

Shifting, Shrinking and Stretching:

Shift formulas: (for $c > 0$)

Vertical shifts

$y = f(x) + c$	or	$y - c = f(x)$	shifts the graph of f up by c units.
$y = f(x) - c$	or	$y + c = f(x)$	shifts the graph of f down by c units.

Horizontal shifts

$y = f(x+c)$	shifts the graph of f left by c units.
$y = f(x-c)$	shifts the graph of f right by c units.

Shrinking, Stretching and Reflecting Formulas:

(for $c > 1$)

$y = c f(x)$ Stretches the graph of f \underline{c} units along y -axis.

$y = \frac{1}{c} f(x)$ Shrinks the graph of f \underline{c} units along y -axis.

$y = f(cx)$ Shrinks the graph of f \underline{c} units along x -axis.

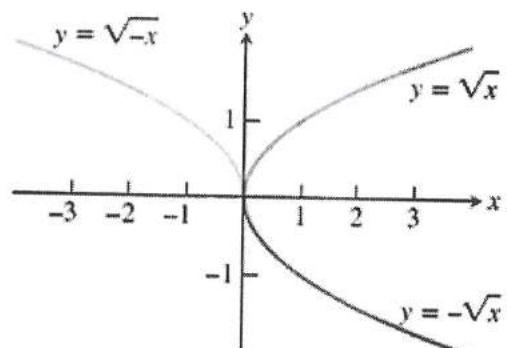
$y = f\left(\frac{x}{c}\right)$ Stretches the graph of f \underline{c} units along x -axis.

(for $c = -1$)

$y = -f(x)$ Reflects the graph of f across the x -axis.

$y = f(-x)$ Reflects the graph of f across the y -axis.

Example 1: The graph of $y = -\sqrt{-x}$ is a reflection of $y = \sqrt{x}$ across the x -axis, and $y = \sqrt{-x}$ is a reflection across the y -axis.



Example 2: Shift the graph of the function

$$f(x) = x^2 ; \text{ if } D_f = \{x: -2 \leq x \leq 3\} \text{ and } R_g = \{y: 0 \leq y \leq 9\}.$$

(a) one unit right. (b) two units left.

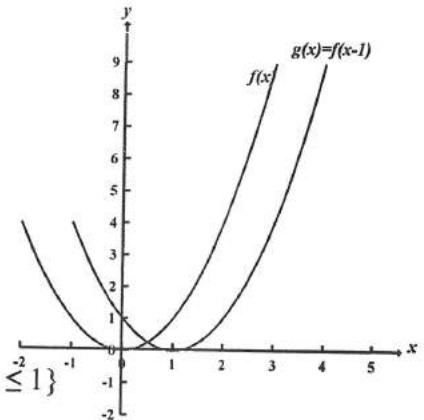
(c) one unit up. (d) two units down.

Sol.: (a) Shifting the function $f(x)$ one unit right:

$$g(x) = f(x-1) = (x-1)^2 \text{ and } D_g = \{x: -2 \leq x-1 \leq 3\} = \{x: -1 \leq x \leq 4\}$$

Note: In case of horizontal shifts, the range of the function will not be changed.

x	$y=f(x)=x^2$	$x-1$	$y=g(x)=(x-1)^2$
-2	4	-	-
-1	1	-2	4
0	0	-1	1
1	1	0	0
2	4	1	1
3	9	2	4
4	-	3	9

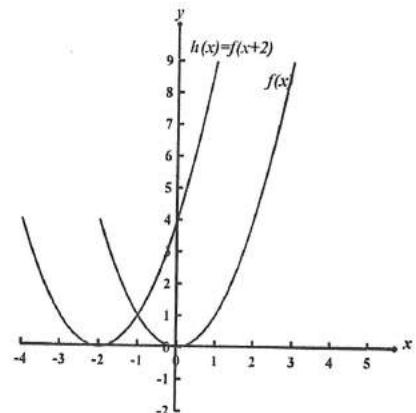


(b) Shifting the function $f(x)$ two units left:

$$h(x) = f(x+2) = (x+2)^2 \text{ and } D_h = \{x: -2 \leq x+2 \leq 3\} = \{x: -4 \leq x \leq 1\}$$

Note: In case of horizontal shifts, the range of the function will not be changed.

