be a complex variable. The q -shifted factorial is defined by

$$(a; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{k=0}^{n-1} (1 - aq^k), & \text{if } n = 1, 2, \dots \end{cases}$$

We

define

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

The following notation is used for the multiple q -shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \quad n = 0, 1, 2, \dots$$

$$(a_1, a_2, \dots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

Definition 1.2 [5]. The generalized basic hypergeometric series is defined by

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} [(-1)^n q^{\binom{n}{2}}]^{1+s-r} x^n,$$

where $r, s \in \mathbb{N}$; $a_1, \dots, a_r \in \mathbb{C}$; $b_1, \dots, b_s \in \mathbb{C} \setminus \{q^{-k}, k \in \mathbb{N}\}$ are assumed to be such that none of the denominator factors evaluate to zero. This series converges absolutely for all x if $r \leq s$ and for $|x| < 1$ if $r = s + 1$.

The case $r = s + 1$ is the most important class of series

$${}_{s+1}\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_{s+1} \\ b_1, b_2, \dots, b_s \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{s+1}; q)_n}{(q, b_1, \dots, b_s; q)_n} x^n, \quad |x| < 1.$$

The general bilateral basic hypergeometric series is given by:

$${}_r\psi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} [(-1)^n q^{\binom{n}{2}}]^{s-r} x^n, \quad |x| < 1.$$

Ramanujan's sum

$${}_1\psi_1(a; b; q, x) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} x^n = \frac{(q, b/a, ax, q/ax; q)_{\infty}}{(b, q/a, x, b/ax)_{\infty}}. \quad (1.1)$$

Definition 1.3 [5]. For $n \in \mathbb{N}$, the q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

In this paper, we will use the following identities (5):

$$(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}. \quad (1.2)$$

$$(aq^{-n}; q)_n = (q/a; q)_n (-a/q)^n q^{-\binom{n}{2}}. \quad (1.3)$$

$$(a; q)_n = (q^{1-n}/a; q)_n (-a)^n q^{\binom{n}{2}}. \quad (1.4)$$

One of the most important identities is the Cauchy identity (5)

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}, \quad |x| < 1. \quad (1.5)$$

Euler found the following special case of Cauchy identity (5):

$$\sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_{\infty}}, \quad |x| < 1. \quad (1.6)$$

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} = (-x; q)_{\infty}. \quad (1.7)$$

Hein's q -Gauss summation formula is (5)

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, c/ab \right) = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}}. \quad (1.8)$$

The q -Chu-Vandermonde summation formula (5):

$${}_2\phi_1 \left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q \right) = \frac{(c/b; q)_n}{(c; q)_n} b^n. \quad (1.9)$$

Definition 1.4 [1, 9]. The q -differential operator θ is defined by

$$\theta\{f(x)\} = \frac{f(q^{-1}x) - f(x)}{q^{-1}x}. \quad (1.10)$$

Definition 1.5 [9]. The Leibniz rule for θ is

$$\theta^n\{f(x)g(x)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \theta^k\{f(x)\}\theta^{n-k}\{g(xq^{-k})\}. \quad (1.11)$$

The following identities are easy to prove:

Theorem 1.6 [4, 12]. Let θ be defined as in (1.10), then

$$\theta^k\{x^n\} = (-1)^k x^{n-k} q^k (q^{-n}; q)_k.$$

$$\begin{aligned} \theta^k\{(xt; q)_\infty\} &= (-t)^k (xt; q)_\infty. \\ \theta^k\left\{\frac{(xv; q)_\infty}{(xt; q)_\infty}\right\} &= t^k q^{-\binom{k}{2}} (v/t; q)_k \frac{(xv; q)_\infty}{(xtq^{-k}; q)_\infty}, \quad |xt| < 1. \end{aligned}$$

Definition 1.7 [4]. The q -exponential operator $E(b\theta)$ is defined by

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q; q)_n}.$$

Theorem 1.8 Let θ be defined as in (1.10), then

$$\begin{aligned} E(b\theta)\{(at; q)_\infty\} &= (at, bt; q)_\infty. \\ E(b\theta)\{(as, at, ; q)_\infty\} &= \frac{(as, at, bs, bt; q)_\infty}{(abst/q; q)_\infty}. \end{aligned} \quad (1.12)$$

Based on the q -Chu-Vandermonde summation formula (1.9), Zhang and Yang [13] considered the finite q -exponential operator ${}_2\mathcal{T}_1\left(q^{-N}, v; q, t\theta\right)$ with two parameters as follows:

