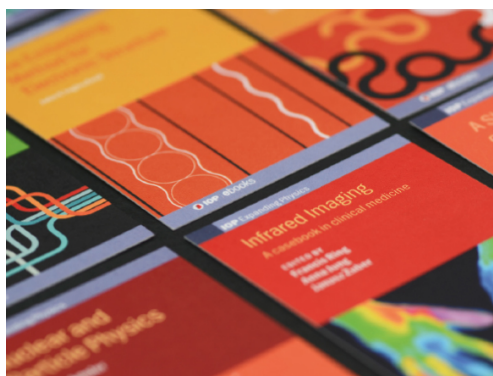**PAPER • OPEN ACCESS**

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On spectral asymptotic for the second-derivative operators

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Abstract. In this work we focus on spectral asymptotic for the second derivative operators. Here we study Schrödinger operator with zero-range potentials, because this operator has great importance for understanding the solvable problems in quantum mechanics and atomic physics. It appears in different models such as the mathematical physics, applied mathematics and theoretical physics. We have two objectives in this work. We first demonstrated that this operator has a continuous spectrum contains an infinite number of bands separated by gaps. We then explained that the bands to gaps ratio tends to zero under certain conditions.

1. Introduction

The differential operators are ubiquitous in many natural systems, ranging from quantum to atomic physics applications. These applications are used to give rise a solvable model of complicated physical phenomena [1,2,5]. Because the method of solid-state physics reproduces the geometry of the problem extremely well, therefore, there is a particular interest in the applications of these models. Kroing and Penney [10] were the first who described this model by the Hamiltonian operator

$$ \mathbf{H} = -\frac{d^2}{dx^2} + \sum_{n \in Z} a_n \, \delta(x - n), $$

where δ is the Dirac delta function and Γ_n are the actual coupling constants that describes each point interactions. They also explained the spectrum of permissible energy values which consists of continuous region separated by finite intervals. Further, this operator is used to solve the complicated physical phenomena. The point interactions found in many different models by considering boundary conditions at the individual points. The generalized point interaction in one dimension with boundary conditions

$$ \begin{pmatrix} \psi(0^+) \\ \psi(0^+) \end{pmatrix} = e^{i\theta} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi(0^-) \\ \psi(0^-) \end{pmatrix}, $$

is studied in [12, 13]. He also discussed the existence and the physical properties of the one-dimensional δ′-interaction Hamiltonian. Bloch theorem is used to explain that any such operator coincides with some self-adjoint extension of the unperturbed second-derivative operator restricted to the set of functions vanishing in a neighbourhood of the origin [7]. Moreover, the

connected extensions of the Schrödinger operator are studied and described by the boundary conditions at the origin in [8],

$$\begin{pmatrix} \psi(0^+) \\ \psi(0^+) \end{pmatrix} = e^{i\theta} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi(0^-) \\ \psi(0^-) \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta$ are real, and $\alpha\delta - \beta\gamma = 1$, $0 \quad \theta < 2\pi$. The spectrum of the generalized Kroing-Penney model has infinitely many gaps and the behaviour depend substantially on the parameters of generalized point interaction [6]. Moreover, the spectral asymptotic for operators with partial derivatives have been the subject of extensive research for over a century. Therefore, it drew the attention of many remarkable mathematicians and physicists. The mathematical framework used to describe this spectral asymptotic was based on the Bloch theorem. In our work we used the transfer matrix to describe this behaviour.

The main result of this paper is contained in three Propositions which describe the asymptotic behaviour of the operator $\mathcal{L}$ corresponding to the values of three independent real parameters. We show that the spectrum of this operator is absolutely continuous and fills in an infinite number of bands separated by gaps.

Let us give a brief outline of the contents of the paper: In section 2, we define the second-derivative operator and discuss the classes of unitary of equivalent of this operator. We also derive the reduction relation in Proposition 2.1. Then, we study the transfer matrix to obtain the dispersion relation which uses to calculate the spectral bands. In section 3, we investigate the spectral asymptotic by three Propositions (3.1), (3.2) and (3.3).