X	$y=f(x)=x^2$	$x+2$	$y=h(x)=(x+2)^2$
-4	-	-2	4
-3	-	-1	1
-2	4	0	0
-1	1	1	1
0	0	2	4
1	1	3	9
2	4	-	-
3	9	-	-

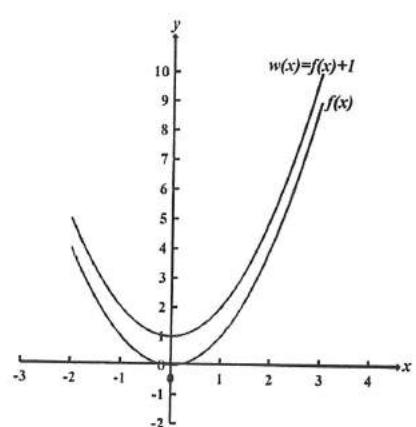


(c) Shifting the function $f(x)$ one unit up:

$$w(x) = f(x)+1 = x^2 + 1 \text{ and } R_w = \{y: 0 \leq y-1 \leq 9\} = \{y: 1 \leq y \leq 10\}$$

Note: In case of vertical shifts, the domain of the function will not be changed.

X	$y=f(x)=x^2$	$y=w(x)=x^2+1$
-2	4	5
-1	1	2
0	0	1
1	1	2
2	4	5
3	9	10

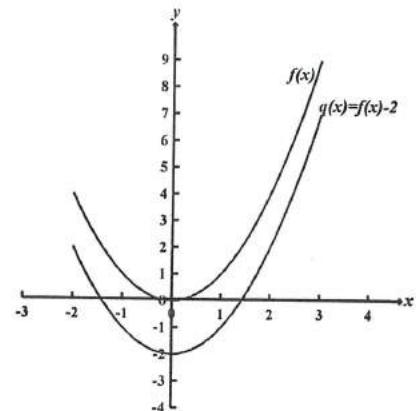


(d) Shifting the function $f(x)$ two units down:

$$q(x) = f(x) - 2 = x^2 - 2 \text{ and } R_q = \{y: 0 \leq y+2 \leq 9\} = \{y: -2 \leq y \leq 7\}$$

Note: In case of vertical shifts, the domain of the function will not be changed.

X	$y=f(x)=x^2$	$y=q(x)=x^2 - 2$
-2	4	2
-1	1	-1
0	0	-2
1	1	-1
2	4	2
3	9	7



Example 3: Sketch the graph of the curve $y=f(x)=|x|$

Sol.: Step1: Find D_f, R_f of the function?

$$y = f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\Rightarrow D_f = (-\infty, \infty) \quad \text{and} \quad R_f = [0, \infty);$$

Step2: Find x and y intercept?

To find x-intercept put $y=0 \Rightarrow x=0$

To find y-intercept put $x=0 \Rightarrow y=0$

So x- and y-intercept is $(0,0)$.

Step 3: check the symmetry:

$$f(-x) = |-x| = |x| = f(x)$$

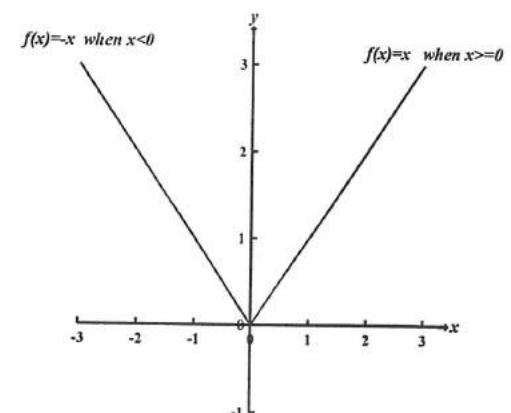
$$-f(x) = -|x| \neq f(x)$$

So it is an even function (it is symmetric about y-axis).

Step 4: Choose some another point on the curve.

x	y
1	1
2	2

Step 5: Draw smooth line through the above points.



