The Fundamental Concepts of Vectors

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2020-2021

Outlines:

1.7 Derivative of a Vector

1.8 Position vector of particle: Velocity and Acceleration in Rectangular Coordinates:

- Examples:

1.7 Derivative of a Vector

Consider a vector A, whose components are function of single variable u . The parameter u is usually time t. The vector may represents position ,velocity and so on.

$$\boxed{A(u) \ = iA_x(u) + jA_y(u) + kA_z(u)}$$

So the derivative of A can be expressed as following

$$\frac{dA}{du} = \lim_{\Delta u \to o} \frac{\Delta A}{\Delta u}$$

$$= \lim_{\Delta u \to o} (i \frac{\Delta A_x}{\Delta u} + j \frac{\Delta A_y}{\Delta u} + k \frac{\Delta A_z}{\Delta u} \ )$$

$$\frac{dA}{du} = \ (i \frac{dA_x}{du} + j \frac{dA_y}{du} + k \frac{dA_z}{du} \ )$$

Ordinary Derivatives.

1.7 Derivative of a Vector

Now, below are the rules of vector differential

$$\frac{d(A+B)}{du} = \frac{dA}{du} + \frac{dA}{du}$$

$$\frac{d(nA)}{du} = \frac{dn}{du}A + n\frac{dA}{du}$$

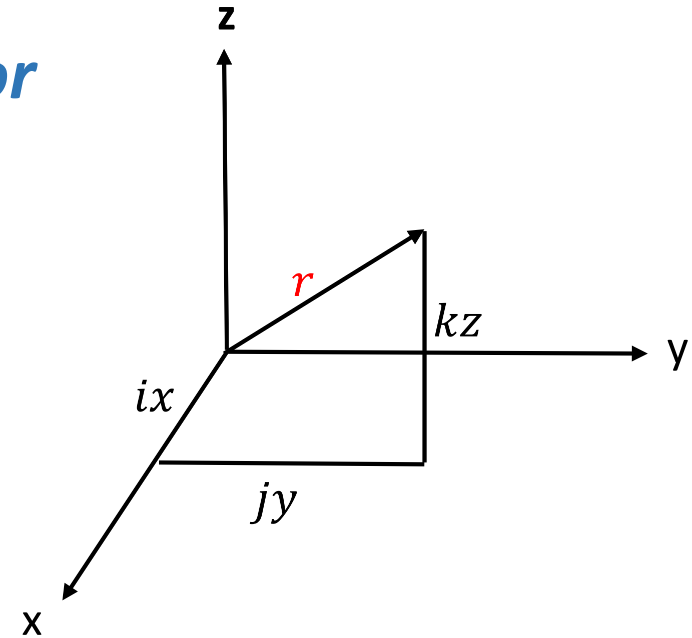
$$\frac{d(A.B)}{du} = \frac{dA}{du}B + A.\frac{dB}{du}$$

$$\frac{d(AxB)}{du} = \frac{dA}{du}xB + Ax\frac{dB}{du}$$

1.8 Position vector of particle: Velocity and Acceleration in Rectangular Coordinates:

$$\boxed{r = ix + jy + kz}$$

Position Vector



$$\boxed{As\ x = x(t), y = y(t)\ and\ z = z(t)}$$

$$\boxed{v = \frac{dr}{dt} = \left(i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}\right)}$$

$$\boxed{\boldsymbol{or}\ \ v = \dot{r} = \dot{x} + \dot{y} + \dot{z}}$$

The velocity value is called **the speed** and defined :

$$v = |v| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

In rectangular components and the second derivative of velocity is called the acceleration

$$a = \dot{v} = \ddot{r} = \ddot{x} + \ddot{y} + \ddot{z}$$

Acceleration

Example1: Projectile Motion

Let us examine the motion represented by the equation

$$r(t) = ibt + j\left(ct - \frac{gt^2}{2}\right) + k0$$

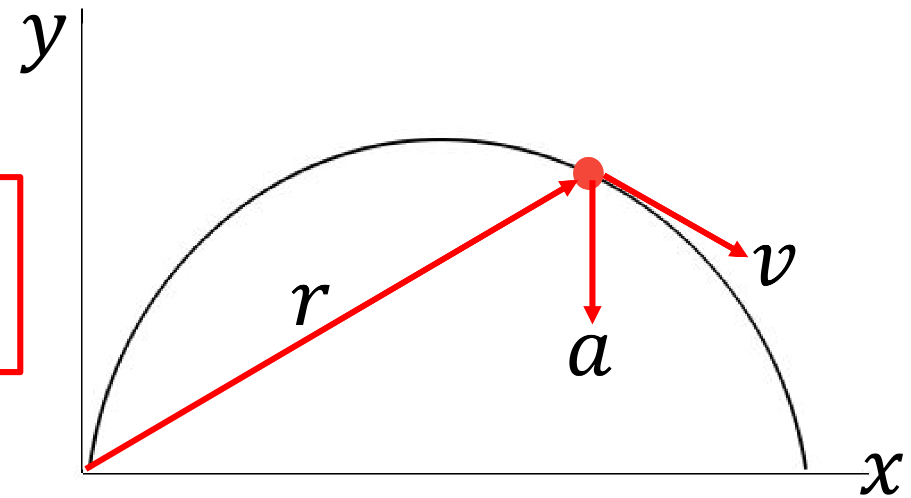
This represents motion xy-plane and the velocity can be obtained by differentiating with respect to (t)

