Mathematics

Contents

1. Number, sets, intervals, inequalities, Absolute value.
2. Function:
   domain, range, method of representation, type of functions and ~~their~~ their graphs.
3. Relations:
   domain, range, symmetry and graph.
   Special functions.

الهندسة تحليلية
4. Analytical geometry
   coordinate system in plane, linear function equations of ~~of~~ line, distance formula, midpoint formula, distance between point and line, parallel and perpendicular lines, angle between two lines.
5. Trigonometric function:
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   علاقات، متطابقات
6. Limits, continuous and derivative, high-order derivatives, chain rule.
7. ~~Integration (anti-derivatives), rules of integration, differential equations, indefinite integration.~~
7. Implicit differentiation, differentials, Parametric equations, related rates, maximum and minimum problems, critical and inflection points. curve sketching →

8. The mean value theorem and their applications.

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تفاضل، تكاملي

11. Integration (Anti-derivatives), rules of integration, differential equations, ~~kd~~ indefinite integration.

12. Area under a curve (as a limit of summation) and their fording by using definite integral.

13. first fundamental theorem of integral, rules of definite integral.

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فائق الدوال السامية

16. Transcendental function ($\ln x$, e^x, a^x, $\log^x$).

17. Method of integration: (by parts, partial fractions, reduction formulas, by substitution) and improper integrals.

18. Application of definite integral : area, volume surfaces area, arc length.

19. Determinates and their applications.

References

1- Calculus, ~~by~~ seventh edition by Anton Bivens Davis (2002)
2- Calculus and Analytic Geometry, fifth edition, (1981) by George B. Thomas, JR.
3- Calculus, Third edition, (2007), by Robert T. Smith and Roland B. Minton
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27/11/2014
(1)

Mathematics

preliminaries تمهيدي 16/11/2013

Sets and Intervals

Set:

$S \cup T$ union

$S \cap T$ intersection

ϕ empty set (null set)

The set A consisting of positive ~~integral~~ integers less than 6 can be expressed as:

$A = \{1, 2, 3, 4, 5\}$.

or $A = \{x : x$ is an integer and $0 < x < 6\}$

مجموعة جزئية **Subset:**

$A \subseteq B$ each element of a set A is a member of a set B

$A \subset B$ set B contains at least one element that is not a member of A.

الحقيقية

Real numbers (R): are numbers that can be expressed as decimals, such as

$-3/4 = -0.75000\ldots,\ 1/3 = 0.3333\ldots,\ \sqrt{2} = 1.4142\ldots$

real number line

There are four special subsets of real numbers:

1- The natural (الطبيعية) numbers (N)

$$N = \{0, 1, 2, 3, \dots, +\infty\}$$

2. The integer (الصحيحة) numbers (I)

$$I = \{-\infty, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, +\infty\}$$

3- The rational (الأعداد النسبية) numbers

$1/3$, $-4/9$, $200/13$, and $57 = 57/1$.

\# The rational numbers are precisely (بالتحديد) the real numbers with decimal (عشري) expansions that are either:

a) terminating

$3/4 = 0.75000\dots$ or

b) eventually repeating

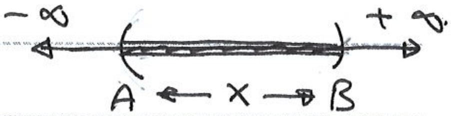
$23/11 = 2.090909\dots = 2.0\overline{9}$

4. The irrational (الأعداد غير النسبية) numbers

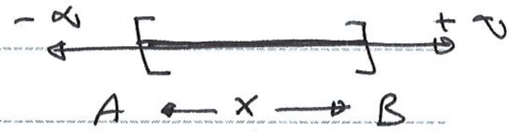
π, $\sqrt{2}$, $\sqrt[3]{5}$, $\log_{10} 3$, $\sin 41^\circ$, $2^{\sqrt{3}}$

Interval: is a set of all real numbers between two points on the real number line (it is a subset of real numbers).

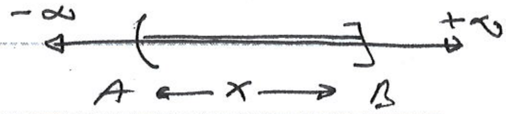
1. open interval

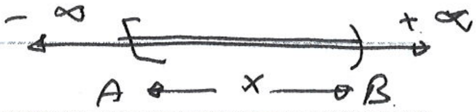


2. closed interval



3. Half-open interval





Example: solve for x ~~the~~

$$\{x : 0 < x < 5\} \cup \{x : 1 < x < 7\}$$

Solve: frome the number line

$(0,5)$

$(1,7)$

$(0,5) \cup (1,7)$

$\therefore$ The interval is $\{0 < x < 7\}$ or $(0,5) \cup (1,7) = (0,7)$.

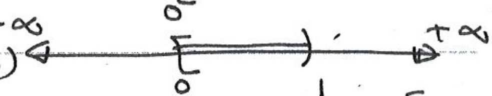
H.W ① $\{x : x < 1\} \cap \{x : x \geq 0\}$

sol. from the number line

$(-\infty, 1)$

$[0, \infty)$

$(-\infty, 1) \cup [0, +\infty)$

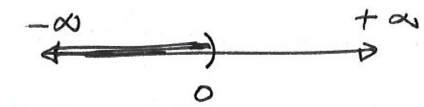


$\therefore$ The interval is $\{x : 0 \leq x < 1\}$ or $[0, 1)$.

② $\{x : x < 0\} \cap \{x : x > 0\}$

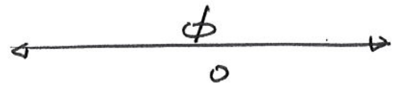
Sol. from the number line.

$(-\infty, 0)$



$(0, \infty)$

$(-\infty, 0) \cap (0, \infty) = \phi$ $(\cancel{-\infty}, 0)$



Inequalities:

rules for inequalities: if a, b, and c are real numbers, then

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$ الضرب بالموجب
4. $a < b$ and $c < 0 \Rightarrow ac > bc$ الضرب بالسالب

   special case $a < b \Rightarrow -a > -b$ الضرب بـ -1
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. if a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

Example: solve the following inequalities and show their solution set.

1. $3 + 7x \leq 2x - 9$

Sol. $3 + 7x - 3 \leq 2x - 9 - 3 \Rightarrow 7x \leq 2x - 12$

$\Rightarrow 7x - 2x \leq 2x - 12 - 2x \Rightarrow 5x \leq -12$

$\Rightarrow x \leq -12/5$

The solution set is $(-\infty, -12/5]$ ~~line~~ number line

2. ~~$x^2 < 4$~~ $7 \leq 2 - 5x < 9$

Sol: $7 - 2 \leq 2 - 5x - 2 < 9 - 2 \Rightarrow 5 \leq -5x < 7$

$\Rightarrow 5/(-5) \geq -5x/(-5) > 7/(-5)$

الضرب بـ -1 عكس الإشارة

$\Rightarrow -1 \geq x > -7/5 \Rightarrow$ ~~$-1 \leq x \leq -7/5$~~

The solution set is $(-7/5, -1]$ $\quad -7/5 < x \leq -1$

H.W

1. $x^2 \leq 4$

Sol. $x^2 \leq 4 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x-2)(x+2) \leq 0$

let $(x-2)(x+2) = 0 \Rightarrow$ either $(x-2) = 0 \Rightarrow x = 2$

or $(x+2) = 0 \Rightarrow x = -2$

sign of $(x-2)$ $\quad -\infty \; - - - - - \; 2 \; + + + \; +\infty$

sign of $(x+2)$ $\quad -\infty \; - - - \; -2 \; + + + + \; +\infty$

sign of $(x-2)(x+2)$ $\quad -\infty \; + + + + \; [-2 \; - - - - \; +2] \; + + + \; +\infty$

$\therefore$ The solution is $\{x : -2 \leq x \leq 2\}$ or $[-2, 2]$.

2. $x^2 - 3x > 10$

Sol. $x^2 - 3x - 10 > 0 \Rightarrow (x-5)(x+2) > 0$

let $(x-5)(x+2) = 0 \Rightarrow$ either $(x-5) = 0 \Rightarrow x = 5$

or $(x+2) = 0 \Rightarrow x = -2$

sign of $(x+2)$ $\quad -\infty \; - - - \; -2 \; + + + + \; +\infty$

sign of $(x-5)$ $\quad -\infty \; - - - - - - \; +5 \; + + + \; +\infty$

sign of $(x-5)(x+2)$ $\quad -\infty \; + + + \; -2) \; - - - \; (+5 \; + + + + + \; \infty$

The solution set is

$(-\infty, -2) \cup (5, \infty)$

3. $\dfrac{(x-1)(x+2)}{x-3} \geq 0$

Sol.

sign of $(x-1)$ $\quad -\infty \; - - - - - - \; +1 \; + + + + + \; +\infty$

sign of $(x+2)$ $\quad -\infty \; - - - \; -2 \; + + + + + + \; +\infty$

sign of $(x-3)$ $\quad -\infty \; - - - - - - - \; +3 \; + + + \; +\infty$

sign of $\dfrac{(x-1)(x+2)}{(x-3)}$ $\quad -\infty \; - - - \; [-2 \; + + + \; +1] \; - - - \; (+3 \; + + + \; +\infty$

[illegible]

$x = 3$

نحول البسط نساويه صفر [illegible]

$(x-1)(x+2) = 0$

$x - 1 = 0 \Rightarrow x = 1$

$x + 2 = 0 \Rightarrow x = -2$

$\therefore$ The solution set is $\{x : -2 \leq x \leq +1\} \cup \{x : x > +3\}$

or $[-2, +1] \cup (+3, \infty)$.

4. $\dfrac{6}{x-1} \geq 5$

<u>sol.</u> this inequality can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative.

$6 \geq 5x-5 \Rightarrow 11 \geq 5x \Rightarrow 11/5 \geq x$ or $x \leq \frac{11}{5}$

$\therefore$ The solution set is the half-open interval $(1, \frac{11}{5}]$.

<u>sol.</u>

$$\frac{6}{x-1} \geq 5$$

$$\Rightarrow \frac{6}{x-1} - 5 \geq 0$$

$$\frac{6-5(x-1)}{x-1} \geq 0$$

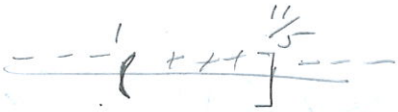
$$\frac{6-5x+5}{x-1} \geq 0$$

$$\frac{11-5x}{x-1} \geq 0$$

$\therefore x-1=0 \Rightarrow x=1$

$11-5x=0$

$\Rightarrow x=\frac{11}{5}$



تعويضا

at $x=0 \Rightarrow \frac{+}{-} = \ominus$

at $x=2 \Rightarrow \frac{+}{+} = \oplus$

at $x=3 \Rightarrow \frac{-}{+} = \ominus$

at $x=1 \Rightarrow \frac{+}{0} \Rightarrow$ undifined Not OK

at $x=\frac{11}{5} \Rightarrow \frac{0}{\frac{11}{5}-1} \Rightarrow 0$ OK

Absolute Value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example: finding absolute values

$|3| = 3$, $|0| = 0$, $|-5| = -(-5) = 5$.