2. Preliminaries

At the beginning let us briefly recall the definition of the second-derivative operator $\mathcal{L}$. We consider here the operator $\mathcal{L} \equiv \mathcal{L}(\mathcal{A}, \theta)$ where $\mathcal{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, R)$ such that $\alpha, \beta, \gamma, \delta$ $\mathbb{R}$ and $0 \quad \theta < 2\pi$, acting in the Hilbert space $L_2(\mathbb{R})$ defined on the functions from $W_2^2\{\mathbb{R}\setminus \{n\}_n \quad \mathbb{Z}$ (Sobolev space) satisfying the boundary conditions,

$$\begin{pmatrix} u_R(n) \\ u'_L(n) \end{pmatrix} = e^{i\theta} \mathcal{A} \begin{pmatrix} u_R(n) \\ u'_L(n) \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{2.1}$$

In addition, this coincides with a self-adjoint operator extension of the operator $\mathcal{L} = -d^2/dx^2$ limited to all functions from $W_2^2(\mathbb{R})$, disappearance in a neighbourhood of the points $x = n$ [9].

Now, in order to illustrate the spectral asymptotic of the second derivative operator, we first are going to describe the classes of unitary equivalent operators of this operator. There are three independent real parameters to describe these classes which are $t = \alpha + \delta$, β and γ. The following proposition explains the relationship between these parameters to each other, as well as determining the values of these parameters to calculate the spectral asymptotic of the second derivative operator.

Proposition 2.1. If t, β and γ be three independent real parameters describing the operator $\mathcal{L}$ such that $t = \alpha + \delta$, then $t > 2\sqrt{\beta\gamma + 1}$.

Proof. Since $t = \alpha + \delta$, then multiplication this equation by α we get:

$$\alpha t = \alpha^2 + \alpha\delta.$$

But

$$\alpha\delta - \beta\gamma = 1,$$

thus

$$\alpha\delta = 1 + \beta\gamma.$$

Implies that

$$\alpha t - \alpha^2 = 1 + \beta\gamma,$$

then

$$\alpha = \frac{t \quad \sqrt{t^2 - 4(\beta\gamma + 1)}}{2}.$$

By the same way we get

$$\delta = \frac{t \quad \sqrt{t^2 - 4(\beta\gamma + 1)}}{2},$$

therefore

$$\alpha + \delta = t \quad \sqrt{t^2 - 4(\beta\gamma + 1)}.$$

Since

$$t^2 - 4(\beta\gamma + 1) \qquad 0,$$

implies that

$$t^2 \quad 4(\beta\gamma + 1).$$

Then

$$t \quad 2\sqrt{\beta\gamma + 1}. \tag{2.2}$$

Now, we are going to study the transfer matrix for the purpose of describing the second derivative operator spectrum. Subsequently, this matrix is given by [3, 4]

$$
\mathcal{T}_\lambda = \begin{pmatrix} \cos\kappa & \frac{1}{\kappa}\sin\kappa \\ -\kappa\sin\kappa & \cos\kappa \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
$$

$$
= \begin{pmatrix} \alpha\cos\kappa + \frac{\gamma}{\kappa}\sin\kappa & \beta\cos\kappa + \frac{\delta}{\kappa}\sin\kappa \\ -\alpha\kappa\sin\kappa + \gamma\cos\kappa & -\beta\kappa\sin\kappa + \delta\cos\kappa \end{pmatrix} \tag{2.3}
$$

where $\kappa = \overline{\lambda}$. And since $\det\mathcal{T}_\lambda = 1$, therefore, the specific determinant of this matrix is given by

$$\det(\mathcal{T}_\lambda - \lambda I) = \lambda^2 - \lambda\mathrm{Tr}\mathcal{T}_\lambda + 1.$$

Furthermore, the operator's spectrum coincides with the set of λ where the spectrum of this operator is calculated as zeros of the following inequality [11],

$$|\mathrm{Tr}\mathcal{T}_\lambda| \le 2.$$

Thus

$$\left|(\alpha + \delta)\cos\kappa + \left(\frac{\gamma}{\kappa} - \beta\kappa\right)\sin\kappa\right| \le 2.$$

Let us now define the function g by

$$g(\kappa) = t\cos\kappa + \left(\frac{\gamma}{\kappa} - \beta\kappa\right)\sin\kappa. \tag{2.4}$$

Consequently, we can be determined the operator's spectrum by solving the following equation

$$|g(\kappa)| \quad 2 \tag{2.5}$$

This equation is called the dispersion relation which used to obtain the spectral bands in the following section.