Definition 1.9 [13]. The finite q -exponential operator ${}_2\mathcal{T}_1\left(q^{-N}, v; q, t\theta\right)$ is defined by

$${}_2\mathcal{T}_1\left(q^{-N}, v; q, t\theta\right) = \sum_{n=0}^N \frac{(q^{-N}, v; q)_n}{(q, w; q)_n} (t\theta)^n. \quad (1.13)$$

Zhang and Yang [13] proved the following result:

Theorem 1.10 [13]. Let ${}_2\mathcal{T}_1\left(q^{-N}, v; q, t\theta\right)$ be defined as in (1.13), then we have

$${}_2\mathcal{T}_1\left(q^{-N}, v; q, t\theta\right)\{(xb; q)_\infty\} = (xb; q)_\infty {}_2\psi_1\left(q^{-N}, v; q, -tb\right).$$

Inspired by the basic hypergeometric series ${}_2\phi_1$, Li and Tan [8] introduced the generalized q -exponential operator $\mathbb{E}\left[\begin{smallmatrix} \mathbf{u}, \mathbf{v} \\ w \end{smallmatrix} \middle| q; t\theta\right]$ with three parameters as follows:

Definition 1.11 [8]. The generalized q -exponential operator $\mathbb{E}\left[\begin{smallmatrix} \mathbf{u}, \mathbf{v} \\ w \end{smallmatrix} \middle| q; t\theta\right]$ is defined by

$$\mathbb{E}\left[\begin{smallmatrix} \mathbf{u}, \mathbf{v} \\ w \end{smallmatrix} \middle| q; t\theta\right] = \sum_{n=0}^{\infty} \frac{(u, v; q)_n}{(q, w; q)_n} (t\theta)^n. \quad (1.14)$$

Li and Tan [8] proved the following result:

Theorem 1.12 [8]. Let $\mathbb{E}\left[\begin{smallmatrix} \mathbf{u}, \mathbf{v} \\ w \end{smallmatrix} \middle| q; t\theta\right]$ be defined as in (1.14), then we have

$$\begin{aligned} \mathbb{E} \left[\frac{\mathbf{u}, \mathbf{v}}{w} \middle| q; \frac{w}{uv\theta} \right] \{x^n\} &= x^n {}_3\phi_2 \left(\begin{matrix} q^{-N}, u, v \\ w, 0 \end{matrix}; q, -\frac{qw}{uvx} \right). \\ \mathbb{E} \left[\frac{\mathbf{u}, \mathbf{v}}{w} \middle| q; t\theta \right] \left\{ \frac{(xa; q)_\infty}{(xb; q)_\infty} \right\} &= \frac{(xa; q)_\infty}{(xb; q)_\infty} {}_3\phi_2 \left(\begin{matrix} u, v, a/b \\ w, q/xb \end{matrix}; q, -qt/x \right), \end{aligned} \tag{1.15}$$

where θ acts on x .

Thomae [10, 11] Jackson [6, 7] introduced the q -integral

$$\int_0^1 f(t) d_q t = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n$$

and Jackson gave the more general definition

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t,$$

where

$$\int_0^a f(t) d_q t = a(1 - q) \sum_{n=0}^{\infty} f(aq^n) q^n.$$

Jackson also defined an integral on $(0, \infty)$ by

$$\int_0^{\infty} f(t) d_q t = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^n.$$

The bilateral q -integral is defined by

$$\int_{-\infty}^{\infty} f(t) d_q t = (1 - q) \sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)] q^n.$$

Askey beta integral is given by ([2]):

$$\int_{-\infty}^{\infty} \frac{(xt, yt; q)_\infty}{(-wt, ut; q)_\infty} d_q t = \frac{2(1 - q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty (-xy/uwq; q)_\infty}. \tag{1.16}$$

Ramanujan's identity (i) is given by ([3])

$$\int_{-\infty}^{\infty} \frac{e^{-x^2 + 2mx}}{(sq^{1/2} e^{2ikx}, tq^{1/2} e^{-2ikx}; q)_\infty} dx = \sqrt{\pi} e^{m^2} \frac{(-sq e^{2mki}, -tq e^{-2mki}; q)_\infty}{(stq; q)_\infty}. \tag{1.17}$$

The Ramanujan's beta integral is given by

$$\int_0^{\infty} t^{x-1} \frac{(-yt; q)_\infty}{(t; q)_\infty} dt = \frac{\pi}{\sin(\pi x)} \frac{(q^{1-x}, y; q)_\infty}{(q, yq^{-x}; q)_\infty}. \tag{1.18}$$

2. The q -Exponential Operator and its Operator Identities

In this section, based on the basic hypergeometric series ${}_3\phi_2$, we define a q -exponential operator with five parameters ${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right)$ and obtain some its operator identities.