Absolute value properties

1. $|-a| = |a|$
2. $|ab| = |a||b|$
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
4. $|a+b| \leq |a| + |b|$

$|4+5| = |4| + |5| = 9$

$|4+(-5)| = |-1| = 1$

$|4| + |-5| = 4 + 5 = 9$

Absolute values and intervals

If a is any possitive number, then

5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$
8. $|x| \leq a$ if and only if $-a \leq x \leq a$
9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

Examples:

$\begin{cases} 2x-3 = 1 \\ -(2x-3) = 1 \Rightarrow 2x-3 = -1 \end{cases}$

1. Solve the equation $|2x-3| = 1$

sol: $2x - 3 = \pm 1 \Rightarrow$ either $2x - 3 = 1 \Rightarrow 2x = 4 \Rightarrow x = 2$

or $2x - 3 = -1 \Rightarrow 2x = 2 \Rightarrow x = 1$

$\therefore$ The solutions are $x = 1$ and $x = 2$.

$$\begin{cases} 2x-3 \le 1 \Rightarrow 2x \le 4 \Rightarrow x \le 2 \\ -(2x-3) \le 1 \Rightarrow 2x-3 \ge -1 \\ \quad \Rightarrow 2x \ge 2 \\ \quad \Rightarrow x \ge 1 \end{cases}$$

2. Solve the inequality $|2x-3| \le 1$

Sol. $|2x-3| \le 1 \Rightarrow -1 \le 2x-3 \le 1$

$\Rightarrow 2 \le 2x \le 4 \Rightarrow 1 \le x \le 2$

$\therefore$ The solution set is the closed interval $[1,2]$.

H.W

[illegible]

Solve the inequality

$$|2x-3| \ge 1 \begin{cases} +(2x-3) \ge 1 \\ -(2x-3) \ge 1 \Rightarrow 2x-3 \le -1 \end{cases}$$

[illegible] -1

1. $|2x-3| \ge 1$

Sol: $|2x-3| \ge 1 \Rightarrow$ either $2x-3 \ge 1 \Rightarrow 2x \ge 4 \Rightarrow x \ge 2$

or $2x-3 \le -1 \Rightarrow 2x \le 2 \Rightarrow x \le 1$

$\therefore$ the solution set is $(-\infty, 1] \cup [2, \infty)$.

2. $\left|5 - \frac{2}{x}\right| < 1$

Sol: $\left|5 - \frac{2}{x}\right| < 1 \Rightarrow -1 < 5 - \frac{2}{x} < 1$

$\Rightarrow -6 < -\frac{2}{x} < -4 \Rightarrow 3 > \frac{1}{x} > 2 \Rightarrow \frac{1}{3} < x < \frac{1}{2}$

$\therefore$ The solution set is the open interval $(\frac{1}{3}, \frac{1}{2})$

3. $|x-3| + |x+2| < 11$

Sol: recall the definition of absolute value

$$y = |x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that: $|x-3|$

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \ge 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \ge 3 \\ -x+3 & \text{if } x < 3 \end{cases}$$

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similarly,

$$|x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0 \\ -(x+2) & \text{if } x+2 < 0 \end{cases} = \begin{cases} x+2 & \text{if } x \geq -2 \\ -x-2 & \text{if } x < -2 \end{cases}$$

These expressions show that we must consider three cases:

$x < -2$, $\quad -2 \leq x < 3 \quad$ and $x \geq 3$

Case 1: if $x < -2$ we have $|x-3| + |x+2| < 11$

$$-x+3-x-2 < 11$$
$$-2x < 10$$
$$-2x < 10$$
$$\cancel{x < -5} \quad x > -5$$

Case 2: if $-2 \leq x < 3$ we have $|x-3| + |x+2| < 11$

$$-x+3+x+2 < 11$$
$$5 < 11 \text{ (always true)}$$

Case 3: if $x \geq 3$ we have $|x-3| + |x+2| < 11$

$$x-3+x+2 < 11$$
$$x < 6$$

cobining cases 1, 2, and 3, we see that the inequality is satisfied when

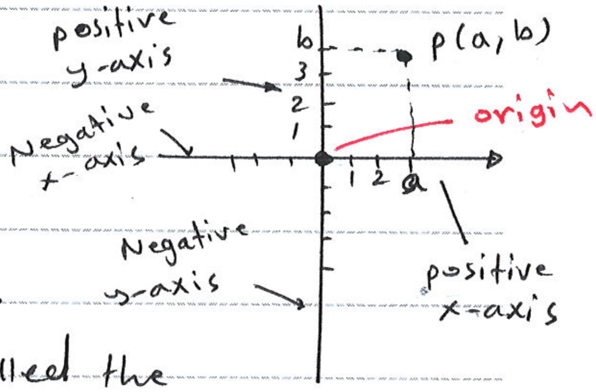
$$-5 < x < 6.$$

so the solution is the interval $(-5, 6)$.

Analytical Geometry

Coordinate system in plane

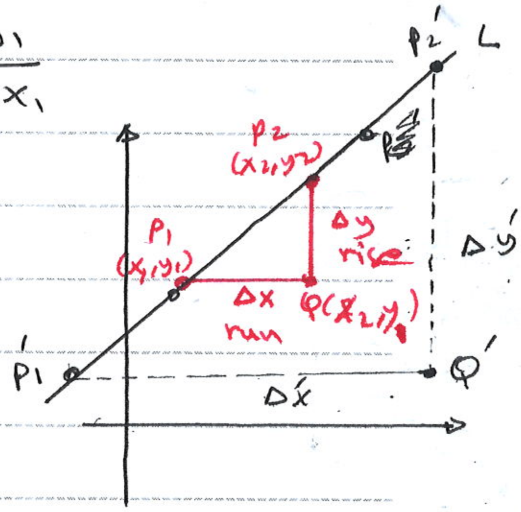
if two perpendicular lines that intersect at the 0-point of each line are drawn, then these lines are called the coordinate axes.



This coordinate system is called the rectangular coordinate system or Cartesian coordinate system.

Slope of a line:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

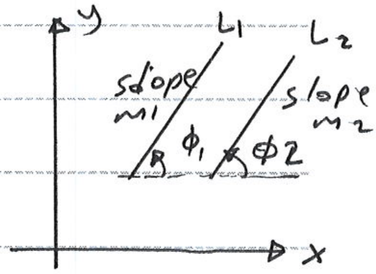


angle of inclination (ϕ) : it is the direction and steepness of a line an be measured with an angle.

horizontal line $\phi = 0°$

vertical line $\phi = 90°$

$0° \leqslant \phi < 180°$



slope

$m = \tan\phi$

note 1 : horizontal lines have $m=0$ ($\Delta y = 0$), and vertical lines have no slope or the slope of a vertical line is undefined ($\Delta x = 0$)

note 2: parallel lines have the same slope.

if $m_1 = m_2$ then $\phi_1 = \phi_2$ and the lines are parallel.

Not3 : if two nonvertical line L_1 and L_2 are perpendicular, then their slops m_1 and m_2 satisfy $m_1 \cdot m_2 = -1$, so each slope is the negative reciprocal of the other.

$$m_1 = -\frac{1}{m_2} \quad , \quad m_2 = -\frac{1}{m_1}$$

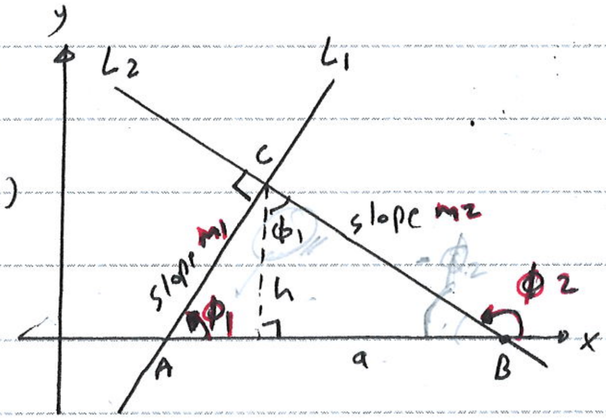
$m_1 = a/h$ and $m_2 = -h/a$

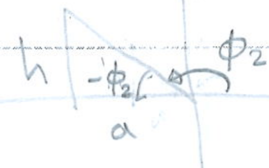
$m_1 = a/h$ and $m_2 = -h/a$

Hence $m_1 \cdot m_2 = (a/h)\cdot(-h/a)$

$= -1$

$m_2 = \frac{h}{-a}$?





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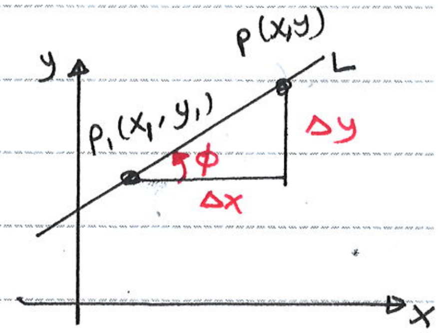
Point-slope equation



$$m = \frac{y-y_1}{x-x_1}$$

$$\Rightarrow \quad y-y_1 = m(x-x_1)$$

$$\therefore \quad y = y_1 + m(x-x_1)$$

Example: write an equation of the line through the point (2,3) with slope $-3/2$.

Sol. $x_1 = 2$, $y_1 = 3$, and $m = -3/2$

$$\therefore y = 3 - \frac{3}{2}(x-2).$$

$$\therefore y = -\frac{3}{2}x + 6$$

when $x = 0$, $y = 6$ so the line intersects the y-axis at $y=6$.

Example: write an equation for the line through $(-2,-1)$ and $(3,4)$.

Sol.

$$m = \frac{-1-4}{-2-3} = \frac{-5}{-5} = 1$$

we can use this slope with of the two given points in the point-slope equation:

with $(x_1, y_1) = (-2,-1)$

$y = -1 + 1\cdot(x-(-2))$

$y = -1 + x + 2$

$y = x+1$

with $(x_1, y_1) = (3,4)$

$y = 4 + 1\cdot(x-3)$

$y = 4 + x - 3$

$y = x+1$

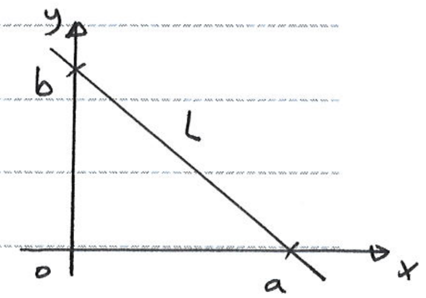
same result

slope – intercept equation

$y = mx + b$

m: slope of the line

b: intercept of the line with y-axis.