Example 4: Use graph of the function $y=|x|$ to sketch the graph of the following functions, then show their domains and range

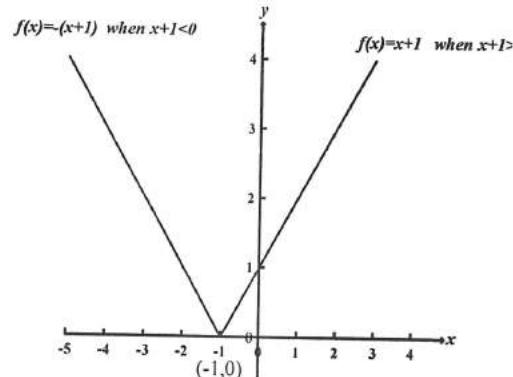
(a) $y=|x+1|$

Sol.

$$y = |x+1| = \begin{cases} (x+1) & \text{if } (x+1) \geq 0 \\ -(x+1) & \text{if } (x+1) < 0 \end{cases}$$

$$= \begin{cases} (x+1) & \text{if } x \geq -1 \\ -x-1 & \text{if } x < -1 \end{cases}$$

Shifting the function $y=|x|$ one unit left.

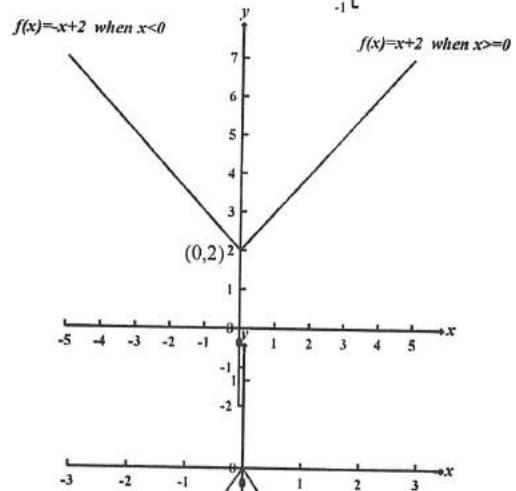


$D_f = (-\infty, \infty)$ and $R_f = [0, \infty)$

(b) $y=|x|+2$

Sol. $y = |x| + 2 = \begin{cases} (x) + 2 & \text{if } (x) \geq 0 \\ (-x) + 2 & \text{if } (x) < 0 \end{cases}$

Shifting the function $y=|x|$ two up.

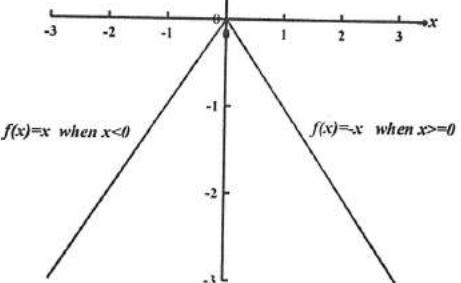


$D_f = (-\infty, \infty)$ and $R_f = [2, \infty)$

(c) $y=-|x|$

Sol. $y = f(x) = -|x| = \begin{cases} -(x) = -x & \text{if } (x) \geq 0 \\ -(-x) = x & \text{if } (x) < 0 \end{cases}$

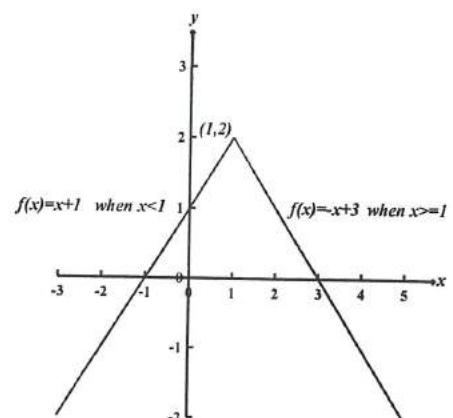
Reflecting the graph of the function $y=|x|$ across x -axis.



(d) $y=2-|1-x|$

Sol. $y=2-|1-x|=-|1-x|+2=-|x-1|+2$

$$= \begin{cases} -(x-1)+2 & \text{if } (x-1) \geq 0 \\ -(-(x-1))+2 & \text{if } x-1 < 0 \end{cases}$$



$$= \begin{cases} -x+3 & \text{if } x \geq 1 \\ x+1 & \text{if } x < 1 \end{cases}$$

Reflecting the graph of the function $y=|x|$ across x -axis, then shifting it one unit right and two units up.