Definition 2.1. The q -exponential operator ${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right)$ is defined as follows:

$${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) = \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_n} (f\theta)^k. \tag{2.1}$$

Note that the finite q -exponential operator defined by Zhang and Yang [13] can be considered as special case of our operator for $a = q^{-N}$, $b = v$, $d = w$ and $c = e = 0$. Also, the generalized q -exponential operator defined by Li and Tan in [8] can be considered as special case of our operator for $a = u$, $b = v$, $c = 0$, $d = w$, $e = 0$ and $f = t$.

Theorem 2.2 We have

$${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \left\{ \frac{(xu; q)_\infty}{(xs; q)_\infty} \right\} = \frac{(xu; q)_\infty}{(xs; q)_\infty} {}_4\phi_3 \left(\begin{matrix} a, b, c, u/s \\ d, e, q/xs \end{matrix}; q, -qf/x \right). \quad (2.2)$$

provided that $\max\{|f/x|, |xs|\} < 1$.

Proof.

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \left\{ \frac{(xu; q)_\infty}{(xs; q)_\infty} \right\} \\ &= \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k \theta^k \left\{ \frac{(xu; q)_\infty}{(xs; q)_\infty} \right\} \\ &= \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k q^{-k(k-1)} s^k \frac{(u/s; q)_k (xu; q)_\infty}{(xsq^{-k}; q)_\infty} \\ &= \frac{(xu; q)_\infty}{(xs; q)_\infty} \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k q^{-k(k-1)} s^k \frac{(u/s; q)_k}{(-xs/q)^k q^{-k(k-1)} (q/xs; q)_k} \\ &= \frac{(xu; q)_\infty}{(xs; q)_\infty} {}_4\phi_3 \left(\begin{matrix} a, b, c, u/s \\ d, e, q/xs \end{matrix}; q, -qf/x \right). \quad \blacksquare \end{aligned}$$

Note that if $c = e = 0$ in (2.2) we get equation (1.15) proved by Li and Tan [8].

Setting $u = 0$ in (2.2), we get the following corollary:

Corollary 2.2.1. We have

$${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \left\{ \frac{1}{(xs; q)_\infty} \right\} = \frac{1}{(xs; q)_\infty} {}_4\phi_3 \left(\begin{matrix} a, b, c, 0 \\ d, e, q/xs \end{matrix}; q, -qf/x \right), \quad (2.3)$$

provided that $\max\{|xs|, |qf/x|\} < 1$.

Setting $s = 0$ in (2.2), we get the following corollary:

Corollary 2.2.2. We have

$${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \{(xu; q)_\infty\} = (xu; q)_\infty {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, -fu \right), \quad (2.4)$$

provided that $|fu| < 1$.

Theorem 2.3 We have

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \{(xs, xt, q)_\infty\} \\ &= (xs, xt, q)_\infty \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, b, c; q)_{k+j} (-ftq^{-j})^k (q/xt; q)_j (ftsx/q)^j q^{-\binom{j}{2}}}{(d, e; q)_{k+j} (q; q)_k (q; q)_j}. \quad (2.5) \end{aligned}$$

Proof. By using Leibniz rule (1.11), we have

$${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \{(xs, xt, q)_\infty\}$$

$$\begin{aligned}
 &= \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \theta_q^j \{ (xs; q)_{\infty} \} \theta^{k-j} \{ (xtq^{-j}; q)_{\infty} \} \\
 &= \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} (-s)^j (xs; q)_{\infty} (-tq^{-j})^{k-j} (xtq^{-j}; q)_{\infty} \\
 &= \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} (-s)^j (xs; q)_{\infty} (-tq^{-j})^{k-j} (xtq^{-j}; q)_j (xt; q)_{\infty}.
 \end{aligned}$$

By using (1.4) and (1.2) we get

$$\begin{aligned}
 &= (xs, xt; q)_{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, b, c; q)_{k+j} f^{k+j} (-tq^{-j})^k (q/xt; q)_j (-xt)^j q^{-j} q^{-\binom{j}{2}}}{(q, d, e; q)_{k+j} (q; q)_k (q; q)_j} \\
 &= (xs, xt, q)_{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, b, c; q)_{k+j} (-ftq^{-j})^k (q/xt; q)_j (ftsx/q)^j q^{-\binom{j}{2}}}{(d, e; q)_{k+j} (q; q)_k (q; q)_j}. \quad \blacksquare
 \end{aligned}$$

Note that, setting $a = 0, b = 0, c = 0, d = 0, e = 0$ and $f = cq^{(k+j-1)/2}$ in (2.5) and by using Euler identity (1.6) and Cauchy identity (1.5), we get the Theorem 2.11. obtained by Chen and Liu [4].