$$v(t) = \frac{dr}{dt} = ib + j(c - gt)$$

$$a(t) = \frac{dv}{dt} = -jg$$

Speed=$v = \sqrt{{v_x}^2 + vy^2}$

$$v = \sqrt{b^2 + (c - gt)^2}$$



Projectile moving in a parabolic path.

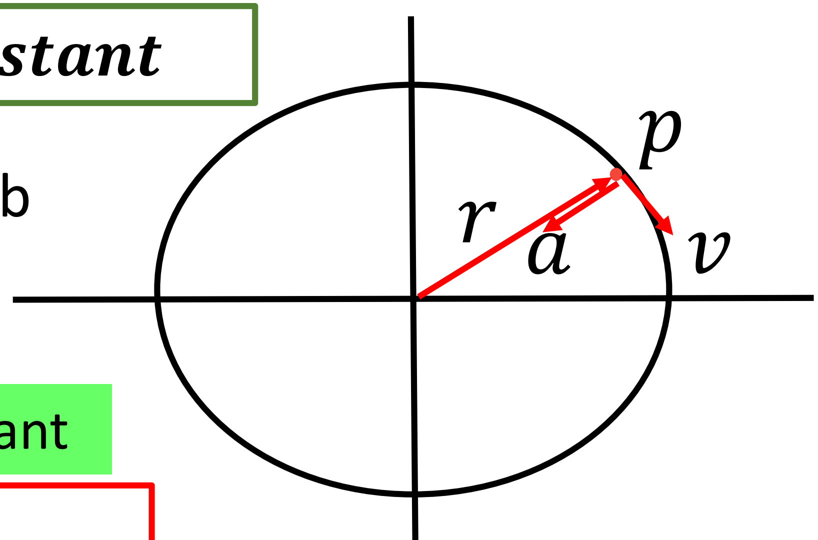
Example2: Circular Motion

Suppose the position vector of a particle is given by

$$r = i\, b \sin \omega t + j\, b\cos \omega t\ ;\ \boldsymbol{\omega}\text{:}\ \boldsymbol{constant}$$

$$|r| = \sqrt{(b \sin \omega t)^2 + (b \cos \omega t\ )^2} = \mathrm{b}$$

The distance from origin remains constant



$$v(t) = \frac{dr}{dt} = i\ b\omega \cos \omega t - j\ b\omega\ sin\ \omega t$$

Velocity Vector

The particle traverses in constant speed

$$\text{Speed} = |v| = \sqrt{(b\omega \cos \omega t)^2 + (b\omega si\,n\, \omega t)^2} = b\omega$$

$$a = \frac{dv}{dt} = -i\ b\omega^2 \sin \omega t - j\ b\omega^2\ cos\ \omega t$$

Acceleration Vector

In this case the (a) is perpendicular to the velocity as $(a.v) = 0$

$$a.v = (-i\ b\omega^2 \sin \omega t - j\ b\omega^2 co\ s\ \omega t).(i\ b\omega \cos \omega t - j\ b\omega\ sin\ \omega t)$$

$$a.v = (-i\ b\omega^2 \sin \omega t).(i\ b\omega \cos \omega t) + (-j\ b\omega^2 co\ s\ \omega t)\ (-j\ b\omega\ sin\ \omega t)$$

$$a.v = -\ b\omega^3 \sin \omega t \cos \omega t + b\omega^3 co\ s\ \omega t\ sin\ \omega t$$