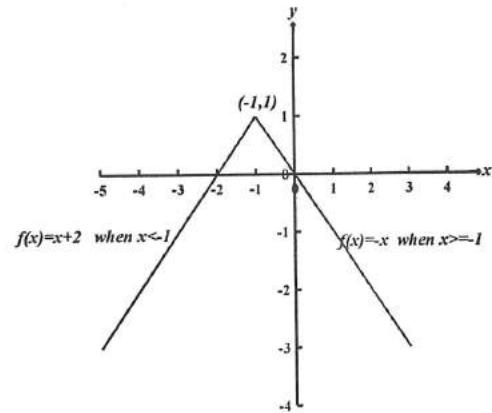
$$D_f=(-\infty, \infty) \text{ and } R_f=(-\infty, 2]$$

$$(e) y=1-|x+1|$$

$$\underline{\text{Sol.}} \quad y=1-|x+1| = -|x+1| + 1$$

$$\begin{aligned} &= \begin{cases} -(x+1)+1 & \text{if } (x+1) \geq 0 \\ -(-(x+1))+1 & \text{if } (x+1) < 0 \end{cases} \\ &= \begin{cases} -x & \text{if } x \geq -1 \\ x+2 & \text{if } x < -1 \end{cases} \end{aligned}$$

Reflecting the graph of the function $y=|x|$ across x -axis, then shifting it one unit left and one unit up.



$$D_f=(-\infty, \infty) \text{ and } R_f=(-\infty, 1]$$

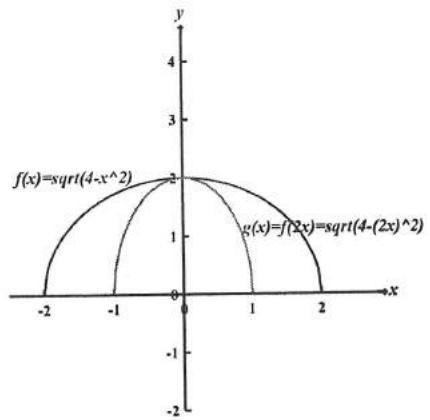
Example 5: If $f(x) = \sqrt{4-x^2}$ which has $D_f=[-2, 2]$ and $R_f=[0, 2]$, shrink and stretch it horizontally by two units and then sketch the original and resulting functions

Sol.: (a) shrinking:

$$g(x) = f(cx) = \sqrt{4-(2x)^2} = \sqrt{4-4x^2} = 2\sqrt{1-x^2}$$

$$D_g = \{x: -2 \leq 2x \leq 2\} = \{x: -1 \leq x \leq 1\}$$

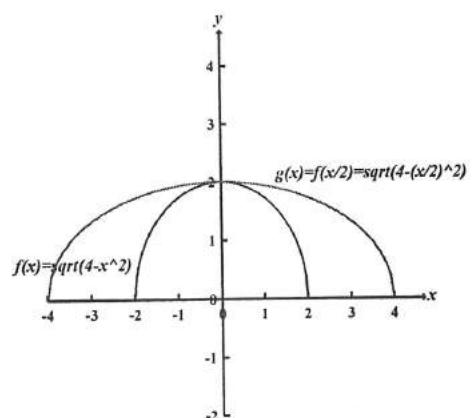
Note: In case of horizontal shrinks, the range of the function will not be changed.