Theorem 2.4. Let the operator ${}_3\phi_2 \left(\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; q, f\theta \right)$ be defined as in (2.1) and n is a nonnegative integer, then

$$\begin{aligned}
 &{}_3\phi_2 \left(\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; q, f\theta \right) \{ (xt, q)_{\infty} x^n \} \\
 &= (xt, q)_{\infty} x^n \sum_{j=0}^{\infty} \sum_{k=0}^n \frac{(a, b, c; q)_{k+j} (q^{-n}; q)_k (-qf/x)^k (-ftq^{-n})^j}{(d, e; q)_{k+j} (q; q)_k (q; q)_j}. \quad (2.6)
 \end{aligned}$$

Proof. From definition of the operator ${}_3\phi_2 \left(\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; q, f\theta \right)$, we have

$${}_3\phi_2 \left(\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; q, f\theta \right) \{ (xt, q)_{\infty} x^n \} = \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k \theta^k \{ x^n (xt, q)_{\infty} \}.$$

By using Leibniz rule (1.11), we have

$$\begin{aligned}
 &\sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k \theta^k \{ (xt, q)_{\infty} x^n \} \\
 &= \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \theta^j \{ (xt; q)_{\infty} \} \theta^{k-j} \{ (xq^{-j})^n \} \\
 &= \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, d, e; q)_k} f^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} (-t)^j (xt; q)_{\infty} q^{-nj} (-1)^{(k-j)} q^{(k-j)} (q^{-n}; q)_{k-j} x^{n-(k-j)} \\
 &= x^n (xt, q)_{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^n \frac{(a, b, c; q)_{k+j} (q^{-n}; q)_k (-qf/x)^k (-ftq^{-n})^j}{(d, e; q)_{k+j} (q; q)_k (q; q)_j}. \quad \blacksquare
 \end{aligned}$$

Note that, setting $a = b = c = d = 0$ and $f = d$ in (2.6) and then using equations (1.7), (1.2) and (1.4), we get Corollary 2.4. obtained by Zhang and Liu [14].

Setting $t = 0$ in (2.6), we get the following corollary:

Corollary 2.4.1. We have

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right)\{x^n\} = x^n {}_4\phi_3\left(\begin{matrix} a, b, c, q^{-n} \\ d, e, 0 \end{matrix}; q, -qf/x\right). \tag{2.7}$$

When $c = e = 0$ and $f = d/ab$ in (2.7) we obtain equation (1.15) obtained by Li and Tan [8].

3. Applications in q -Identities

In this section, we use the operator ${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right)$ to give an extension to some wellknown q -identities such as: Euler identities (1.6), (1.7), Ramanujan’s sum (1.1), q -Chu-Vanermonde summation formula (1.9) and we give some other identities.

3.1 Extension of Euler Identities

Theorem 3.1. (Extension of Euler identity (1.7)). *We have*

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} {}_4\phi_3\left(\begin{matrix} a, b, c, q^{-n} \\ d, e, 0 \end{matrix}; q, -qf/x\right) = (-x; q)_{\infty} {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\right). \tag{3.1}$$

Proof. Recalling Euler identity, we have

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} = (-x; q)_{\infty}.$$

Applying the operator ${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right)$ to both sides of the above equation with respect to the parameter x , we get

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right)\{x^n\} = {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right)\{(-x; q)_{\infty}\}.$$

By using (2.7) and (2.4), we get

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} {}_4\phi_3\left(\begin{matrix} a, b, c, q^{-n} \\ d, e, 0 \end{matrix}; q, -qf/x\right) = (-x; q)_{\infty} {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\right). \quad \blacksquare$$

Theorem 3.2. (Cauchy identity). *We have*

$$\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}.$$

Proof. Setting $b = c = e = 0$, $f = -x$, in equation (3.1), we get

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} {}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ d \end{matrix}; q, q\right) = (-x; q)_{\infty} {}_2\phi_1\left(\begin{matrix} a, 0 \\ d \end{matrix}; q, -x\right).$$