3. Spectral asymptotic for the periodic operator

In this section, we study the spectral asymptotic for the second derivative operator $\mathcal{L}$. There are infinite numbers of bands in this operator, which has a continuous spectrum (i.e. consist of all eigenvalues such that the resolvent of operator $\mathcal{L}$ exists and defined on a set which is dense in $L_2(\mathbb{R})$) and it is tending to   . The following three Propositions give an explicit description of the spectral asymptotic corresponding to the parameters of this operator.

Proposition 3.1. Assume that β and γ are arbitrary satisfying the equation (2.2). If $\beta \quad 0$, then there are infinite numbers of bands $\Delta_n = [A_n^2, B_n^2]$ of the operator $\mathcal{L}$, which has a continuous spectrum and located in the intervals $[(\pi n - \pi/2)^2, (\pi n + \pi/2)^2]$ for large values of n. And their edges are asymptotically which are given by

$$A_n = \pi n + \frac{t-2}{\beta\pi}\frac{1}{n} + \left(-\frac{1}{3\beta^3\pi^3}t^3 + \frac{1-\beta}{\beta^3\pi^3}t^2 + \frac{\gamma+4}{\beta^2\pi^3}t - \frac{4}{3\beta^3\pi^3} - \frac{4+2\gamma}{\beta^2\pi^3}\right)\frac{1}{n^3} + O\left(\frac{1}{n^5}\right),$$
$$as\, n \quad ,$$

$$\tag{3.1}$$

$$B_n = \pi n + \frac{t+2}{\beta\pi}\frac{1}{n} + \left(-\frac{1}{3\beta^3\pi^3}t^3 - \frac{1+\beta}{\beta^3\pi^3}t^2 + \frac{\gamma-4}{\beta^2\pi^3}t + \frac{4}{3\beta^3\pi^3} + \frac{2\gamma-4}{\beta^2\pi^3}\right)\frac{1}{n^3} + O\left(\frac{1}{n^5}\right),$$
$$as\, n \quad ,$$

In addition, the length and the midpoint of the band are asymptotically which given by:

$$|\quad_n| = \frac{8}{|\beta|} + \frac{4}{\pi^2}\left(-\frac{1}{|\beta|\beta^2}t^2 - \frac{2}{|\beta|\beta}t + \frac{4}{3\beta^3} + \frac{2\gamma}{|\beta|\beta}\right)\frac{1}{n^2} + O\left(\frac{1}{n^4}\right), \qquad as\, n$$
$$,\quad (3.4)$$

and

$$M_n = \pi^2 n^2 + \frac{2t}{\beta} + \frac{1}{\pi^2}\left(-\frac{2}{3\beta^3}t^3 - \frac{1}{\beta^2}t^2 + \frac{2\gamma}{\beta^2}t - \frac{4}{\beta^2}\right)\frac{1}{n^2} + O\left(\frac{1}{n^4}\right),$$
$$as\, n \quad ,\quad (3.5)$$

respectively.

Proof. At first, let us to prove that there is only one band n of continuous spectrum in each interval I_n for the large enough values of κ.

Now, by the equation (2.4) we get

$$g(\pi n + \pi/2) = t\cos(\pi n + \pi/2) + (\frac{\gamma}{\pi n + \pi/2} - \beta(\pi n + \pi/2)\sin(\pi n + \pi/2)$$

$$= (-1)^{n+1}\beta\pi n + O(1) \ as \ n$$

This equation determines the values of the end points of each interval I_n. Since it has alternating signs, and when n is sufficiently large, thus $|g(\pi n + \pi/2)| > 2$. Consequently, that means there is one spectral band when the interval is considered.

Let $g'(\kappa) = 0$ we get:

$$0 = g'(\kappa) = -\left(t + \frac{\gamma}{\kappa^2} + \beta\right)\sin\kappa + \left(\frac{\gamma}{\kappa} - \beta\kappa\right)\cos\kappa,$$

implies that

$$\tan\kappa = \kappa(\gamma - \beta\kappa^2)/(\kappa^2(t + \beta) + \gamma.$$
$$(3.6)$$

This function is rational and by the comparison test it tends to $\pm$ as κ .

Note that

1- if $t + \beta = 0, \gamma \quad 0$, then $(\kappa(\gamma - \beta\kappa^2))/((t + \beta)\kappa^2 + \gamma) = \kappa - \beta/\gamma \kappa^3$.

2- if $t + \beta \neq 0$, γ arbitrary, then $\frac{\kappa(\gamma - \beta\kappa^2)}{(t+\beta)\kappa^2+\gamma} = -\beta/(t + \beta) \kappa + (\gamma(t + 2\beta))/((t + \beta)^2) \ 1/\kappa + O(1/\kappa^2)$.