(b) stretching:

$$g(x) = f\left(\frac{x}{c}\right) = \sqrt{4-\left(\frac{x}{2}\right)^2} = \sqrt{4-\frac{x^2}{4}} = \sqrt{\frac{16-x^2}{4}} = \frac{1}{2}\sqrt{16-x^2}$$

$$D_g = \{x: -2 \leq x/2 \leq 2\} = \{x: -4 \leq x \leq 4\}$$



$$(i) \quad y = 3 - \sqrt{x+1}$$

$$(iii) \quad y = \frac{1}{2}\sqrt{x} + 1$$

$$(ii) \quad y = 1 + \sqrt{x-4}$$

$$(iv) \quad y = -\sqrt{3x}$$

(c) The given function $y = \frac{1}{x}$

$$(i) \quad y = \frac{1}{x-3}$$

$$(iii) \quad y = 2 - \frac{1}{x+1}$$

$$(ii) \quad y = \frac{1}{1-x}$$

$$(iv) \quad y = \frac{x-1}{x}$$

(d) The given function $y = |x|$

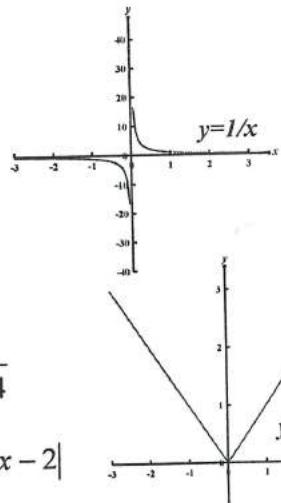
$$(i) \quad y = |x+2|-2$$

$$(iii) \quad y = |2x-1|+2$$

$$(ii) \quad y = 1 - |x-3|$$

$$(iv) \quad y = \sqrt{x^2 - 4x + 4}$$

$$= \sqrt{(x-2)^2} = |x-2|$$



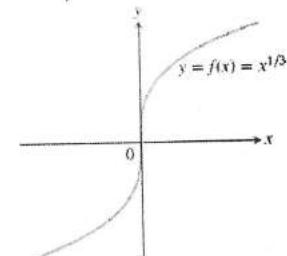
(e) The given function $y = \sqrt[3]{x}$

$$(i) \quad y = 1 - 2\sqrt[3]{x}$$

$$(iii) \quad y = 2 + \sqrt[3]{x+1}$$

$$(ii) \quad y = \sqrt[3]{x-1} - 3$$

$$(iv) \quad y = -\sqrt[3]{x-2}$$



2. Shrink and stretch the following functions along both x -axis and y -axis by $(3/2)$ units then sketch the resulting function.

$$(a) \quad x^2 + y^2 = 4,$$

$$D_f = \{x: -2 \leq x \leq 2\}$$

$$R_f = \{y: -2 \leq y \leq 2\}$$

$$(b) \quad 2x^2 + y^2/2 = 6,$$

$$D_f = \{x: -2 \leq x \leq 3\}$$

$$R_f = \{y: -2 \leq y \leq 2\sqrt{6}\}$$

$$(c) \quad y = 3x^2 - 2x + 1,$$

$$D_f = \{x: -1 \leq x \leq 2\}$$

$$R_f = \{y: \frac{6}{9} \leq y \leq 9\}$$

LIMITS

Limits

Definition:

If the value of $f(x)$ can be made as close as we like to L by taking the value of x sufficiently close to a (but not equal a), then we write:

$$\lim_{x \rightarrow a} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches a is L ".

Properties of limits:

1. If $f(x) = k$, then $\lim_{x \rightarrow a} f(x) = k$, where a and k are real numbers.

2. Sum rule: $\lim_{x \rightarrow a} [f_1(x) + f_2(x)] = \lim_{x \rightarrow a} f_1(x) + \lim_{x \rightarrow a} f_2(x).$

3. Difference rule: $\lim_{x \rightarrow a} [f_1(x) - f_2(x)] = \lim_{x \rightarrow a} f_1(x) - \lim_{x \rightarrow a} f_2(x).$

4. Product rule: $\lim_{x \rightarrow a} [f_1(x) \cdot f_2(x)] = \lim_{x \rightarrow a} f_1(x) \cdot \lim_{x \rightarrow a} f_2(x).$

5. Constant multiple rule: $\lim_{x \rightarrow a} k \cdot f(x) = k \cdot \lim_{x \rightarrow a} f(x)$, where k is a constant.

6. Quotient rule: $\lim_{x \rightarrow a} \frac{f_1(x)}{f_2(x)} = \lim_{x \rightarrow a} f_1(x) / \lim_{x \rightarrow a} f_2(x)$, $\lim_{x \rightarrow a} f_2(x) \neq 0$.