By using (1.9), we get

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} a^n \frac{(d/a; q)_n}{(d; q)_n} = (-x; q)_{\infty} {}_2\phi_1\left(\begin{matrix} a, 0 \\ d \end{matrix}; q, -x\right)$$

$$\sum_{n=0}^{\infty} \frac{(d/a; q)_n q^{\binom{n}{2}} (xa)^n}{(q, d; q)_n} = (-x; q)_{\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q, d; q)_n} (-x)^n.$$

Replacing (d, x) by $(0, -x)$ in above equation to have:

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-xa)^n}{(q; q)_n} = (x; q)_{\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n.$$

by using Euler identity we obtain

$$\frac{(xa; q)_{\infty}}{(x; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n. \quad \blacksquare$$

Setting $c = e = 0, f = d/ab$ in (3.1) and then using the Hein’s q -Gauss summation formula (1.8), we get the following corollary:

Corollary 3.2.1. *We have*

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}x^n}{(q; q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, a, b \\ d, 0 \end{matrix}; q, -dq/abx\right) = (-x; q)_{\infty} \frac{(d/a, d/b; q)_{\infty}}{(d, d/ab; q)_{\infty}}.$$

Theorem 3.3. (Extension of Euler identity (1.6)). *We have*

$$\sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} {}_4\phi_3\left(\begin{matrix} a, b, c, q^{-n} \\ d, e, 0 \end{matrix}; q, -qf/x\right) = \frac{1}{(x; q)_{\infty}} {}_4\phi_3\left(\begin{matrix} a, b, c, 0 \\ d, e, \frac{q}{x} \end{matrix}; q, -\frac{fq}{x}\right). \quad (3.2)$$

Proof. From Euler identity (1.6), we have

$$\sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_{\infty}}.$$

Applying the operator ${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right)$ with respect to x , we have

$$\sum_{n=0}^{\infty} \frac{1}{(q; q)_n} {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right) \{x^n\} = {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta\right) \left\{ \frac{1}{(x; q)_{\infty}} \right\}.$$

By using (2.7) and (2.3) the proof is complete. \blacksquare

3.2. Extension of Ramanujan’s Sum

Theorem 3.4. (Extension of Ramanujan’s sum(1.1)). *We have*

$$\begin{aligned} \sum_{k=0}^n \frac{(s; q)_n}{(t; q)_n} x^n {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, -fq^n\right) \\ = \frac{(q, sx, q/sx, t/s; q)_{\infty}}{(t, q/s, x, t/sx; q)_{\infty}} {}_4\phi_3\left(\begin{matrix} a, b, c, x \\ d, e, qsx/t \end{matrix}; q, -qf/t\right). \end{aligned} \quad (3.3)$$

Proof. From Ramanujan’s sum(1.1) we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(s; q)_n}{(t; q)_n} x^n &= \frac{(q, t/s, sx, q/sx; q)_{\infty}}{(t, q/s, x, t/sx; q)_{\infty}} \\ \sum_{n=0}^{\infty} (s; q)_n x^n \frac{(tq^n; q)_{\infty}}{(t; q)_{\infty}} &= \frac{(q, t/s, sx, q/sx; q)_{\infty}}{(t, q/s, x, t/sx; q)_{\infty}} \\ \sum_{n=0}^{\infty} (s; q)_n x^n (tq^n; q)_{\infty} &= \frac{(q, t/s, sx, q/sx; q)_{\infty}}{(q/s, x, t/sx; q)_{\infty}}. \end{aligned}$$

Applying the operator ${}_3\phi_2\left(\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; q, f\theta\right)$ to both sides of the above equation with respect to the parameter t , we get

$$\begin{aligned} & \sum_{n=0}^{\infty} (s; q)_n x^n {}_3\phi_2\left(\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; q, f\theta\right) \{(tq^n; q)_{\infty}\} \\ &= \frac{(q, sx, q/sx; q)_{\infty}}{(q/s, x; q)_{\infty}} {}_3\phi_2\left(\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; q, f\theta\right) \left\{ \frac{(t/s, q)_{\infty}}{(t/sx; q)_{\infty}} \right\}. \end{aligned}$$

Now by using the relation (2.4) and (2.2) we get (3.3). ■

Corollary 3.4.1. *We have*

$$\sum_{n=0}^{\infty} \frac{(s; q)_n}{(t; q)_n} (tq^{n-m}; q)_m x^n = \frac{(q, sx, q/sx, t/s; q)_{\infty} (qs/t; q)_m}{(t, q/s, x, t/sx; q)_{\infty} (qsx/t; q)_m} x^m.$$

