The Fundamental Concepts of Vectors

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Outlines:

1.5 Triple Products

1.6 Change of Coordinate System: The Transformation Matrix

1.5 Triple Products

The expression

$$A.(B \times C) \tag{1.36}$$

we can see that the scalar triple product may be written as matrix

$$A.(B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (A \times B).\vec{C} \tag{1.37}$$

H.W: can you prove that ?

Additionally, we can write

$$\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B}) \tag{1.38}$$

which represents the triple cross products .

Example:

Given the three vectors $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$, and $\mathbf{C} = \mathbf{k}$, find $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

Solution:

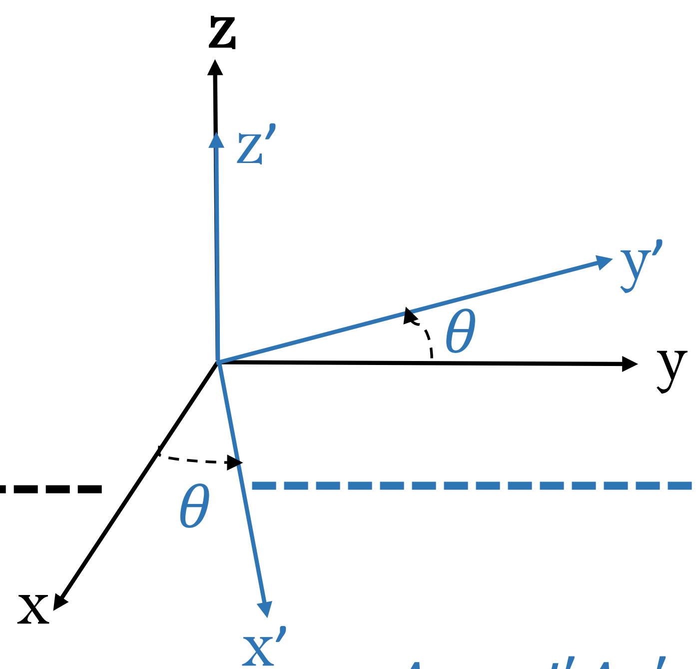
Using the determinant expression, Equation 1.7.1, we have

$$A.(B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(-1+0) = -1$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{i} - \mathbf{j})0 - \mathbf{k}(1-0) = -\mathbf{k}$$

1.6 Change of Coordinate System: The Transformation Matrix



$$A = iAx + jAy + kAz$$

$$A = i'Ax' + j'Ay' + k'Az'$$

How we can transfer between ijk and i′j′k′

Consider the vector A expressed relative to the triad ijk:

$$A = iAx + jAy + kAz$$

Relative to a new triad i'j'k' having a different orientation from that of ijk, the same vector A is expressed as

$$A = i'Ax' + j'Ay' + k'Az'$$

Now the dot product $A.i'$ is just $\boldsymbol{Ax'}$, that is, the projection of $\boldsymbol{A}$ on the unit Vector i'. Now the dot product is just $\boldsymbol{Ax'}$ as

$$A.i' = (i.i')Ax + (j.i')Ay + (k.i')Az = Ax'$$

$$A.j' = (i.j')Ax + (j.j')Ay + (k.j')Az = Ay'$$

$$A.i' = (i.k')Ax + (j.k')Ay + (k.k')Az = Az'$$

In similar way, the unprimed components are similarly expressed

$$A.i = (i'.i)Ax' + (j'.i)Ay' + (k'.i)Az' = Ax$$

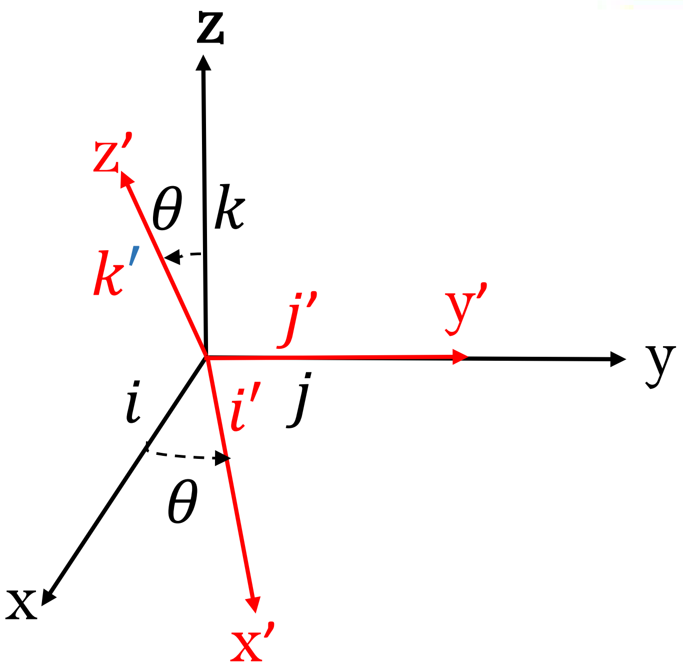
$$A.j = (i'.j)Ax' + (j'.j)Ay' + (k'.j)Az' = Ay$$