3- if $t + \beta = 0$, $\gamma = 0$, then the relation (3.6) takes the form:
$$(t + \gamma/\kappa^2 + \beta)\sin\kappa = (\gamma/\kappa - \beta\kappa)\cos\kappa,$$
implies that $-\beta\kappa\cos\kappa = 0$.
But $\cos\kappa = 0$ when $\kappa = n\pi + \pi/2$, hence, there is one extreme point in each interval I_n for the function g when $n \to \infty$. Consequently, because the function g is continuous and monotonically between these points, then for n is sufficiently large, there is only one band where $|g(\kappa)|$ 2 in each interval I_n.
In order to calculate the end points of each band n, let us to solve the equation $|g(\kappa)| = 2$ [11]. Consider the first case $\beta > o$, then the left and right end points of the intervals Δ_n satisfy the following equations
$$t\cos A_n + (\gamma/A_n - \beta A_n)\sin A_n = (-1)^n 2, \tag{3.7}$$
$$t\cos B_n + (\gamma/B_n - \beta B_n)\sin B_n = (-1)^n 2, \tag{3.8}$$
respectively.
On the other hand, due to the points A_n and B_n are closed to πn for large n, then let us to use the following representation of the asymptotic
$$A_n = \pi n + \frac{a}{n} + \frac{a'}{n^3} + O\left(\frac{1}{n^5}\right), \qquad B_n = \pi n + \frac{b}{n} + \frac{b'}{n^3} + O\left(\frac{1}{n^5}\right) \ as \ n \to \infty.$$

Substituting these representations into (3.7) and (3.8), we get:

$$A_n = \pi n + \frac{1}{\pi}\left[\frac{t}{\beta} - \frac{2}{|\beta|}\right]\frac{1}{n} + \left[-\frac{t^3}{3\beta^3\pi^3} - \left(1 - \frac{1}{|\beta|}\right)\frac{t^2}{\beta^2\pi^3} + +(\frac{\gamma}{\beta^2\pi^3} + \frac{\gamma}{\beta^2\pi^3})t - \frac{4}{3|\beta|^3\pi^3}\right.$$
$$\left. - \frac{2}{\beta^3\pi^3}(2\beta - \gamma|\beta|)\right]\frac{1}{n^3} + O\left(\frac{1}{n^5}\right), as \ n \to \infty$$

$$B_n = \pi n + \frac{1}{\pi}\left[\frac{t}{\beta} + \frac{2}{|\beta|}\right]\frac{1}{n} + \left[-\frac{t^3}{3\beta^3\pi^3} - (1 + \frac{1}{|\beta|})\frac{t^2}{\beta^2\pi^3} + (\frac{\gamma}{\beta^2\pi^3} - \frac{4|\beta|}{\beta^3\pi^3})t + \frac{4}{3|\beta|^3\pi^3}\right.$$
$$\left. - \frac{2}{\beta^3\pi^3}(2\beta - \gamma|\beta|)\right]\frac{1}{n^3} + O(\frac{1}{n^5}) \quad as \ n \to \infty.$$

In the similar way we can be analysed of the case when $\beta < 0$, which leads to formula (3.1). Finally, the $|\Delta_n|$ and M_n of the band are given by

$$|\Delta_n| = B_n^2 - A_n^2 = \frac{8}{|\beta|} + \frac{4}{\pi^2}\left(-\frac{1}{|\beta|\beta^2}t^2 - \frac{2}{|\beta|\beta}t + \frac{4}{3|\beta|^3} + \frac{2\gamma}{|\beta|\beta}\right)\frac{1}{n^2} + O\left(\frac{1}{n^4}\right), \quad as \ n \to \infty,$$

And

$$M_n = \frac{A_n^2 + B_n^2}{2} = \pi^2 n^2 + \frac{2t}{\beta} + \frac{1}{\pi^2}\left(-\frac{2}{3\beta^3}t^3 - \frac{1}{\beta^2}t^2 + \frac{2\gamma}{\beta^2}t - \frac{4}{\beta^2}\right)\frac{1}{n^2} + O\left(\frac{1}{n^4}\right), as \ n \to \infty,$$

respectively. ∎

Additionally, the length of the gaps $\mathcal{G}_n$ is calculated as the following

$$|\mathcal{G}_n| = A_{n+1}^2 - B_n^2 = \pi^2(2n + 1) - \frac{8}{\beta} + O\left(\frac{1}{n^2}\right).$$

Implies that

$$\frac{|\Delta_n|}{|\mathcal{G}_n|} = \frac{4}{\pi^2|\gamma|n} + O\left(\frac{1}{n^2}\right), \qquad as \ n \to \infty. \tag{3.9}$$

As a result, we conclude that the bands to gaps ratio tends to zero at high energies.