Proof. letting $a = q^{-m}, b = c = d = e = 0$ and $f = -t$ in (3.3), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(s; q)_n}{(t; q)_n} x^n \sum_{k=0}^m \frac{(q^{-m}; q)_k}{(q; q)_k} (tq^n)^k = \frac{(q, sx, q/sx, t/s; q)_{\infty}}{(t, q/s, x, qsx/t; q)_{\infty}} {}_2\phi_1\left(\begin{smallmatrix} q^{-m}, x \\ qsx/t \end{smallmatrix}; q, q\right) \\ & \sum_{n=0}^{\infty} \frac{(s; q)_n}{(t; q)_n} x^n \frac{(tq^{n-m}; q)_{\infty}}{(tq^n; q)_{\infty}} = \frac{(q, sx, q/sx, t/s; q)_{\infty} (qs/t; q)_m}{(t, q/s, x, t/sx; q)_{\infty} (qsx/t; q)_m} x^m \\ & \sum_{n=0}^{\infty} \frac{(s; q)_n}{(t; q)_n} (tq^{n-m}; q)_m x^n = \frac{(q, sx, q/sx, t/s; q)_{\infty} (qs/t; q)_m}{(t, q/s, x, t/sx; q)_{\infty} (qsx/t; q)_m} x^m. \end{aligned}$$
■

Setting $b = c = e = 0, f = -x$, in (3.2) and using the q -Chu-Vandermonde summation formula (1.9), we get the following corollary:

Corollary 3.4.2. *We have*

$${}_2\phi_1\left(\begin{smallmatrix} d/a, 0 \\ d \end{smallmatrix}; q, xa\right) = \frac{1}{(x; q)_{\infty}} {}_3\phi_2\left(\begin{smallmatrix} a, 0, 0 \\ d, q/x \end{smallmatrix}; q, q\right).$$

3.3 Extension of the q -Chu-Vandermonde Summation Formula

Theorem 3.5. (Extension of the q -Chu-Vandermonde summation formula (1.9)). *We have*

$$\begin{aligned} & \sum_{j=0}^n \frac{(q^{-n}, x; q)_j}{(q, y; q)_j} q^j {}_4\phi_3\left(\begin{smallmatrix} a, b, c, q^j x \\ d, e, qx/y \end{smallmatrix}; q, -qf/y\right) \\ &= x^n \frac{(y/x; q)_n}{(y; q)_n} {}_4\phi_3\left(\begin{smallmatrix} a, b, c, xq^{2n} \\ d, e, q^{1+n}x/y \end{smallmatrix}; q, -qf/y\right). \end{aligned} \tag{3.4}$$

Proof. Recalling the q -Chu-Vandermonde summation formula

$$\begin{aligned} & \sum_{j=0}^n \frac{(q^{-n}, x; q)_j}{(q, y; q)_j} q^j = x^n \frac{(y/x; q)_n}{(y; q)_n} \\ & \sum_{j=0}^n \frac{(q^{-n}, x; q)_j}{(q; q)_j} q^j \frac{(yq^j; q)_{\infty}}{(y/x; q)_{\infty}} = x^n \frac{(yq^n; q)_{\infty}}{(y/xq^n; q)_{\infty}}. \end{aligned}$$

Applying the operator ${}_3\phi_2\left(\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; q, f\theta\right)$ with respect to y , we obtain

$$\sum_{j=0}^n \frac{(q^{-n}, x; q)_j}{(q; q)_j} q^j {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \left\{ \frac{(yq^j; q)_\infty}{(y/x; q)_\infty} \right\} = x^n {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \left\{ \frac{(yq^n; q)_\infty}{(y/xq^n; q)_\infty} \right\}.$$

Then by using (2.2) the proof is complete . ■

4. Applications q -Integrals

In this section, we use the operator ${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right)$ to obtain an extension of the Askey beta integral (1.16), the Ramanujan’s identity (1.17), , Ramanujans beta integral (1.18) and we give some other integrals formulas.

4.1. Extension of the Askey Beta Integral

Theorem 4.1. (Extension of the Askey Beta Integral (1.16)). *We have*

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{(xt, yt; q)_\infty}{(-wt, ut; q)_\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, b, c; q)_{k+j}}{(d, e; q)_{k+j}} \frac{(fyq^{-1-j}/wu)^k}{(q; q)_k} \frac{(-uwq^2/xy; q)_j}{(q; q)_j} \left(\frac{-fyxt}{wuwq^2}\right)^j q^{-\binom{j}{2}} d_q t \\ &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} \\ & \quad \times \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, b, c; q)_{k+j}}{(d, e; q)_{k+j}} \frac{(fq^{-j}/w)^k}{(q; q)_k} \frac{(-qw/x; q)_j}{(q; q)_j} \left(\frac{-fx}{uwq}\right)^j q^{-\binom{j}{2}}. \end{aligned} \tag{4.1}$$