$$A.i = (i'.k)Ax' + (j'.k)Ay' + (k'.k)Az' = Az$$

The above two groups of equations can be written as matrix as following:

$$\begin{vmatrix} A_{\acute{x}} \\ A_{\acute{y}} \\ A_{\acute{z}} \end{vmatrix} = \begin{vmatrix} (i.\acute{i}) & (j.\acute{i}) & (k.\acute{i}) \\ (i.\acute{j}) & (j.\acute{j}) & (k.\acute{j}) \\ (i.\acute{k}) & (j.\acute{k}) & (k.\acute{k}) \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix}$$

The above matrix is called a **TRANSFORMATION MATRIX.**



$i.\,i' = (1)(1)\cos(\theta) = \cos(\theta)$

j. $i' = (1)(1)\cos(90) = 0$

k. $i' = (1)(1)\cos(90+\theta)$

=cos(90)cos(θ)-sin(90)sin(θ)

=-sin(θ)

$$\boxed{\begin{pmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{pmatrix}}$$

H. W.

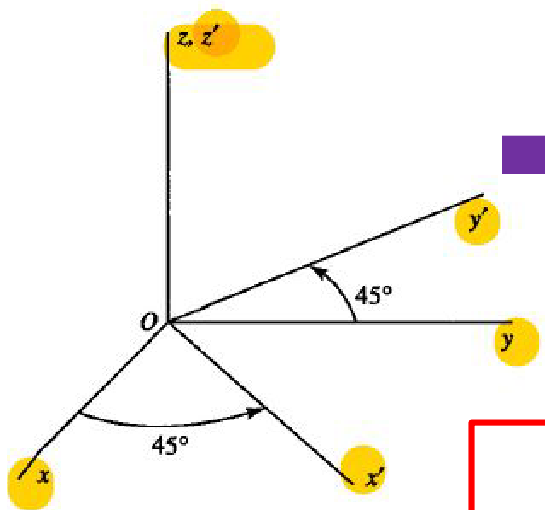
1- Prove that the transformation matrix around z-axis equal to

$$\boxed{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

2- Prove that the transformation matrix around x-axis equal to

EXAMPLE 1.8.1

Express the vector $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ in terms of the triad $\mathbf{i'j'k'}$, where the $x'y'$–axes are rotated 45° around the z-axis, with the z- and z'-axes coinciding, as shown in Figure 1.8.1. Referring to the figure, we have for the coefficients of transformation $\mathbf{i}\cdot\mathbf{i'} = \cos 45°$ and so on; hence,



$$\begin{vmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{vmatrix} = \begin{vmatrix} (i.i') & (j.i') & (k.i') \\ (i.j') & (j.j') & (k.j') \\ (i.k') & (j.k') & (k.k') \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix}$$

$$i.i' = 1 * 1 * \cos 45 = 1/\sqrt{2}$$

$$\begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{j}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{i}' = 0 \\ \mathbf{i}\cdot\mathbf{j}' = -1/\sqrt{2} & \mathbf{j}\cdot\mathbf{j}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{j}' = 0 \\ \mathbf{i}\cdot\mathbf{k}' = 0 & \mathbf{j}\cdot\mathbf{k}' = 0 & \mathbf{k}\cdot\mathbf{k}' = 1 \end{array}$$

$$\begin{pmatrix} Ax' \\ Ay' \\ Az \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} Ax' \\ Ay' \\ Az' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad A_{y'} = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad A_{z'} = 1$$

so that, in the primed system, the vector **A** is given by

$$\mathbf{A} = \frac{5}{\sqrt{2}}\mathbf{i}' - \frac{1}{\sqrt{2}}\mathbf{j}' + \mathbf{k}'$$

Thanks for your attention

1.7 Derivative of a Vector

- Consider a vector A, whose components are function of single variable u . The parameter u is usually time t. The vector may represents position ,velocity and so on.
- Hence $A(u) \; = iA_x(u) + jA_y(u) + kA_z(u)$
- So the derivative of A can be expressed as following:
- $\frac{dA}{du} = \lim_{\Delta u \to o} \frac{\Delta A}{\Delta u} =$

$$= \lim_{\Delta u \to o} (i \frac{\Delta A_x}{\Delta u} + j \frac{\Delta A_y}{\Delta u} + k \frac{\Delta A_z}{\Delta u} \;)$$

$$\frac{dA}{du} = \; (i \frac{dA_x}{du} + j \frac{dA_y}{du} + k \frac{dA_z}{du} \;)$$

- This mean the derivative of vector is a vector whose Cartesian components are ordinary derivatives.
- Now, below are the rules of vector differential.