General linear equation

$Ax + By + C = 0$

where A, B, and C are constants and A and B are not both 0

Example. Find the slope and y-intercept of the line

$8x + 5y = 20$

sol. $8x + 5y = 20$

$5y = -8x + 20$ $\qquad y = mx + b$

$y = -\frac{8}{5}x + 4$

The slope is $m = -8/5$. the y-intercept is $b = 4$

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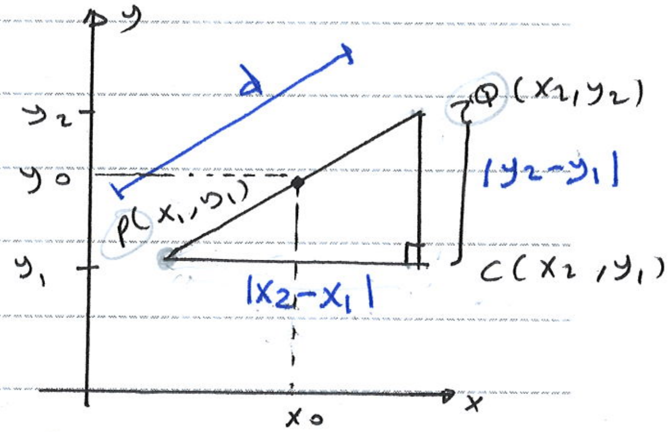
Distance between two points

The distance between $p(x_1, y_1)$ and $Q(x_2, y_2)$



$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\therefore d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

the mid-point formula

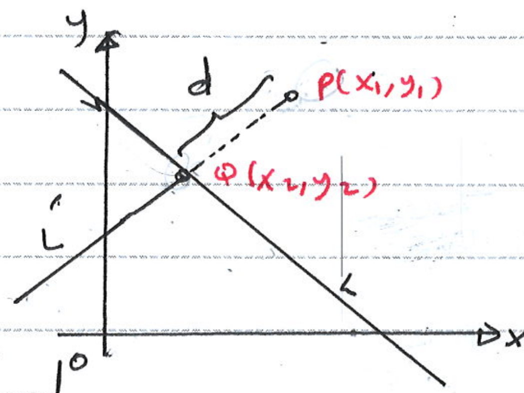
$$x_0 = \frac{x_1 + x_2}{2} \; ; \quad y_0 = \frac{y_1 + y_2}{2}$$

Example: The distance between $p(-1, 2)$ and $Q(3, 4)$ is

$$\sqrt{(3 - (-1))^2 + (4 - 2)^2} = \sqrt{(4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$$

Distance ~~to~~ from a point to a line

To calculate the distance d from a certain point $p(x_1, y_1)$ to a line L:



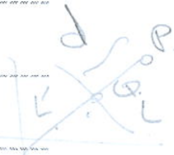
1. Find an equation for the line L' that pass through a point $p(x_1, y_1)$ and perpendicular to the line L.
2. Find the point $Q(x_2, y_2)$ where L' meet with L.

3. Calculate the distance between p and Q

4. To find d use the distance between two points formula.

or use the distance (d) between the line L $(Ax+By+C=0)$ and the point $p(x_1, y_1)$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$$



Example: find the distance from the point $p(2,1)$ to the line $\boxed{y = x+2}$ y_1 $\quad y = mx+b$

Sol. The slope of L is $m=1$ (from slope-intercept form $\boxed{y = mx+b}$).

the slope of L' is $m' = -1/1 = -1$

$\boxed{y-y_1 = m(x-x_1)} \Rightarrow$ from point $p(2,1)$ and $m=-1$

$y-1 = -1(x-2) \Rightarrow y = -x+2+1 \Rightarrow \boxed{y = -x+3}$ y_2

To find Q (the point of L and L' intersection

$\boxed{\text{put } y_L = y_{L'}}$

$\boxed{x+2 = -x+3} \Rightarrow 2x = 1 \Rightarrow x = 1/2 \Rightarrow y = 5/2$

$\therefore$ The coordinate of the point Q is $(1/2, 5/2)$

$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(2-1/2)^2+(1-5/2)^2}$

$= \frac{3}{2}\sqrt{2} = \frac{3}{\sqrt{2}}$ unit length.

المسافة مستقيمة

another Solution: put the equation of the line L in the general form $Ax+By+C=0$

$\therefore \boxed{y = x+2} \Rightarrow x-y+2=0$; $\therefore A=1, B=-1$ and $C=2$

also $x_1 = 2$ and $y_1 = 1$

$$d = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}} = \frac{|1\times(2)+(-1)\times(1)+2|}{\sqrt{1^2+(-1)^2}} = \frac{3}{\sqrt{2}} \text{ unit length.}$$

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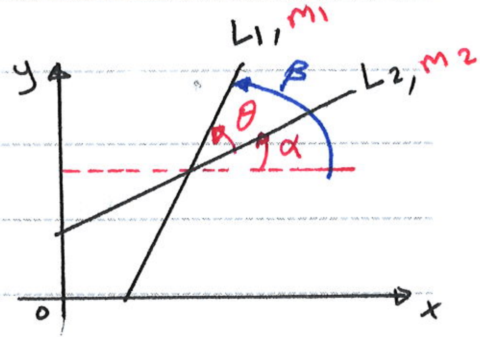
Angles between two lines



$\theta = \beta - \alpha$

$\Rightarrow \tan\theta = \tan(\beta - \alpha)$

$= \dfrac{\tan\beta - \tan\alpha}{1 + \tan\beta \cdot \tan\alpha}$

where: $m_1 = \tan\beta$, $m_2 = \tan\alpha$

$$\Rightarrow \quad \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Example: Find

1- The slope of the line $2x + 3y = 5$?

2- The distance from the above line to the point $p(-1, 0)$.

sol.

1. put the equation in the form $y = mx + b$

$\therefore 3y = 5 - 2x \Rightarrow y = -\frac{2}{3}x + \frac{5}{3}$

$\Rightarrow m = -2/3$ and $b = 5/3$

2- put the equation in the general form $Ax + By + C = 0$

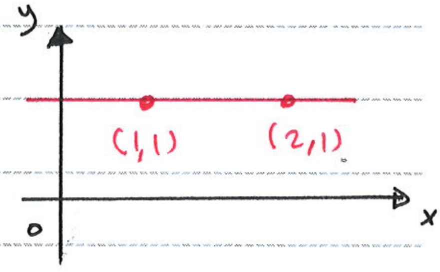
$\therefore 2x + 3y - 5 = 0 \Rightarrow A = 2,\ B = 3$ and $C = -5$,

but $x_1 = -1$ and $y_1 = 0$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|2(-1) + 3(0) + (-5)|}{\sqrt{2^2 + 3^2}}$$

$= \dfrac{7}{\sqrt{13}}$ unit length.

x Example: plote the given pair of points and find the equation for the line determined by them. the points (1,1) and (2,1).



sol.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-1}{2-1} = 0$$

$$y = y_1 + m(x - x_1)$$

from point (1,1)

$$\Rightarrow y = 1 + 0(x-1)$$

$$\Rightarrow y = 1$$

H.W Example: find the slope of the line $3x - 4y = -8$ and then find the distance from the point $p(3,-2)$ to this line.

sol. a) put the equation in the form $y = mx + b$

$$\therefore 3x - 4y = -8 \Rightarrow 4y = 3x + 8 \Rightarrow y = \frac{3}{4}x + 2$$

$$\therefore m = \frac{3}{4} \text{ and } b = 2$$

b) put the equation in the form $Ax + By + C = 0$

$$\Rightarrow 3x - 4y + 8 = 0 \Rightarrow A = 3,\ B = -4 \text{ and } C = 8$$

$$x_1 = 3 \text{ and } y_1 = -2$$

$$\therefore d = \frac{|3(3) + (-2)(-4) + 8|}{\sqrt{3^2 + (-4)^2}} = \frac{|9+8+8|}{\sqrt{9+16}} = \frac{|25|}{\sqrt{25}} = 5$$

unit lengths

P

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Functions

A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

Domain (D_f) : is the set of all possible inputs (x-values)

Range (R_f) : is the set of all possible outputs (y-values).

To find domain and range the following points must be noticed:

1. The denominator in a function must not equal to zero
2. The values under even roots must be positive.

Examples: Find the domain and range of the following functions:

1. $y = f(x) = \frac{1}{x}$

Sol. $x \neq 0 \Rightarrow D_f = \{x : x \neq 0\}$

To find R_f : convert $y = f(x)$ into $x = f(y)$

$\therefore x = \frac{1}{y} \Rightarrow R_f = \{y : y \neq 0\}$, or $R_f = R/\{0\}$

or $R_f = (-\infty, 0) \cup (0, \infty)$

H.W 2. $y = f(x) = \sqrt{4-x}$

sol. $4 - x \geq 0 \Rightarrow 4 \geq x$

$\therefore D_f = \{x : x \leq 4\}$

convert $f(x) \Rightarrow x = f(y)$

$\Rightarrow y^2 = 4 - x \Rightarrow x = 4 - y^2 \quad \therefore R_f = R$.

But the values of y must be always positive, we must exclude negative values,

$\Rightarrow R_f = \{y : y \geq 0\}$, or $R_f = [0, \infty)$.

~~H.W~~ ③ $y = f(x) = \sqrt{1-x^2}$

Sol. $1-x^2 \geqslant 0 \Rightarrow -x^2 \geqslant -1 \Rightarrow x^2 \leq 1$

$\Rightarrow$ either $x \leq 1$, or $-x \leq 1 \Rightarrow x \geqslant -1$

$\therefore D_f = \{x : -1 \leq x \leq 1\}$.

To find R_f: convert the function from $y = f(x)$ into $x = f(y)$.

$y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$

$\Rightarrow x^2 = 1-y^2 \Rightarrow x = \pm\sqrt{1-y^2}$

$\therefore 1-y^2 \geqslant 0 \Rightarrow -y^2 \geqslant -1 \Rightarrow y^2 \leq 1$

$\Rightarrow$ either $y \leq 1$

or $-y \leq 1 \Rightarrow y \geqslant -1$

$\therefore R_f = \{y : -1 \leq y \leq 1\}$.

But the values of y must be always positive, we must exclude negative values,

$\Rightarrow R_f = \{y : 0 \leq y \leq 1\}$.