Proof. From Askey beta integral (1.16) we have:

$$\int_{-\infty}^{\infty} \frac{(xt, yt; q)_\infty}{(-wt, ut; q)_\infty} d_q t = \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty (-xy/uwq; q)_\infty}.$$

multiplying both sides of the above equation by $(-xy/uwq; q)_\infty$ and then applying the operator

${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right)$ with respect to x , we get

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{(yt; q)_\infty}{(-wt, ut; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \{(xt, -xy/uwq; q)_\infty\} d_q t \\ &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \{(x/u, -x/w; q)_\infty\}. \end{aligned}$$

Using equation (2.5) on both sides of above equation, we get the required result. ■

Note that setting $a = b = c = d = e = 0$ and $f = cq^{(k+j-1)/2}$ in (4.1) and then using Cauchy identity and Euler identity we get Theorem 6.3 obtained by Chen and Liu [4].

Corollary 4.4.1. *We have*

$$\int_{-\infty}^{\infty} \frac{(xt, yt; q)_\infty}{(-wt, ut; q)_\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, b; q)_{k+j}}{(u; q)_{k+j}} \frac{(yq^{-1-j}/ab)^k}{(q; q)_k} \frac{(-uwq^2/xy; q)_j}{(q; q)_j} \left(\frac{-yxt}{abq^2}\right)^j q^{-\binom{j}{2}} d_q t$$

$$\begin{aligned}
 &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} \frac{(u/a, u/b, q)_\infty}{(u, u/ab; q)_\infty} \\
 &\times {}_3\phi_2 \left(\begin{matrix} a, b, -\frac{qw}{u} \\ \frac{d}{u}qab, 0 \end{matrix}; q, \frac{x}{u} \right). \tag{4.2}
 \end{aligned}$$

Proof. Setting $c = e = 0, d = u$ and $f = uw/ab$ in (4.1) we have

$$\begin{aligned}
 L.H.S. &= \int_{-\infty}^{\infty} \frac{(xt, yt; q)_\infty}{(-wt, ut; q)_\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, b; q)_{k+j}}{(u; q)_{k+j}} \frac{(yq^{-1-j}/ab)^k}{(q; q)_k} \frac{(-uwq^2/xy; q)_j}{(q; q)_j} \left(\frac{-yxt}{abq^2}\right)^j q^{-\binom{j}{2}} d_q t. \\
 R.H.S. &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} \\
 &\times \sum_{j=0}^{\infty} \frac{(a, b, -qw/x; q; q)_j}{(q, u; q)_j} \left(\frac{-x}{abq}\right)^j q^{-j_2} \sum_{k=0}^{\infty} \frac{(aq^j, bq^j; q)_k}{(q, q^j u; q)_k} (uq^{-j}/ab)^k \\
 &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} \\
 &\times \sum_{j=0}^{\infty} \frac{(a, b, -qw/x; q; q)_j}{(q, u; q)_j} \left(\frac{-x}{abq}\right)^j q^{-j_2} {}_2\phi_1 \left(\begin{matrix} aq^j, bq^j \\ uq^j \end{matrix}; q, uq^{-j}/ab \right) \\
 &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} \\
 &\times \sum_{j=0}^{\infty} \frac{(a, b, -qw/x; q; q)_j}{(q, u; q)_j} \left(\frac{-x}{abq}\right)^j q^{-j_2} \frac{(u/a, u/b; q)_\infty}{(uq^j, uq^{-j}/ab; q)_\infty} \quad (\text{by using (1.8)}) \\
 &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} \frac{(u/a, u/b, q)_\infty}{(u, u/ab; q)_\infty} \\
 &\times \sum_{j=0}^{\infty} \frac{(a, b, -qw/x; q; q)_j}{(q; q)_j} \left(\frac{-x}{abq}\right)^j q^{-j_2} \frac{1}{(uq^{-j}/ab; q)_j} \quad (\text{by using (1.2)}) \\
 &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} \frac{(u/a, u/b, q)_\infty}{(u, u/ab; q)_\infty} \\
 &\times \sum_{j=0}^{\infty} \frac{(a, b, -qw/x; q; q)_j}{(q; q)_j} \left(\frac{-x}{abq}\right)^j q^{-j_2} \frac{1}{(-1)^j q^{-j_2} \left(\frac{u}{abq}\right)^j (qab/u; q)_j} \\
 &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} \frac{(u/a, u/b, q)_\infty}{(u, u/ab; q)_\infty} \\
 &\times \sum_{j=0}^{\infty} \frac{(a, b, -qw/x; q; q)_j}{(q, qab/u; q)_j} \left(\frac{x}{u}\right)^j \\
 &= \frac{2(1-q)(q^2; q^2)_\infty^2 (wu, q/wu, x/u, -x/w, y/u, -y/u; q)_\infty}{(q; q)_\infty (w^2, u^2, q^2/w^2; q^2)_\infty} \frac{(u/a, u/b, q)_\infty}{(u, u/ab; q)_\infty} \\
 &\times {}_3\phi_2 \left(\begin{matrix} a, b, -qw/x \\ qab/u, 0 \end{matrix}; q, x/u \right). \quad \blacksquare
 \end{aligned}$$