$$\frac{d(A+B)}{du} = \frac{dA}{du} + \frac{dA}{du}$$

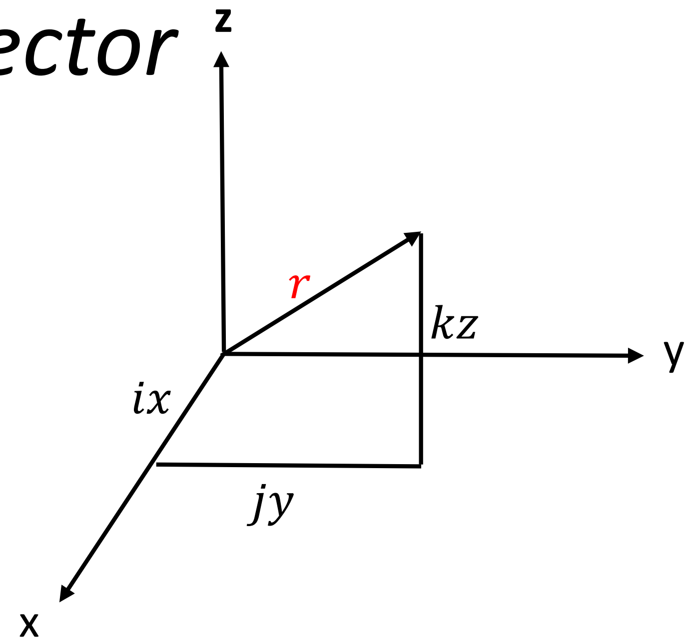
$$\frac{d(nA)}{du} = \frac{dn}{du}A + n\frac{dA}{du}$$

$$\frac{d(A.B)}{du} = \frac{dA}{du}B + A.\frac{dB}{du}$$

$$\frac{d(AxB)}{du} = \frac{dA}{du}xB + Ax\frac{dB}{du}$$

1.8 Position vector of particle: Velocity and Acceleration in Rectangular Coordinates:

r = ix + jy + kz ---Position Vector



As x=x(t),y= y(t) and z=z(t) are the components of the position vector of moving particle . So , the particle velocity vector can be written as the following :

$$v = \frac{dr}{dx} = \left(i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}\right)$$
$$\boldsymbol{or} \;\; v = \dot{r} + \dot{x} + \dot{y} + \dot{z}$$

The velocity value is called **the speed** and defined :

$$v = |v| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

In rectangular components and the second derivative of velocity is called the acceleration

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \frac{d\dot{r}}{dx} = \ddot{\boldsymbol{x}} + \ddot{\boldsymbol{y}} + \ddot{\boldsymbol{z}}$$

1.2.3 Multiplication by a Scalar

If c is a scalar and A is a vector, then:

$$\boxed{cA = c(Ax + Ay + Az) = cAx + cAy + cAz = A\,c}$$

1.2.4 Vectors Subtraction

$$\boxed{A - B = (Ax - Bx) + (Ay - By) + (Az - Bz)}$$

1.2.6 The Commutative Law of Addition

This law holds for vectors; that is,

$$\boxed{\mathbf{A + B = B + A}}$$

Because $A_x + B_x = B_x + A_x$ and similarly for the y and z components.

1.2.7 The Associative Law

$$A + (B + C) = (A_x + (B_x + C_x)\ A_y + (B_y + C_y)\ A_z + (B_z + C_z))$$

$$= ((A_x + B_x) + C_x\ ;\ (A_x + B_x) + C_x\ ;\ (A_x + B_x) + C_x)$$

$$= (A + B) + C$$

1.2.8 The Distributive Law

$$c(A + B) = c\ (Ax + Bx;\ Ay + By;\ Az + Bz)$$

$$= c\ (Ax + Bx);\ c(Ay + By);\ c(Az + Bz)$$

$$= c\,A + c\,B$$

1.2.9 Magnitude of a Vector

$$A = |A| = (A^2x + A^2y + A^2z\ )^{(1/2)}$$

Example

A helicopter flies 100 m vertically upward, then 500 m horizontally east, then 1000 m horizontally north. How far is it from a second helicopter that started from the same point and flew 200 m upward, 100 m west, and 500 m north?

Solution:

Choosing up, east, and north as basis directions, the final position of the first helicopter is expressed vertically as A = (100,500,1000) and the second as B =(200,-100, 500), in meters. Hence, the distance between the final positions is given by the expression

|A – B| = ((100 -200); (500 + 100); (1000-500)) = 787.4m

1.3 Scalar Product

Given two vectors A and B, the scalar product or "dot" product, A.B, is the scalar defined by the equation

$$\boxed{A.B = A_xB_x + A_yB_y + A_zB_z}$$

From the above dentition,

1-Scalar multiplication is commutative(A.B=B.A)

2-It is also distributive (A.(B+C)=A.B+A.C)