7. Power rule: $\lim_{x \rightarrow a} [f(x)]^{r/s} = [\lim_{x \rightarrow a} f(x)]^{r/s}$, provided that $\lim_{x \rightarrow a} f(x)$ is a real number (if s is even, we assume $\lim_{x \rightarrow a} f(x) \geq 0$).

* Polynomials: $\lim_{x \rightarrow a} (c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n) = c_0 + c_1 a + c_2 a^2 + \dots + c_n a^n$.

* $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

* Sandwich theorem:

If $g(x) \leq f(x) \leq h(x)$ are three functions such that:

$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} f(x) = L$.

Note: Indeterminate quantities: $(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 * \infty)$.

Example: Find the limits of the following:

$$1. \lim_{x \rightarrow 2} x^2 - 4x = 2^2 - 4 * 2 = 4 - 8 = -4.$$

$$2. \lim_{x \rightarrow 1} x^3 + 2x^2 - 3x + 4 = 1^3 + 2 * 1^2 - 3 * 1 + 4 = 4.$$

$$3. \lim_{x \rightarrow 1} \frac{(3x-1)^2}{(x+1)^3} = \frac{(3*1-1)^2}{(1+1)^3} = \frac{2^2}{2^3} = \frac{1}{2}.$$

$$4. \lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 5x + 6} = \frac{2^2 - 4}{2^2 - 5*2 + 6} = \frac{0}{0}$$
 (Indeterminate quantities)

$$\text{So } \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x \rightarrow 2} \frac{(x+2)}{(x-3)} = \frac{2+2}{2-3} = \frac{4}{-1} = -4.$$

$$5. \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x^2-4}} = \frac{2-2}{\sqrt{2^2-4}} = \frac{0}{0} \quad (\text{Indeterminate quantities})$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{x-2}\sqrt{x-2}}{\sqrt{(x-2)(x+2)}} = \lim_{x \rightarrow 2} \frac{\sqrt{x-2}\sqrt{x-2}}{\sqrt{x-2}\sqrt{x+2}} = \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{\sqrt{x+2}} = \frac{\sqrt{2-2}}{\sqrt{2+2}} = \frac{0}{\sqrt{4}} = 0$$

$$6. \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{x^2-4} = \frac{\sqrt{2-2}}{2^2-4} = \frac{0}{0} \quad (\text{Indeterminate quantities})$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{\sqrt{x-2}\sqrt{x-2}(x+2)}$$

$$= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x-2}(x+2)} = \frac{1}{\sqrt{2-2}(2+2)} = \frac{1}{0*4} = \frac{1}{0} = \infty \Rightarrow \text{the limit does not exist.}$$

$$7. \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2+3}-2} = \frac{1-1}{\sqrt{1^2+3}-2} = \frac{0}{0} \quad (\text{Indeterminate quantities})$$

$$= \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2+3}-2} * \frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} \quad (\text{Multiplying both the numerator and denominator by the conjugate factor})$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{x^2+3-4} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{x^2-1} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{(x-1)(x+1)}$$

$$= \lim_{x \rightarrow 1} \frac{(\sqrt{x^2+3}+2)}{(x+1)} = \frac{\sqrt{1^2+3}+2}{x+1} = \frac{4}{2} = 2.$$

$$8. \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{x} * \frac{3}{3} = 3 * \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} = 3 * 1 = 3$$

$$9. \lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sin x/\cos x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} * \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1 * \frac{1}{1} = 1$$

$$10. \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2}-x} \quad \text{Let } z = \frac{\pi}{2}-x, \quad \text{so as } x \rightarrow \frac{\pi}{2} \Rightarrow z \rightarrow 0$$

$$\therefore \lim_{z \rightarrow 0} \frac{\cos(\frac{\pi}{2}-z)}{z} = \lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$$

$$11. \lim_{x \rightarrow 0} \frac{1-\cos x}{x} = \lim_{x \rightarrow 0} \frac{1-\cos x}{x} * \frac{1+\cos x}{1+\cos x} = \lim_{x \rightarrow 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} * \lim_{x \rightarrow 0} \frac{\sin x}{(1+\cos x)} = 1 * \frac{0}{1+1} = 0$$

Example: Given that $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$, find $\lim_{x \rightarrow 0} u(x)$.