4.2. Extension of Ramanujan’s Identity

Theorem 4.2. (Extension of the Ramanujan’s Identity (1.17)). *We have*

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(sqe^{1/2}e^{2ikx}, tqe^{1/2}e^{-2ikx}, q)_\infty} {}_4\phi_3 \left(\begin{matrix} a, b, c, 0 \\ d, e, \frac{q^{1/2}}{s}e^{-2kix} \end{matrix}; q, -qf \right) dx$$

$$= \frac{\sqrt{\pi} e^{m^2} (-sqe^{2mki}, -tqe^{-2mki}; q)_{\infty}}{(stq; q)_{\infty}} {}_4\phi_3 \left(\begin{matrix} a, b, c, \frac{-e^{2mki}}{t} \\ d, e, 1/ts \end{matrix}; q, -qf/s \right), \tag{4.3}$$

provided that $\max\{|stq|, |qf/s|\} < 1$.

Proof. Recalling Ramanujan’s identity (1.17), then applying the operator ${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right)$ on both sides of the above equation with respect to the parameter s , we get

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(tq^{1/2}e^{-2ikx}; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \left\{ \frac{1}{(sq^{1/2}e^{2ikx}; q)_{\infty}} \right\} dx$$

$$= \sqrt{\pi} e^{m^2} (-tqe^{-2mki}; q)_{\infty} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right) \left\{ \frac{(-sqe^{2mki}; q)_{\infty}}{(stq; q)_{\infty}} \right\}.$$

Now by using equations (2.3) and (2.2) the proof is complete. ■

4.3 Extension The Ramanujan’s Beta Integral

Theorem 4.3 (Extension The Ramanujan’s beta integral (1.17))

$$\int_0^{\infty} t^{x-1} \frac{(-yt; q)_{\infty}}{(t; q)_{\infty}} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, b; q)_{k+j} (-fq^{-x-j})^k (q^{x+1}/y; q)_j (-ftyq^{-x-1})^j}{(d, e; q)_{k+j} (q; q)_k (q; q)_j} q^{-\binom{j}{2}} d_q t$$

$$= \frac{\pi}{\sin(\pi x)} \frac{(q^{1-x}; q)_{\infty}}{(q, yq^{-x}; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, -f \right). \tag{4.4}$$

Proof. Multiplying The Ramanujan’s beta integral (1.17) by $(yq^{-x}; q)_{\infty}$ we get

$$\int_0^{\infty} t^{x-1} \frac{1}{(t; q)_{\infty}} (-yt, yq^{-x}; q)_{\infty} dt = \frac{\pi}{\sin(\pi x)} \frac{(q^{1-x}; q)_{\infty}}{(q; q)_{\infty}} (y; q)_{\infty}.$$

Now applying the operator ${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, f\theta \right)$ with respect to y , and using tow relations (2.5), (2.4) the proof is complete. ■

Setting $c = e, f = -d/ab$ in (4.4) and then using Hein’s q -Gauss summation formula (1.8) we can obtain the following corollary:

Corollary 4.3.1. *We have*

$$\int_0^{\infty} t^{x-1} \frac{(-yt; q)_{\infty}}{(t; q)_{\infty}} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, b; q)_{k+j} (dq^{-x-j}/ab)^k (q^{x+1}/y; q)_j (dtyq^{-x-1}/ab)^j}{(d; q)_{k+j} (q; q)_k (q; q)_j} q^{-\binom{j}{2}} d_q t$$

$$= \frac{\pi}{\sin(\pi x)} \frac{(q^{1-x}, y; q)_{\infty} (d/a, d/b; q)_{\infty}}{(q, yq^{-x}; q)_{\infty} (d, d/ab; q)_{\infty}}.$$

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