Proposition 3.2. Assume that $\beta = 0$, $t > 2$, and γ is an arbitrary, then there are infinite numbers of bands $\Delta_n = [A_n^2, B_n^2]$ of the operator $\mathcal{L}$, which has a continuous spectrum and located in the intervals $I_n = [\pi^2 n^2, \pi^2(n + 1)^2]$ for large values of n. And their edges are asymptotically which are given by

$$A_n = \pi n + \cos^{-1}\frac{2}{t} + \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right), \quad as \ n \to \infty,$$
$$\tag{3.10}$$
$$B_n = \pi(n + 1) - \cos^{-1}\frac{2}{t} + \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right), \quad as \ n \to \infty.$$

In addition, the length and the midpoint of the band are asymptotically which given by:

$$|\Delta_n| = 2\pi\left(\pi - 2\cos^{-1}\frac{2}{t}\right)n + \left(\pi^2 - 2\pi\cos^{-1}\frac{2}{t}\right) + O\left(\frac{1}{n}\right), as \ n \to \infty, \tag{3.11}$$

and

$$M_n = \pi^2(n + \frac{1}{2})^2 + (\cos^{-1}\frac{2}{t} - \frac{\pi}{2})^2 + \frac{2\gamma}{t} + O\left(\frac{1}{n}\right), as \ n \tag{3.12}$$

respectively.

Proof. At first, let us to prove that there is only one band Δ_n of continuous spectrum in each interval I_n for the large enough values of κ.

Now, since $\beta = 0$, and by the equation (2.4) we get

$$g(\kappa) = t\cos\kappa + \frac{\gamma}{\kappa}\sin\kappa, \tag{3.13}$$

and

$$g(n\pi) = t\cos n\pi + \frac{\gamma}{n\pi}\sin n\pi = (-1)^n t.$$

Since the function $g(\kappa)$ is continuous and $g(n\pi)$ has alternating signs, moreover, when n is sufficiently large, $|g(n\pi)| > 2$, then we conclude that there is only one spectral band in each interval.

The zeroes of $g'(\kappa)$ we get

$$0 = g'(\kappa) = -t\sin\kappa + \frac{\gamma}{\kappa}\cos\kappa - \frac{\gamma}{\kappa^2}\sin\kappa.$$

Impels that the equation for extreme points is given by

$$\tan\kappa = \frac{\gamma\kappa}{\kappa^2 t + \gamma},$$

and because this function is decreasing if κ is sufficiently large, then there is only one solution in each interval.

Note that if $\gamma = 0$, then $g(\kappa) = t\cos\kappa$. Also, since $g(\kappa) = (-1)^n t$, $t = 2$, then $t\cos\kappa = 2$. Consequently,

$$\kappa = \cos^{-1}\frac{2}{t} + n\pi.$$

Hence, there is one extreme point in each interval I_n for the function g when $n \to \infty$.

Consequently, because the function g is continuous and monotonically between these points, therefore, for n is sufficiently large, there is only one band where $|g(\kappa)| \quad 2$ in each interval I_n.

Now, when $t > 2$ then $\cos^{-1}\frac{2}{t}$ satisfies

$$0 < \cos^{-1}\frac{2}{t} < \pi/2.$$

On the other hand, due to the A_n and B_n points are closed to $\pi n + \cos^{-1}\frac{2}{t}$ and $\pi(n+1) - \cos^{-1}\frac{2}{t}$ respectively, then let us to use the following representation of the asymptotic

$$A_n = n\pi + \cos^{-1}\frac{2}{t} + a_n \quad , \quad B_n = (n+1)\pi - \cos^{-1}\frac{2}{t} + b_n,$$

where a_n, b_n are real constant.