$y = \sqrt{1-x^2}$

$\Rightarrow y^2 = 1-x^2$

$\Rightarrow y^2 + x^2 = 1$

هذه معادلة equation لدائرة

∴ رسمها

ولكن هنا $y = f(x)$

∴ هنا هي ~~معادلة~~ دالة function

وليست معادلة

∴ يكون رسمها محدود ب $y = \sqrt{1-x^2}$

ورسمها هو

∴ القيم السالبة غير مقبولة بالدالة

∴ هي تختلف عن المعادلة

$y^2 + x^2 = 1$

H.W ④ $y = f(x) = \dfrac{1}{\sqrt{9-x^2}}$

Sol. $9-x^2 > 0 \Rightarrow -x^2 > -9 \Rightarrow x^2 < 9$

$\Rightarrow$ either $x < 3$, or $-x < 3 \Rightarrow x > -3$

$\therefore D_f = \{x : -3 < x < 3\}$.

convert $y = f(x)$ ~~into~~ into $x = f(y)$.

$$y = \frac{1}{\sqrt{9-x^2}} \Rightarrow y^2 = \frac{1}{9-x^2} \Rightarrow 9-x^2 = \frac{1}{y^2}$$

$$\Rightarrow x^2 = 9 - \frac{1}{y^2} \Rightarrow x^2 = \frac{9y^2-1}{y^2} \Rightarrow x = \pm\sqrt{\frac{9y^2-1}{y^2}}$$

$$\Rightarrow x = \pm\frac{\sqrt{9y^2-1}}{y}$$

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$\frac{\sqrt{9y^2-1}}{y}$

$9y^2 - 1 \geq 0 \Rightarrow y^2 \geq \frac{1}{9} \Rightarrow$ either $y \geq \frac{1}{3}$

or $-y \geq \frac{1}{3} \Rightarrow y \leq -\frac{1}{3}$

$\Rightarrow R_f = (-\infty, -\frac{1}{3}] \cup [\frac{1}{3}, \infty).$

هنا الجذر فقط على البسط ∴ لم نأخذ الحالة كاملة السؤال لحالة أن $y \neq$ صفر وإنما فقط على البسط الجزء أن $y \neq$ صفر

$-\infty \quad -\frac{1}{3} \quad 0 \quad +\frac{1}{3} \quad +\infty$

The denominator must not equal zero $\Rightarrow y \neq 0$

But the values of y must be always positive: we must exclude negative values.

$\Rightarrow R_f = [\frac{1}{3}, \infty)$, or $R_f \{y : \frac{1}{3} \leq y \leq \infty\}$.

لم؟

(5) $y = f(x) = \frac{1}{x^2 - 9}$

Sol. $x^2 - 9 \neq 0 \Rightarrow x \neq \pm 3 \Rightarrow D_f = R / \{-3, 3\}$.

To find R_f:

$y(x^2 - 9) = 1 \Rightarrow yx^2 - 9y = 1 \Rightarrow x = \pm\sqrt{\frac{1+9y}{y}}$

$\frac{1+9y}{y} \geq 0$

حسب الجذر يكون على البسط والمقام ∴ أخذنا البسط والمقام إلا أن $y \neq$ صفر

sign of $(1+9y)$: $-\infty$ $- - -$ $-1/9$ $+ + + +$ $+\infty$

~~sign of~~ (y): $-\infty$ $- - - -$ 0 $+ + + +$ $+\infty$

sign of $\frac{(1+9y)}{y}$: $-\infty$ $+ + +]$ $-1/9$ $- - -$ 0 $(+ + +$ $+\infty$

$\therefore R_f = (-\infty, -1/9] \cup (0, \infty)$.

or $R_f = R / (\frac{-1}{9}, 0]$.

عندما نأخذ الصفر غير معرف $\frac{1+0}{0}$ ∴ 0 لا ينتمي إلا الفترة

عندما نأخذ $\frac{-1}{9}$

$\frac{1+9(-\frac{1}{9})}{-\frac{1}{9}} = \frac{0}{-\frac{1}{9}} = 0$ ∴ $\frac{-1}{9}$ تنتمي للفترة لأنها ≥ 0 ∴ موجودة

H.W 6) $y = f(x) = -\sqrt{1-x^2}$

Sol. $1-x^2 \geq 0 \Rightarrow (1-x)(1+x) \geq 0$

$\therefore D_f = [-1, +1]$

$1-x \geq 0$
$\Rightarrow -x \geq -1$
$\Rightarrow x \leq 1$

$1+x \geq 0$
$\Rightarrow x \geq -1$

$-1 \quad +1$

To find R_f:

$y = -\sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$

$\Rightarrow x^2 = 1-y^2 \Rightarrow x = \mp\sqrt{1-y^2}$

$\therefore 1-y^2 \geq 0 \Rightarrow (1-y)(1+y) \geq 0$

$\Rightarrow R_f = [-1, +1]$

But the values of y must be always negative; we must exclude positive values,

$\Rightarrow R_f = [-1, 0]$

30/11
9.

Types of functions

Linear functions : $y = mx + b$

power functions : $y = x^a$

Polynomials functions : $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$,

Algebraic functions :

[illegible] polynomial [illegible] algebraic [illegible]

Trigonometric functions : sin, cos, tan, csc, sec, and cot.

Exponential functions : $y = a^x$

logarithmic functions : $y = \log_a x$.

Transcendental functions ;

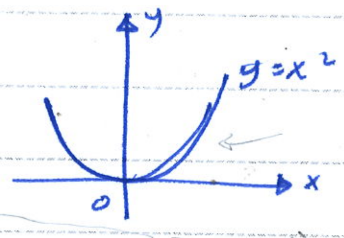
Rational function : is ratio of two polynomials $f(x) = \dfrac{P(x)}{Q(x)}$

Even and odd functions

even function of x if $f(-x) = f(x)$; ~~symm~~ symmetric with respect to the y-axis

odd function of x if $f(-x) = -f(x)$

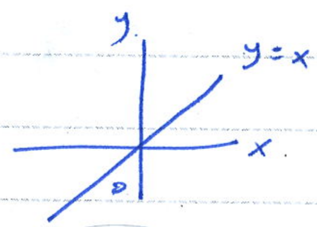
symmetric about the origin



for $x \geq 0$

even

الرسم يكون مع الشكل



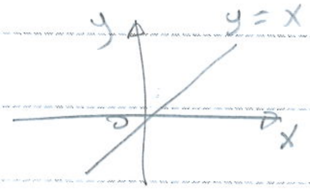
for $x \geq 0$

odd

الرسم يكون مع الشكل

Examples : check the symmetry of the graphs:

① $y = x$

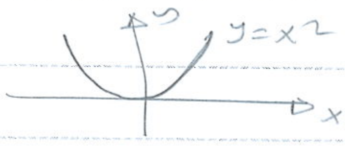


sol. $f(-x) = -x \neq f(x)$

$= -f(x)$

So the function has symmetry about the origin (it is an odd function).

② $y = x^2$



sol. $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$.

so the function has symmetry about y-axis (it is an even function).

③ $y = x^3 + 4$

sol. $f(x) = x^3 + 4$

$f(-x) = (-x)^3 + 4 = -x^3 + 4 \neq f(x)$

$= -(x^3 - 4) \neq -f(x)$.

so the function has no symmetry.

17/12 8.
2015

Asymptotes :

Example: find the asymptotes of $xy - y - 1 = 0$.

sol. solving for y in terms of $x \Rightarrow y = \dfrac{1}{x-1}$

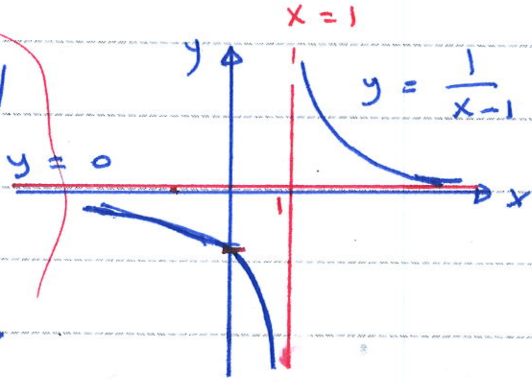
$x - 1 = 0 \Rightarrow x = 1$

Thus the vertical asymptotes is the vertical line through the point $(1, 0)$.

solving for x in terms of $y \Rightarrow x = \dfrac{y+1}{y}$

$y = 0$

Thus the horizontal asymptote is the x-axis.



هذه غير موجودة في الأوراق العادية

or to find the horizontal asymptote :

$$\lim_{x \to \infty} \frac{1}{x-1} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{x}{x} - \frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1 - \frac{1}{x}}$$

$$\therefore = \frac{\frac{1}{\infty}}{1 - \frac{1}{\infty}} = \frac{0}{1-0}$$

$$= \frac{0}{1} = 0$$

$\therefore y = 0$

$$\lim_{x \to \infty} \frac{1}{x-1} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{x}{x} - \frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1 - \frac{1}{x}}$$

$$\therefore = \frac{\frac{1}{\infty}}{1 - \frac{1}{\infty}} = \frac{0}{1-0}$$

$$= \frac{0}{1} = 0$$

$\therefore$ horizontal asymptote is $y = 0$

ممكن بالبداية

$$\lim_{x \to \infty} \frac{1}{x-1} = \frac{1}{\infty - 1} = \frac{1}{\infty} = 0$$

$\therefore y = 0$

Example: find an equation of the linear oblique asymptote for curve?

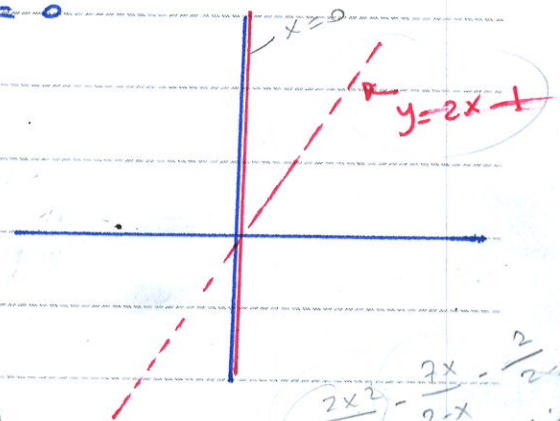
ت ① $y = \dfrac{2x^3 - x^2 + 3}{x^2}$

sol.

$$y = 2x - 1 + \frac{3}{x^2}$$

if the absolute value of x is very large, the value of $\frac{3}{x^2}$ approaches 0 and becomes insignificant in comparsion to the ~~order~~ other two terms. Thus, the curve approaches the line $y = 2x - 1$.

vertical asymptote at $x = 0$



ت

② $y = \dfrac{2x^2 - 7x - 2}{2 - x}$

$$\begin{array}{r|l} & -2x + 3 \\ \hline -x + 2 & 2x^2 - 7x - 2 \\ & \mp 2x^2 \pm 4x \\ \hline & -3x - 2 \\ & \pm 3x \mp 6 \\ \hline & -8 \end{array}$$

$\dfrac{2x^2}{2-x} - \dfrac{7x}{2-x} - \dfrac{2}{2-x}$

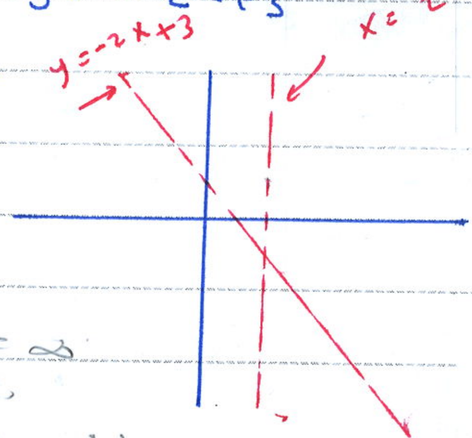
عندما نوزع البسط على المقام فإن المتغير x لا يزول من المقام لذلك نقسم قسمة طويلة

thus

$= -2x + 3 - \dfrac{8}{2 - x}$

the oblique asymptote is $y = -2x + 3$

vertical asymptote at $x = 2$



$\lim\limits_{x \to \infty} \dfrac{2x^2 - 7x - 2}{2 - x}$

$= \lim\limits_{x \to \infty} \dfrac{2 - \frac{7}{x} - \frac{2}{x^2}}{\frac{2}{x^2} - \frac{1}{x}} = \dfrac{2 - 0 - 0}{0 - 0} = \dfrac{2}{0} = \infty$