$a.v = 0$ → (a) is perpendicular to the velocity

$$r = i\ b \sin \omega t + j\ b\cos \omega t$$

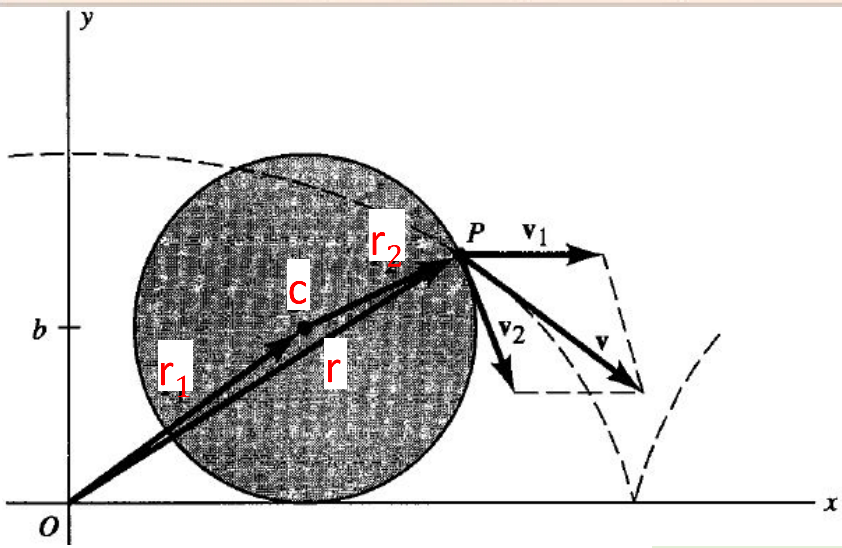
$$a = \frac{dv}{dt} = -\omega^2 (ib \sin \omega t + j\ b\omega^2\ cos\ \omega t)$$

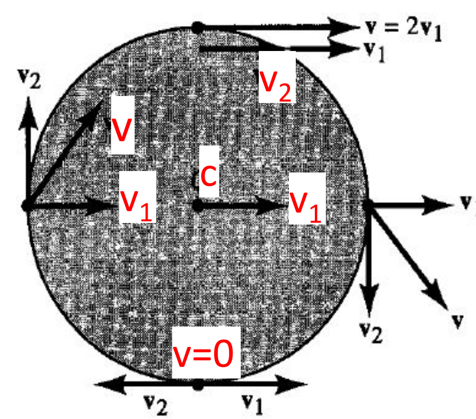
$$a = \frac{dv}{dt} = -\omega^2 r$$

→ So a and r in opposite direction , that is a always points toward the center of the circular path.

Example3: Rolling Wheel

Consider a rolling wheel following position vector $r = r_1 + r_2$ in which $r_1 = ib\omega t + jb$ and $r_2 = ib\sin\omega t + jb\cos\omega t$, where r_1 represents a point moving along the line $y = b$ at constant velocity and r_2 is just the position vector for the circular motion (see example 1.8.2). Find the velocity of the point P .



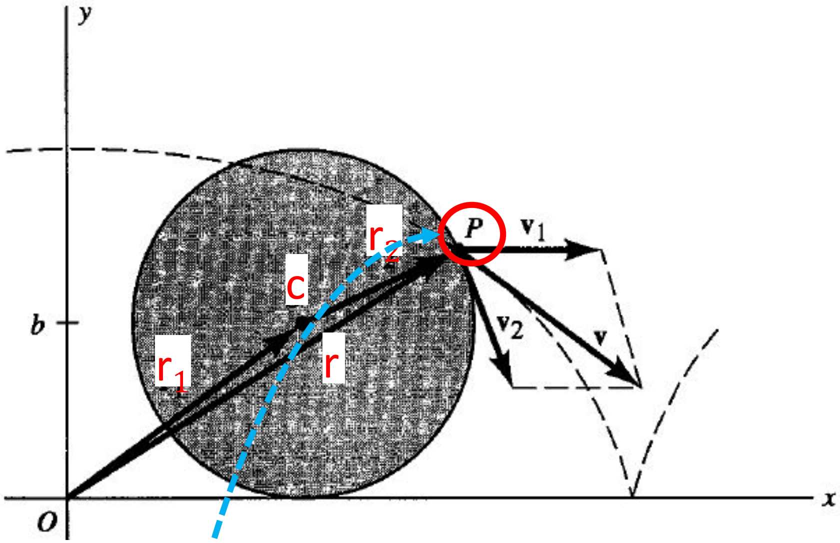


$$r_1 = i\, b \sin\omega t + j\, b$$

r1 represents a point moving along line ($y = b$) at constant velocity ω.

$$r = r_1 + r_2$$

$$r_2 = i\, b \sin\omega t + j\, b\cos\omega t$$



$$r_1 = i\ b\omega t + j\ b$$

$$r_2 = i\ b\sin\omega t + j\ b\cos\omega t$$

$$\boxed{v_1 = \frac{dr_1}{dt} = i\ b\omega}$$

$$\boxed{v_2 = \frac{dr_2}{dt} = i\ b\omega\cos\omega t - j\ b\omega\sin\omega t}$$

$$v = v_1 + v_2 = i\ (b\omega + b\omega\cos\omega t) - j\ b\omega\sin\omega t$$

Velocity vector of P

Home Work

Accelaration vector of P

Home Work:

(a) A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

(b) A bee goes out from its hive in a spiral path given in plane polar coordinates by

$$r = be^{kt} \qquad \theta = ct$$

where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (*Hint: Find* $\mathbf{v} \cdot \mathbf{a}/va$.)

(c) A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \cos \omega t + \mathbf{j}2b \sin \omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and at $t = \pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

Thanks for your attention