Sol.: Since $\lim_{x \rightarrow 0} 1 - \frac{x^2}{4} = 1$ and $\lim_{x \rightarrow 0} 1 + \frac{x^2}{2} = 1$, then

the Sandwich Theorem implies that $\lim_{x \rightarrow 0} u(x) = 1$.

Right-hand limits and left-hand limits

One sided vs. two sided limits

Definition:

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbol:

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^+} f(x) = L \text{ and } \lim_{x \rightarrow c^-} f(x) = L$$

Example: Discuss the limit properties of the function $f(x)$ which shown in figure.

Sol.:

- At $x = 0$ $\lim_{x \rightarrow 0^+} f(x) = 1$

$\lim_{x \rightarrow 0^-} f(x)$ does not exist (because the function is not defined to the left of $x = 0$)

- At $x = 1$ $\lim_{x \rightarrow 1^-} f(x) = 0$ even though $f(1) = 1$

$\lim_{x \rightarrow 1^+} f(x) = 1$

$\lim_{x \rightarrow 1} f(x)$ does not exist, because the right-hand and left-hand limits are not equal.

- At $x = 2$ $\lim_{x \rightarrow 2^-} f(x) = 1$

$\lim_{x \rightarrow 2^+} f(x) = 1$

$\lim_{x \rightarrow 2} f(x) = 1$ even though $f(2) = 2$

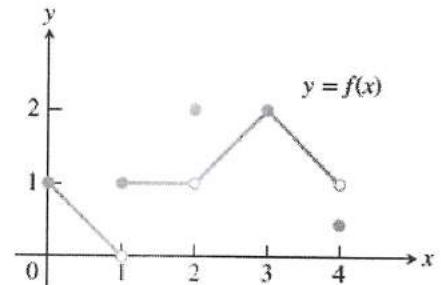
- At $x = 3$ $\lim_{x \rightarrow 3^-} f(x) = 2$

$\lim_{x \rightarrow 3^+} f(x) = 2$

$\lim_{x \rightarrow 3} f(x) = f(3) = 2$

- At $x = 4$ $\lim_{x \rightarrow 4^-} f(x) = 1$ even though $f(4) = 0.5$

$\lim_{x \rightarrow 4^+} f(x)$ does not exist, because the function is not defined to the right of $x = 4$.



Example: Check the existence of the limit of the function $f(x)$ at $x = 1$,

$$f(x) = \begin{cases} 2x + 1 & -1 < x < 1 \\ x^2/2 - 3 & 1 < x < 4 \end{cases}$$

Sol.: $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2x + 1 = 3.$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2/2 - 3 = -2.5.$$

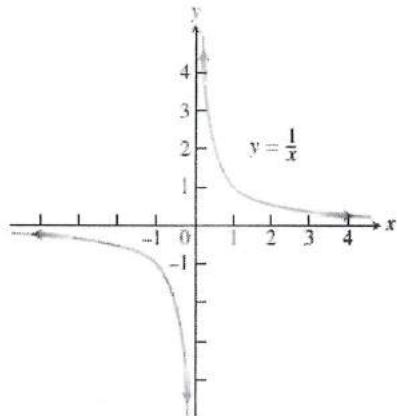
Since the right-hand and left-hand limits are not equal, thus the limit does not exist at $x = 1$.

Limits Involving Infinity:

These are the limits that include $x \rightarrow \infty$ or $x \rightarrow -\infty$ and $\lim f(x) = \infty$ or $\lim f(x) = -\infty$.