∴ لا يوجد

~~vertical~~ horizontal asymptote

9.

$$\lim_{x\to\infty} \frac{2x^2-7x-2}{2-x}$$

$$= \lim_{x\to\infty} \frac{2-\frac{7}{x}-\frac{2}{x^2}}{\frac{2}{x^2}-\frac{1}{x}}$$

$$= \frac{2-0-0}{0} = \frac{2}{0} = \infty$$

vertical ~~zero~~ asymptote

H.W **Example**: ~~determing~~ determine the asymptotes to a curve

$$y = \frac{x^2 - x - 2}{x - 1}$$

Sol.

$$y = \frac{(x+1)(x-2)}{x-1}$$

$x - 1 = 0 \Rightarrow x = 1 \Leftarrow$ vertical asymptote.

to find linear oblique asymptote.

thus:

$$y = \frac{x^2 - x - 2}{x - 1}$$

$$\begin{array}{r|l} & x \\ \hline x-1 & x^2 - x - 2 \\ & \mp x^2 \pm x \\ \hline & -2 \end{array}$$

$$\therefore y = x - \frac{2}{x-1}$$

$\therefore$ the equation of oblique asymptote is $\boxed{y = x}$

Example: find a horizontal asymptotes for the following equation:

① $y = \dfrac{1}{10x^2}$

$$\lim_{x \to \infty} \frac{1}{10x^2} = \frac{1}{10\infty} = \frac{1}{\infty} = 0$$

$\therefore y = 0$

or $y = \dfrac{1}{10x^2} \Rightarrow x^2 = \dfrac{1}{10y} \Rightarrow y = 0$

H.w الدكتور
ا كلام عجيب

تمارين
H.w ② $y = \dfrac{2x^2+1}{x^2+3}$

$\lim_{x\to\infty} \dfrac{2x^2+1}{x^2+3} \Rightarrow$

$\Rightarrow \lim_{x\to\infty} \dfrac{\frac{2x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}} = \lim_{x\to\infty} \dfrac{2+\frac{1}{x^2}}{1+\frac{3}{x^2}} = \dfrac{2+\frac{1}{\infty}}{1+\frac{3}{\infty}}$

$= \dfrac{2+0}{1+0} = 2 \quad \therefore y = \underline{\underline{2}} \Leftarrow$ ~~vertic~~ horizontal asymptote

تمارين ③ $y = \dfrac{5x^4-4}{3x^2+2}$

Example: find a horizontal asymptote for the curve

sol. $\lim_{x\to\infty} \dfrac{\frac{5x^4}{x^4}-\frac{4}{x^4}}{\frac{3x^2}{x^4}+\frac{2}{x^4}} = \lim_{x\to\infty} \dfrac{5-\frac{4}{x^4}}{\frac{3}{x^2}+\frac{2}{x^4}}$

$= \dfrac{5-\frac{4}{\infty}}{\frac{3}{\infty}+\frac{2}{\infty}} = \dfrac{5-0}{0} = \infty$ no horizontal asymptotes

H.w
محاضرة
Example ④ find a vertical and a horizontal asymptotes of the function $y = \dfrac{(x+1)^2}{1+x^2}$

sol.

1. $1+x^2 \neq 0 \Rightarrow$ there is no vertical asymptote.

2. to find horizontal asymptote.

$\lim_{x\to\infty} \dfrac{(x+1)^2}{1+x^2} = \lim_{x\to\infty} \dfrac{(x+1)^2 \cdot \frac{1}{x^2}}{1+x^2 \cdot \frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{\left(\frac{x}{x^2}+\frac{1}{x^2}\right)^2}{\frac{1}{x^2}+\frac{x^2}{x^2}}$

$= \lim_{x\to\infty} \dfrac{(1+\frac{1}{x^2})^2}{\frac{1}{x^2}+1} = \dfrac{(1+\frac{1}{\infty})^2}{\frac{1}{\infty}+1} = \dfrac{(1+0)^2}{0+1} = \dfrac{1}{1} = 1 \Rightarrow \boxed{y=1}$

or $y = \dfrac{(x+1)^2}{1+x^2} \Rightarrow y + yx^2 = x^2+2x+1 \Rightarrow (yx^2-x^2)+y-2x-1=0$

$x^2(y-1)-2x+(y-1)=0$

17/12/2015

$$\frac{-b \pm \sqrt{b^2-4AC}}{2A}$$

$$\therefore \frac{+2 \pm \sqrt{4-4(y-1)(y-1)}}{2(y-1)}$$

$$\frac{2 \pm 2\sqrt{1-(y-1)^2}}{2(y-1)}$$

$$= \frac{1 \pm \sqrt{1-(y-1)^2}}{y-1}$$

$\Rightarrow y-1=0 \Rightarrow y=1$ is a horizontal asymptote.

Key Concepts

* A summary of the possible end behaviour of a rational function:
  - if degree of numerator < degree of denominator, the graph has a horizontal asymptote y=0
  - if degree of numerator = degree of denominator, the graph has a horizontal asymptote other than y=0
  - if degree of numerator ~~de~~ = degree of denominator +1, the graph has ~~a~~ a linear oblique asymptote.
  - if the degree of the numerator exceeds the degree of the denominator by more than 1, the graph will have neither a horizontal asymptote nor a linear oblique asymptote.

V, P

* 9.

2012
2015

18/12/2014

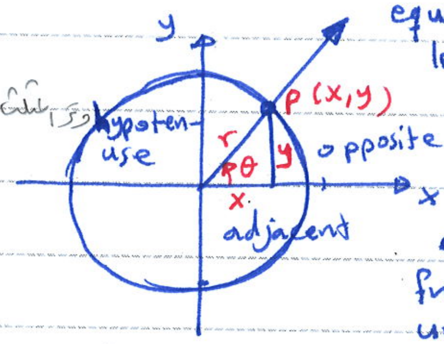
Trigonometric Functions

1/12/13

1- degree measure: one degree (1°) is the measure of angle generated by 1/360 of revolution

2- radian measure: The radian measure of ~~an~~ the angle at the center of the unit circle (circle with radius equals one unit) equals to the length of the arc that the angle cuts from the unit center.

$$1^\circ = \frac{\pi}{180}\text{ rad} \simeq 0.02\text{ rad}$$



sine: $\sin\theta = \frac{y}{r}$

cosine: $\cos\theta = \frac{x}{r}$

tangent: $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$

cosecant: $\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$

secant: $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$

cotangent: $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

$$\cot^2\theta + 1 = \csc^2\theta$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\cdot\sin\theta\cdot\cos\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Identities:

-P.

- Periodicity (دورية)

$\cos(\theta \pm 2\pi) = \cos\theta$

$\sin(\theta \pm 2\pi) = \sin\theta$

$\tan(\theta \pm 2\pi) = \tan\theta$

$\cot(\theta \pm 2\pi) = \cot\theta$

$\sec(\theta \pm 2\pi) = \sec\theta$

$\csc(\theta \pm 2\pi) = \csc\theta$

- Symmetry

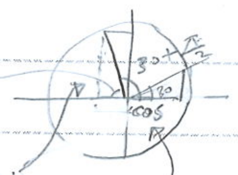
even function	odd function
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\cot(-x) = -\cot x$
	$\csc(-x) = -\csc x$

- shift formulas

$\sin(x + \frac{\pi}{2}) = \cos(x)$ $\qquad$ $\cos(x + \frac{\pi}{2}) = -\sin(x)$

$\sin(x - \frac{\pi}{2}) = -\cos(x)$ $\qquad$ $\cos(x - \frac{\pi}{2}) = \sin(x)$

$30 + \frac{\pi}{2}$



$\sin(x + \frac{\pi}{2}) = \cos 30$

20/12/2015

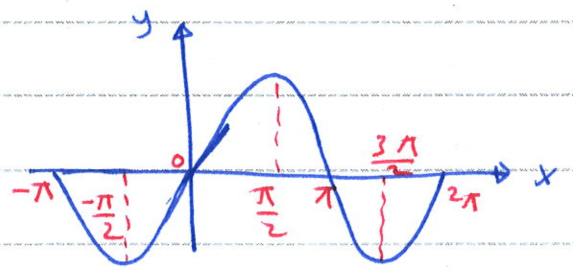
Graph of trigonometric functions:

1. $y = \sin x$

domain: $-\infty < x < \infty$

range: $-1 \leq y \leq 1$

Period: 2π

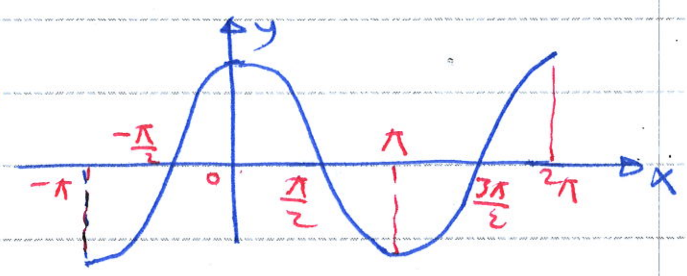


2. $y = \cos x$

domain: $-\infty < x < \infty$

range: $-1 \leq y \leq 1$

period: 2π

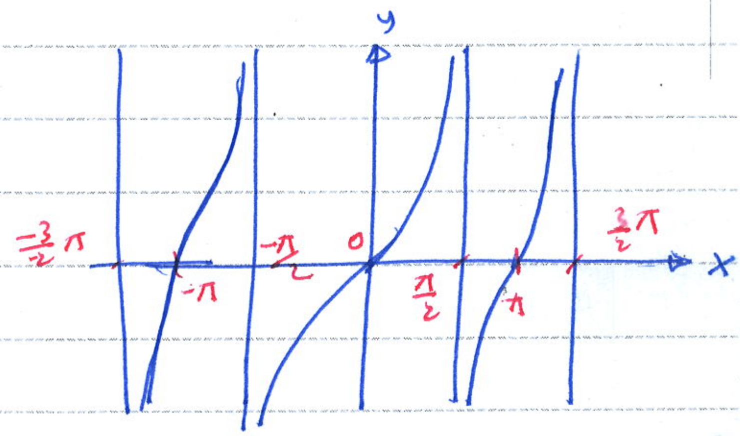


3. $y = \tan x = \dfrac{\sin x}{\cos x}$

domain: $x \neq \pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, ----

range: $-\infty < y < \infty$

Period: π

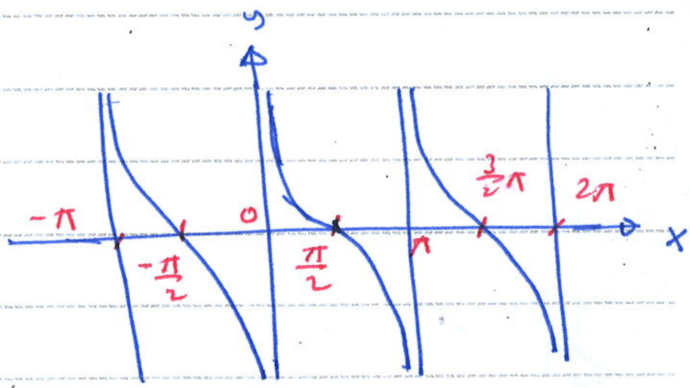


4. $y = \cot x = \dfrac{\cos x}{\sin x}$

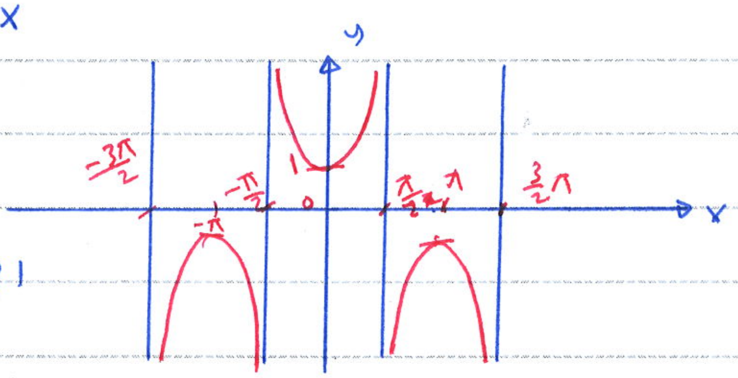
domain: $x \neq 0, \pm\pi, \pm 2\pi, --$

range: $-\infty < y < \infty$

Period: π



5. $y = \sec x = \dfrac{1}{\cos x}$

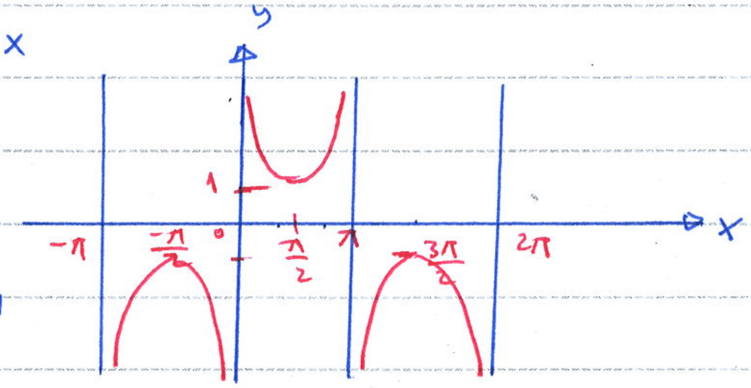


domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2} \cdots$

range: $y \leq -1$ and $y \geq 1$

period: 2π

6. $y = \csc x = \dfrac{1}{\sin x}$



domain: $x \neq 0, \pm\pi, \pm 2\pi, \cdots$

range: $y \leq -1$ and $y \geq 1$

period: 2π

Limits and continuity

limits

if the value of $f(x)$ can be made as close as we like to L by taking the value of x sufficiently close to a (but not equal a), then we write:

$$\lim_{x \to a} f(x) = L.$$

properties of limits:

1. if $f(x) = K$, then $\lim_{x \to a} f(x) = K$, where a and K are real numbers.
2. $\lim_{x \to a} [f_1(x) + f_2(x)] = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x)$.
3. $\lim_{x \to a} [f_1(x) - f_2(x)] = \lim_{x \to a} f_1(x) - \lim_{x \to a} f_2(x)$.
4. $\lim_{x \to a} [f_1(x) \cdot f_2(x)] = \lim_{x \to a} f_1(x) \cdot \lim_{x \to a} f_2(x)$.
5. $\lim_{x \to a} K \cdot f(x) = K \cdot \lim_{x \to a} f(x)$, where K is a constant.
6. $\lim_{x \to a} \frac{f_1(x)}{f_2(x)} = \frac{\lim_{x \to a} f_1(x)}{\lim_{x \to a} f_2(x)}$, $\lim_{x \to a} f_2(x) \neq 0$.
7. $\lim_{x \to a} [f(x)]^{r/s} = [\lim_{x \to a} f(x)]^{r/s}$, provided that $\lim_{x \to a} f(x)$ is a real number (if s is even, we assume $\lim_{x \to a} f(x) \geq 0$).

* $\lim_{x \to a} (c_0 + c_1 x + c_2 x^2 --- c_n x^n) = c_0 + c_1 a + c_2 a^2 --- c_n a^n$.

* $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

* **sandwich theorem:** if $g(x) \leq f(x) \leq h(x)$ ~~are three~~ $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.

27/12 2015

24/12 2015

Note: In determinate quantities

$\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 * \infty\right)$.

Example: find the limits of the following: 7/12/2013

1. $\lim_{x \to 2} x^2 - 4x = 2^2 - 4 * 2 = 4 - 8 = -4$

2. $\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 5x + 6} = \frac{2^2 - 4}{2^2 - 5 * 2 + 6} = \frac{0}{0}$ (indeterminate quantities).

$\lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-3)} = \lim_{x \to 2} \frac{(x+2)}{(x-3)} = \frac{2+2}{2-3} = \frac{4}{-1} = -4$

3. $\lim_{x \to 2} \frac{x-2}{\sqrt{x^2-4}} = \frac{2-2}{\sqrt{2^2-4}} = \frac{0}{0}$ (indeterminate quantities).

$= \lim_{x \to 2} \frac{\sqrt{x-2}\,\sqrt{x-2}}{\sqrt{(x-2)(x+2)}} = \lim_{x \to 2} \frac{\cancel{\sqrt{x-2}}\,\sqrt{x-2}}{\cancel{\sqrt{x-2}}\,\sqrt{x+2}}$

$= \lim_{x \to 2} \frac{\sqrt{x-2}}{\sqrt{x+2}} = \frac{\sqrt{2-2}}{\sqrt{2+2}} = \frac{0}{\sqrt{4}} = 0$

4. $\lim_{x \to 2} \frac{\sqrt{x-2}}{x^2-4} = \frac{\sqrt{2-2}}{2^2-4} = \frac{0}{0}$ (indeterminate quantities).

$\lim_{x \to 2} \frac{\sqrt{x-2}}{(x-2)(x+2)} = \lim_{x \to 2} \frac{\cancel{\sqrt{x-2}}}{\cancel{\sqrt{x-2}}\,\sqrt{x-2}\,(x+2)}$

$= \lim_{x \to 2} \frac{1}{\sqrt{x-2}\,(x+2)} = \frac{1}{\sqrt{2-2}\,(2+2)} = \frac{1}{0 * 4} = \frac{1}{0} = \infty$ the limit does not exist.

5. $\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{\sin 3x}{x} * \frac{3}{3} = 3 \cdot \lim_{x \to 0} \frac{\sin 3x}{3x}$

$= 3 * 1 = 3$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

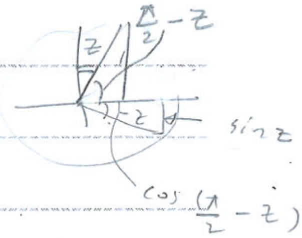
6. $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x}$

$= 1 * \frac{1}{1} = 1$

H.W 7. $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x}$ let $z = \frac{\pi}{2} - x$; so as $x \to \frac{\pi}{2} \Rightarrow z \to 0$

$\boxed{x = \frac{\pi}{2} - z}$

$\therefore \lim_{z \to 0} \frac{\cos(\frac{\pi}{2} - z)}{z} = \lim_{z \to 0} \frac{\sin z}{z} = 1$



H.W 8. $\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x}$

$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)}$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{(1 + \cos x)} = 1 \cdot \frac{0}{1+1} = 0$

H.w ~~8~~ 9. $\lim_{x \to 1} \frac{x - 1}{\sqrt{x^2 + 3} - 2} = \frac{1 - 1}{\sqrt{1^2 + 3} - 2} = \frac{0}{0}$ (indeterminate quantities).

$= \lim_{x \to 1} \frac{x - 1}{\sqrt{x^2 + 3} - 2} \cdot \frac{\sqrt{x^2 + 3} + 2}{\sqrt{x^2 + 3} + 2}$ (multiplying both the numerator and denominator by the conjugate factor)

$= \lim_{x \to 1} \frac{(x - 1)(\sqrt{x^2 + 3} + 2)}{x^2 + 3 - 4}$

$= \lim_{x \to 1} \frac{(x - 1)(\sqrt{x^2 + 3} + 2)}{x^2 - 1} = \lim_{x \to 1} \frac{(x - 1)(\sqrt{x^2 + 3} + 2)}{(x - 1)(x + 1)}$

$= \lim_{x \to 1} \frac{(\sqrt{x^2 + 3} + 2)}{(x + 1)} = \frac{\sqrt{1^2 + 3} + 2}{x + 1} = \frac{4}{2} = 2.$

24/12

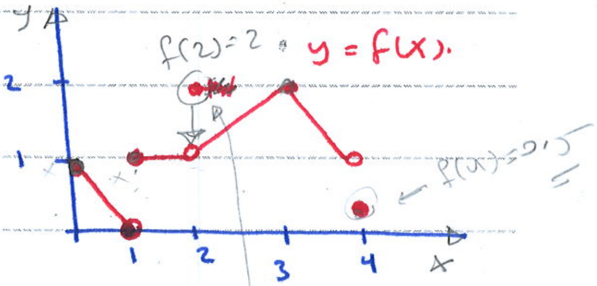
[illegible] من الص 34 الى 41

Right-hand limits and left-hand limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and ~~and~~ are equal.

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = L \text{ and } \lim_{x\to c^-} f(x) = L$$

Example: discuss the limit properties of the function $f(x)$ which shown in figure.



Sol.

- at $x = 0$

$\lim_{x\to 0^+} f(x) = 1$

$\lim_{x\to 0^-} f(x)$ does not exist (because the function is not defined to the left of $x=0$)

- at $x = 1$

ولو أن ، بالرغم من

$\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$

$\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1} f(x)$ does not exist, because the right-hand and left-hand limits are not equal.