The equation for the left end point,

$$(-1)^n 2 = (-1)^n t[(\frac{2}{t}\cos a_n - \sin(\cos^{-1}\frac{2}{t})\sin a_n] + \frac{\gamma}{n\pi + \cos^{-1}(2/t)}$$
$$[(\sin(\cos^{-1}\frac{2}{t})\cos a_n) + \frac{2}{t}\sin a_n)].$$

By using the perturbation theory to keep the first terms, we get

$$a_n = \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right), \quad as\ n \to \infty,$$

thus

$$A_n = n\pi + \cos^{-1}\frac{2}{t} + \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right) \quad as\ n \to \infty.$$

By the same way we can prove the representation for B_n, i.e.

$$B_n = (n+1)\pi - \cos^{-1}\frac{2}{t} + \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right), \quad as\ n \to \infty.$$

Furthermore,

$$|\ _n| = 2\pi\left(\pi - 2\cos^{-1}\frac{2}{t}\right)n + \left(\pi^2 - 2\pi\cos^{-1}\frac{2}{t}\right) + O\left(\frac{1}{n}\right), \quad as\ n \to \infty,$$

and

$$M_n = \pi^2(n+\frac{1}{2})^2 + (\cos^{-1}\frac{2}{t} - \frac{\pi}{2})^2 + \frac{2\gamma}{t} + O(\frac{1}{n}), \quad as\ n \to \infty,$$

respectively.

In addition, the length of the gaps $\mathcal{G}_n$ is calculated as the following

$$|\mathcal{G}_n| = 4 \cdot \cos^{-1}\frac{2}{t}(n+1) + O\left(\frac{1}{n}\right) \text{ as } n \to \infty .$$

Implies that

$$\frac{|\Delta_n|}{|\mathcal{G}_n|} = \frac{\pi/2 - 2\cos^{-1}\frac{2}{t}}{\cos^{-1}\frac{2}{t}} + O\left(\frac{1}{n}\right) \text{ as } n \to \infty . \tag{3.14}$$

As a result, we conclude that the bands to gaps ratio tends to the finite non-zero limit depending on the parameter t only at high energies.

Proposition 3.3. Assume that $\beta = 0$, $t = 2$, and $\gamma \neq 0$; then there are infinite numbers of bands $\mathcal{A}_n = [A_n^2, B_n^2]$ of the operator $\mathcal{L}$, which has a continuous spectrum and located in the intervals $I_n = [\pi^2 n^2, \pi^2 (n+1)^2]$. And their edges are asymptotically which given by

if $\gamma > 0$, then $A_n = \pi n + \frac{\gamma}{n\pi} + O\left(\frac{1}{n^2}\right), \quad B_n = \pi(n+1), \text{ as } n \to \infty,$ (3.15)

if $\gamma < 0$, then $A_n = \pi n, \ B_n = \pi(n+1) - \frac{|\gamma|}{\pi n} + O\left(\frac{1}{n}\right), \text{ as } n \to \infty$ (3.16)

In addition, the length and the midpoint of the band are asymptotically which given by:

$$|\Delta_n| = 2\pi^2 n + (\pi^2 - 2|\gamma|) + O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty, \tag{3.17}$$

and

$$M_n = \pi^2 n^2 + \pi^2 n + \frac{\pi^2}{2} + \gamma + O\left(\frac{1}{n}\right), \text{ as } n \to \infty, \tag{3.18}$$

respectively.

Proof. By using the similar way which used in the previous two propositions we can prove this proposition.

Furthermore, the length of the gaps $\mathcal{G}_n$ is calculated as the following

$$|\mathcal{G}_n| = 2|\gamma| + O\left(\frac{1}{n}\right) \text{ as } n \to \infty .$$

Implies that

$$\frac{|\Delta_n|}{|\mathcal{G}_n|} = \frac{\pi^2}{|\gamma|}n + O\left(\frac{1}{n}\right) \text{ as } n \to \infty . \tag{3.19}$$

As a result, we conclude that the bands to gaps ratio tends to infinity at high energies.

4. Conclusions

As mentioned in the introduction, the goal of this study was to describe a spectral asymptotic of the second derivative operator corresponding to the values of three independent real parameters. We first used the transfer matrix method to obtain the dispersion relation which allowed to describe the spectrum of this operator. Then, we observed there are three different spectral asymptotics for this operator depending on independent parameters which are described in three propositions. More importantly, we proved analytically that there are infinite numbers of bands of this operator $\mathcal{L}$ filled with a pure absolutely continuous spectrum. Furthermore, we proved analytically that the bands to gaps ratio tends to zero at particular case when $\beta \neq 0$.

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