Let $y = \frac{1}{x}$ then

1. $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$
 2. $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$
 3. $\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$
 4. $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$
- $\left. \begin{array}{l} \text{One-sided limits} \Rightarrow \text{the} \\ \text{limit does not exit} \end{array} \right\}$



Example: Find the limits of the following:

$$1. \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x}\right) = \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} = 5 + 0 = 5$$

$$2. \lim_{x \rightarrow \infty} \frac{x}{7x+4} = \lim_{x \rightarrow \infty} \frac{x/x}{7x/x+4/x} = \lim_{x \rightarrow \infty} \frac{1}{7+4/x} = \frac{1}{7+0} = \frac{1}{7}$$

Note: In rational functions divide both the numerator and denominator by the largest power of x in the denominator.

$$3. \lim_{x \rightarrow \infty} \frac{2x^2 - x + 3}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2x^2/x^2 - x/x^2 + 3/x^2}{3x^2/x^2 + 5/x^2} = \lim_{x \rightarrow \infty} \frac{2 - 1/x + 3/x^2}{3 + 5/x^2} = \frac{2 - 0 + 0}{3 + 0} = \frac{2}{3}$$

$$4. \lim_{x \rightarrow \infty} \frac{4x^2 - 3}{3x} = \lim_{x \rightarrow \infty} \frac{4x^2/x - 3/x}{3x/x} = \lim_{x \rightarrow \infty} \frac{4x - 3/x}{3} = \frac{4 * \infty - 0}{3} = \infty \Rightarrow \text{the limit does not exist.}$$

$$5. \lim_{x \rightarrow \infty} \frac{5x+3}{2x^2-1} = \lim_{x \rightarrow \infty} \frac{5x/x^2 + 3/x^2}{2x^2/x^2 - 1/x^2} = \lim_{x \rightarrow \infty} \frac{5/x+3/x^2}{2-1/x^2} = \frac{0-0}{2-0} = \frac{0}{2} = 0.$$

Summary for Rational Functions

a) $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = 0$ if $\deg(f) < \deg(g)$

b) $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)}$ is finite if $\deg(f) = \deg(g)$

c) $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)}$ is infinite if $\deg(f) > \deg(g)$

$$6. a \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} \text{ (since } \sqrt{x^2} = |x|).$$

As $x \rightarrow +\infty$, the values of x under consideration are positive, so we can replace $|x|$ by x :

$$\Rightarrow \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/x} = \lim_{x \rightarrow +\infty} \frac{\sqrt{1+2/x^2}}{(3-6/x)} = \frac{1}{3}.$$

$$.b \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} \text{ (since } \sqrt{x^2} = |x|).$$

As $x \rightarrow -\infty$, the values of x under consideration are negative, so we can replace $|x|$ by $-x$:

$$\Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/(-x)} = \lim_{x \rightarrow -\infty} \frac{\sqrt{1+2/x^2}}{(-3+6/x)} = -\frac{1}{3}.$$

$$7. \lim_{x \rightarrow \infty} \frac{\sin x}{x}$$

Remember that $-1 \leq \sin x \leq 1$. Dividing the inequality by x yields

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} -\frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0 \quad (\text{Sandwich theorem})$$

$$8. \lim_{x \rightarrow \infty} x \sin \frac{1}{x}$$

$$\text{Let } x = \frac{1}{z} \Rightarrow z = \frac{1}{x}$$

$$\text{When } x \rightarrow \infty \Rightarrow z \rightarrow 0$$

$$\therefore \lim_{z \rightarrow 0} \frac{1}{z} \sin z = 1$$

9. $\lim_{x \rightarrow \infty} \sqrt{x^2 + 6x + 1} - \sqrt{x^2 + x}$ $(\infty - \infty)$

$$\begin{aligned} \lim_{x \rightarrow \infty} \sqrt{x^2 + 6x + 1} - \sqrt{x^2 + x} &= \lim_{x \rightarrow \infty} \sqrt{x^2 + 6x + 1} - \sqrt{x^2 + x} * \frac{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}}{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}} \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 + 6x + 1) - (x^2 + x)}{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{x^2 + 6x + 1 - x^2 - x}{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{5x + 1}{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}} \\ &= \lim_{x \rightarrow \infty} \frac{5x/x + 1/x}{\sqrt{x^2/x^2 + 6x/x^2 + 1/x^2} + \sqrt{x^2/x^2 + x/x^2}} = \frac{5}{\sqrt{1+0+0} + \sqrt{1+0}} = \frac{5}{1+1} = \frac{5}{2} = 2.5. \end{aligned}$$