- at $x = 2$ $\lim_{x\to 2^-} f(x) = 1$

$\lim_{x\to 2^+} f(x) = 1$

دالة عند $x=2$ فقط

$\boxed{\lim_{x\to 2} f(x) = 1}$ even though $f(2) = 2$

24/12/201

- at $x=3$ $\lim\limits_{x \to 3^-} f(x) = 2$

$\lim\limits_{x \to 3^+} f(x) = 2$

$$\lim_{x \to 3} f(x) = f(3) = 2$$

- at $x=4$ $\lim\limits_{x \to 4^-} f(x) = 1$ even though $f(4) = 0.5$

$\lim\limits_{x \to 4^+} f(x)$ does not exist, because the function is not defined to the right of $x=4$

<u>Example</u>: check the existence of the limit of the function $f(x)$ at $x=1$.

$$f(x) = \begin{cases} 2x+1 & -1 \leq x \leq 1 \\ \cancel{\frac{x^2}{2-3}} \quad \frac{x^2}{2} - 3 & 1 < x < 4 \end{cases}$$

Sol. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 2x+1 = 3$

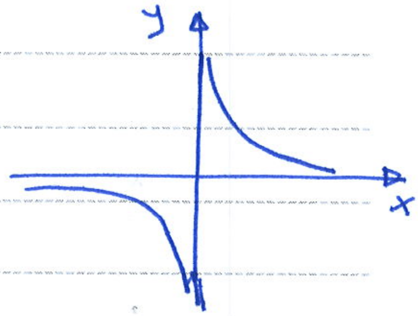
$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \frac{x^2}{2} - 3 = -2.5$

Since the right-hand and left-hand limits are not equal, thus the limit does not exist at $x=1$.

limits involving infinity

These are the limits that include $x \to \infty$ or $x \to -\infty$ and $\lim f(x) = \infty$ or $\lim f(x) = -\infty$

Examples: ① $y = \frac{1}{x}$



$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$\left.\right\}$ one-sided limits $\Rightarrow$ limit does not exist

الغاية عندما تكون الناتج ∞ نحتاج الى اختبار الاشارة يمينه ويسار

② $\displaystyle\lim_{x \to 3} \frac{x+3}{x-3}$ $\qquad \frac{3+3}{3-3} = \frac{6}{0} = \infty$

$$\lim_{x \to 3^+} \frac{x+3}{x-3} = \frac{+}{+} = +\infty$$

$x=4$ [illegible] $4-3=1$ +ve

$$\lim_{x \to 3^-} \frac{x+3}{x-3} = \frac{+}{-} = -\infty$$

$-2-3=-5$ -ve

limit does not exist at $x=3$

③ $\displaystyle\lim_{x \to 1} \frac{|x+1|}{x-1}$ $\qquad \frac{1+1}{1-1} = \frac{2}{0} = \infty$

$$\lim_{x \to 1^+} \frac{|x+1|}{x-1} = \frac{+}{+} = +\infty$$

$2-1=2$ +ve

$$\lim_{x \to 1^-} \frac{|x+1|}{x-1} = \frac{+}{-} = -\infty$$

1 [illegible] $0-1=-1$ -ve

limit does not exist at $x=1$

$\lim_{x\to\infty}\left(5+\frac{1}{x}\right)=\lim_{x\to\infty}\left(\frac{5x+1}{x}\right)$

$\therefore \deg(f)=\deg(g)$

يعني أس البسط يساوي أس المقام

Examples: find the limits of the following:

① $\lim_{x\to\infty}\left(5+\frac{1}{x}\right)=\lim_{x\to\infty}5+\lim_{x\to\infty}\frac{1}{x}=5+0=5$

② $\lim_{x\to\infty}\frac{x}{7x+4}=\lim_{x\to\infty}\frac{\frac{x}{x}}{\frac{7x}{x}+\frac{4}{x}}=\lim_{x\to\infty}\frac{1}{7+\frac{4}{x}}$

$=\frac{1}{7+0}=\frac{1}{7}$

H.W ③ $\lim_{x\to\infty}\frac{2x^2-x+3}{3x^2+5}=\lim_{x\to\infty}\frac{\frac{2x^2}{x^2}-\frac{x}{x^2}+\frac{3}{x^2}}{3\frac{x^2}{x^2}+\frac{5}{x^2}}$

$=\lim_{x\to\infty}\frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x^2}}=\frac{2-0+0}{3+0}=\frac{2}{3}$

هل نقسم على اكبر أس موجود بالبسط أو بالمقام؟ أو نقسم على اكبر أس بالبسط

~~H.W~~ ④ $\lim_{x\to\infty}\frac{4x^2-3}{3x}=\lim_{x\to\infty}\frac{\frac{4x^2}{x}-\frac{3}{x}}{\frac{3x}{x}}=\lim_{x\to\infty}\frac{4x-\frac{3}{x}}{3}$

$\frac{\frac{4x^2}{x^2}-\frac{3}{x^2}}{\frac{3x}{x^2}}=\frac{4-\frac{3}{x^2}}{\frac{3}{x}}=\frac{4-0}{0}=\infty$

$=\frac{4\cdot\infty-0}{3}=\infty \Rightarrow$ the limit does not exit.

H.W ⑤ $\lim_{x\to\infty}\frac{5x+3}{2x^2-1}=\lim_{x\to\infty}\frac{\frac{5x}{x^2}+\frac{3}{x^2}}{\frac{2x^2}{x^2}-\frac{1}{x^2}}$

$=\lim_{x\to\infty}\frac{\frac{5}{x}+\frac{3}{x^2}}{2-\frac{1}{x}}=\frac{0-0}{2-0}=\frac{0}{2}=0.$

تمارين 3,11

⑥ a. $\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to+\infty}\frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|}$ (since $\sqrt{x^2}=|x|$).

as $x\to+\infty$, the values of x under consideration are positive, so we can replace $|x|$ by x:

$\Rightarrow \lim_{x\to+\infty}\frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|}=\lim_{x\to+\infty}\frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/x}=\lim_{x\to+\infty}\frac{\sqrt{1+\frac{2}{x^2}}}{\left(3-\frac{6}{x}\right)}$

$=\frac{1}{3}.$

$=\lim_{x\to+\infty}\frac{\sqrt{\frac{x^2+2}{x^2}}}{\frac{3x-6}{x}}=\lim_{x\to+\infty}\frac{\sqrt{1+\frac{2}{x^2}}}{3-\frac{6}{x}}$

$=\frac{\sqrt{1+\frac{2}{\infty}}}{3-\frac{6}{\infty}}=\frac{\sqrt{1+0}}{3-0}=\frac{1}{3}$

* 9.

b. $$\lim_{x \to -\infty} \frac{\sqrt{x^2+2}}{3x-6}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} \quad (\text{since } \sqrt{x^2} = |x|).$$

as $x \to -\infty$, the values of x under consideration are negative, so we can replace $|x|$ by $-x$:

$$\Rightarrow \lim_{x \to -\infty} \frac{\frac{\sqrt{x^2+2}}{|x|}}{\frac{(3x-6)}{|x|}} = \lim_{x \to -\infty} \frac{\frac{\sqrt{x^2+2}}{\sqrt{x^2}}}{(3x-6)/(-x)}$$

$$\frac{\sqrt{\frac{x^2+2}{x^2}}}{\frac{3x-6}{-x}}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{1+(2/x^2)}}{[-3+(6/x)]} = -\frac{1}{3}$$

$$\frac{\sqrt{1+\frac{2}{x^2}}}{\frac{3x}{-x} - \frac{6}{-x}} = \frac{\sqrt{1+\frac{2}{x^2}}}{-3+\frac{6}{x}} = \frac{1+0}{-3+0}$$

بالمرافق =

7 $$\lim_{x \to \infty} \sqrt{x^2+6x+1} - \sqrt{x^2+x} \qquad (\infty - \infty)$$

sol. $$\lim_{x \to \infty} \sqrt{x^2+6x+1} - \sqrt{x^2+x}$$

$$= \lim_{x \to \infty} \left(\sqrt{x^2+6x+1} - \sqrt{x^2+x}\right) * \frac{\sqrt{x^2+6x+1} + \sqrt{x^2+x}}{\sqrt{x^2+6x+1} + \sqrt{x^2+x}}$$

$$= \lim_{x \to \infty} \frac{(x^2+6x+1)-(x^2+x)}{\sqrt{x^2+6x+1} + \sqrt{x^2+x}}$$

$$= \lim_{x \to \infty} \frac{\cancel{x^2}+6x+1-\cancel{x^2}-x}{\sqrt{x^2+6x+1} + \sqrt{x^2+x}} = \lim_{x \to \infty} \frac{5x+1}{\sqrt{x^2+6x+1} + \sqrt{x^2+x}}$$

$$= \lim_{x \to \infty} \frac{\frac{5x}{x} + \frac{1}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{6x}{x^2} + \frac{1}{x^2}} + \sqrt{\frac{x^2}{x^2} + \frac{x}{x^2}}}$$

$$= \frac{5+0}{\sqrt{1+0+0} + \sqrt{1+0}} = \frac{5}{1+1} = \frac{5}{2} = 2.5$$

summery for rational functions

① $\lim_{x \to \pm \infty} \frac{f(x)}{g(x)} = 0$ if $\deg(f) < \deg(g)$

② $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite if $\deg(f) = \deg(g)$.

③ $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is infinite if $\deg(f) > \deg(g)$.

Note

$c > 1$

$c^{\infty} = \infty$

$c^{-\infty} = 0$

$0 < c < 1$

$c^{\infty} = 0$

$c^{-\infty} = +\infty$

⑧ $\lim_{x \to 0} \dfrac{1}{3 + 2^{1/x}}$

[illegible] $2^{1/x}$ [illegible]

Sol. 1- let $x \to 0^-$

$\Rightarrow \dfrac{1}{x} \to -\infty$

$\therefore \lim_{x \to 0^-} \dfrac{1}{3 + 2^{1/x}} = \dfrac{1}{3 + 0} = \dfrac{1}{3}$

$2^{-\infty}$

2- let $x \to 0^+$

$\Rightarrow \dfrac{1}{x} \to +\infty$

$\therefore \lim_{x \to 0^+} \dfrac{1}{3 + 2^{1/x}} = \dfrac{1}{3 + \infty} = 0$

2^{∞}

the limit does not exit.

⑨ $\lim_{x \to 0} \dfrac{1 + 2^{1/x}}{3 + 2^{1/x}}$

$2^{\frac{1}{0^+}} = 2^{-\infty} = 0$

Sol. 1- let $x \to 0^- \Rightarrow 2^{1/x} \to 0$

$\therefore \lim_{x \to 0^-} \dfrac{1 + 2^{1/x}}{3 + 2^{1/x}} = \dfrac{1 + 0}{3 + 0} = \dfrac{1}{3}$

$2^{\frac{1}{0^-}} = 2^{-\infty} = 0$

هنا [illegible]

2- let $x \to 0^+$; where $\lim_{x \to 0^+} 2^{-1/x} = 0$

$\Rightarrow \dfrac{1 + 2^{1/x}}{3 + 2^{1/x}} = \dfrac{\dfrac{1}{2^{1/x}} + \dfrac{2^{1/x}}{2^{1/x}}}{\dfrac{3}{2^{1/x}} + \dfrac{2^{1/x}}{2^{1/x}}} = \dfrac{2^{-1/x} + 1}{3 * 2^{-1/x} + 1} \quad x \neq 0$

$\therefore \lim_{x \to 0^+} \dfrac{2^{-1/x} + 1}{3 * 2^{-1/x} + 1} = \dfrac{0 + 1}{0 + 1} = 1$

the limit does not exist.

عند تطبيق limit ل [illegible] مش رح [illegible] يكون الناتج $\dfrac{\infty}{\infty}$

$\lim_{x \to 0^+} \dfrac{1 + 2^{1/x}}{3 + 2^{1/x}}$

$= \dfrac{1 + 2^{1/0}}{3 + 2^{1/0}}$

$= \dfrac{1 + 2^{\infty}}{3 + 2^{\infty}}$

$= \dfrac{1 + \infty}{3 + \infty} = \dfrac{\infty}{\infty}$

$\therefore$ indeterminate quantites لذلك نقسم [illegible]

الى هنا لم نقف بالمقرر
ص 43، 4/41

في كتاب Calculus يشرح الدالة فوق الـ x في أعلاها (بسط المقام) والبسط في صفحة 134

H.W

Example 4 (b)

① سؤال P.21 Q18

$$\lim_{x \to \infty} \frac{x^2+5x+6}{x+1}$$

$$= \lim_{x \to \infty} \frac{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{6}{x^2}}{\frac{x}{x^2} + \frac{1}{x^2}} = \lim_{x \to \infty} \frac{1 + \frac{5}{x} + \frac{6}{x^2}}{\frac{1}{x} + \frac{1}{x^2}}$$

$$= \frac{1+0+0}{0+0} = \frac{1}{0} = \infty$$

② سؤال P.21 Q.18

$$\lim_{x \to +\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$$

$$= \lim_{x \to +\infty} \frac{\frac{3^x}{3^x} - \frac{3^{-x}}{3^x}}{\frac{3^x}{3^x} + \frac{3^{-x}}{3^{x}}} = \lim_{x \to +\infty} \frac{1 - 3^{-x} \cdot 3^{-x}}{1 + 3^{-x} \cdot 3^{-x}}$$

$$= \lim_{x \to \infty} \frac{1-3^{-2x}}{1+3^{-2x}} = \frac{1-3^{-\infty}}{1+3^{-\infty}} = \frac{1-0}{1+0} = 1$$

③ سؤال P.21 Q.18

$$\lim_{x \to -\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$$

هنا نقسم على 3^x $\Rightarrow$ $\cancel{\lim_{x \to \infty} \frac{\frac{3^x}{3^x} - 3^{-x}}{\ldots}}$

$$= \lim_{x \to -\infty} \frac{\frac{3^x}{3^x} - \frac{3^{-x}}{3^x}}{\frac{3^x}{3^x} + \frac{3^{-x}}{3^x}} = \lim_{x \to -\infty} \frac{1-3^{-2x}}{1+3^{-2x}} = \frac{1-3^{\infty}}{1+3^{\infty}} = \frac{1-\infty}{1+\infty}$$

$$= \frac{\infty}{\infty}$$

لكن هنا نقسم على 3^{-x}

$$\therefore \lim_{x \to -\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$$

$$= \lim_{x \to -\infty} \frac{\frac{3^x}{3^{-x}} - \frac{3^{-x}}{3^{-x}}}{\frac{3^x}{3^{-x}} + \frac{3^{-x}}{3^{-x}}} = \lim_{x \to -\infty} \frac{3^x \cdot 3^x - 1}{3^x \cdot 3^x + 1} = \lim_{x \to -\infty} \frac{3^{2x}-1}{3^{2x}+1}$$

$$\therefore = \frac{3^{-\infty}-1}{3^{-\infty}+1} = \frac{0-1}{0+1} = -1$$

المثال لم يعطا بالمحاضرة

Continuity

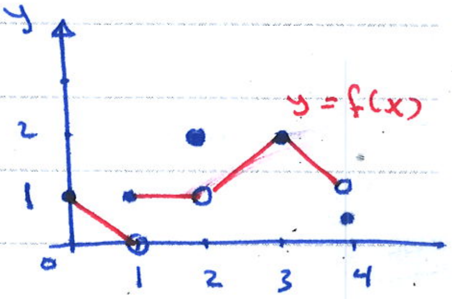
the continuity test

A function $y = f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions:

1. $f(c)$ is defined (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value at c).

Example: discuss the continuity conditions of the function $f(x)$ which shown in figure at $x=0$, $x=1$, $x=2$, $x=3$, $x=1.5$, and $x=4$.

sol.



- at $x = 0$:

$f(0) = 1$

$\lim_{x \to 0^+} f(x) = 1$

$\therefore \lim_{x \to 0^+} f(x) = f(0) = 1$

so it is continuous at $x = 0$

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- at $x = 1$

$f(1) = 1$

$\lim_{x \to 1^-} f(x) = 0$, $\lim_{x \to 1^+} f(x) = 1$

$\therefore \lim_{x \to 1} f(x)$ does not exist, so it is discontinuous at $x = 1$

- at $x = 2$

$f(2) = 2$

$\lim_{x \to 2^-} f(x) = 1$, $\lim_{x \to 2^+} f(x) = 1$, $\therefore \lim_{x \to 2} f(x) = 1$

$\because \lim_{x \to 2} f(x) \neq f(2)$, so it is discontinuous at $x = 2$.

- at $x = 3$

$f(3) = 2$

$\lim_{x \to 3^-} f(x) = 2$, $\lim_{x \to 3^+} f(x) = 2$

$\therefore \lim_{x \to 3} f(x) = 2$

$\because \lim_{x \to 3} f(x) = f(3) = 2$, So it is continuous at $x = 3$

- at $x = 1.5$

$f(1.5) = 1.$

$\lim_{x \to 1.5^-} f(x) = 1$, $\lim_{x \to 1.5^+} f(x) = 1$

$\therefore \lim_{x \to 1.5} = 1$

$\because \lim_{x \to 1.5} f(x) = f(1.5) = 1$, so it is continuous at $x = 1.5$

- at $x = 4$

$f(4) = 0.5$

$\lim_{x \to 4^-} f(x) = 1$, $\because \lim_{x \to 4^-} f(x) \neq f(4)$

$\therefore$ so it is discontinous at ~~right~~ $x = 4$

<u>Example</u> : determine whether the following functions are continuous at $x = 2$.

① $f(x) = \dfrac{x^2-4}{x-2}$

sol.

$f(2)$ is not found $(D \neq 2)$

so the function is discontinuous at $x = 2$

② $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$

sol. $f(2) = 3$

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2} (x+2) = 2+2 = 4$$

$\therefore f(2) \neq \lim_{x \to 2} f(x)$ so the function is discontinuous at $x = 2$

③ $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$

$\dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = x+2$

$\therefore \lim_{x \to 2} (x+2) = 4$

sol. $f(2) = 4$

$\lim_{x \to 2} \dfrac{x^2-4}{x-2} = 4$ $\therefore f(2) = \lim_{x \to 2} f(x)$

so the function is continuous at $x = 2$

6th w Example: test the continuity of the following function at $x=1$

$$f(x) = \begin{cases} x^2 & x<1 \\ \frac{x}{2} & x \geq 1 \end{cases}$$

sol. $f(1) = \frac{1}{2}$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x}{2} = \frac{1}{2}$$

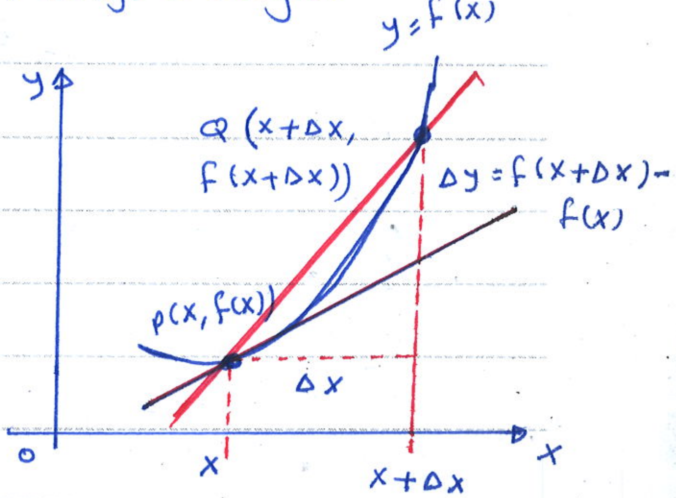
$\lim_{x \to 1} f(x)$ is not found, so the function is discontinuous at $x=1$.

14/12/2013
25/12/2014

Differentiation

derivatives

derivatives are the functions which are used to measure rates at which things change.



If $y = f(x)$

$\therefore \Delta y = f(x+\Delta x) - f(x)$

So, slope of secant

$$PQ = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

As $Q \to p$ then slope of secant pQ will equal to slope of tangent of the curve $f(x)$ at p and $\Delta x \to 0$

$$\therefore \lim_{Q \to p} \text{slope of secant } pQ = \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

= slope of tangent of the curve $f(x)$ at p.

And this is called the definition of derivative of the function $f(x)$ and this denoted by y', $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx} f(x)$, and $D_x f(x)$.

$$\therefore f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example: find the derivative of the function $f(x) = x^2$ using the definition of derivative.

sol.

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x \cdot \Delta x + \Delta x^2 - \cancel{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x (2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

Example: using the definition of derivative, differentiate $f(x) = \sqrt{x}$

sol. $$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} * \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{1}{(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \frac{1}{(\sqrt{x} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

30/12 2015

30/12 2015

27/12 2015

Law of derivatives

1. $\frac{d}{dx} c = 0$ where c is a constant

2. $\frac{d}{dx} x^n = n . x^{n-1}$

3. If u and v are two functions of x then:

constant

a) $\frac{d}{dx}(c * u) = c * \frac{du}{dx}$

b) $\frac{d}{dx}(u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx}$

c) $\frac{d}{dx}(u * v) = u \frac{dv}{dx} + v \frac{du}{dx}$

d) $\frac{d}{dx}(u^n) = n u^{n-1} * \frac{du}{dx}$

e) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Example: if $y = x^3 + 7x^2 - 5x + 4$, find $\frac{dy}{dx}$.

Sol.: $\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(7x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(4)$

$= 3x^2 + 2 * 7x - 5 + 0 = 3x^2 + 14x - 5$

higher order derivatives :

$$y' = \frac{dy}{dx}$$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} ;$$

$$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

the names continue as you imagine with

$$y^{n} = \frac{d}{dx}(y^{n-1}) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} ,$$

is the n^{th} derivative of y with respect to x for any positive integer n.

Example : $y = x^3 - 3x^2 + 2$

first derivative : $y' = 3x^2 - 6x$

second " : $y'' = 6x - 6$

third " : $y''' = 6$

fourth " : $y'''' = 0$

the function has derivatives of all orders, but the ~~fifth~~ fourth and subsequent order derivatives are all zero.

implicit differentiation

In some cases, it is difficult or impossible to solve $y = f(x)$, so to find $\frac{dy}{dx}$ for such cases, implicit differentiation will be use.

Example: find $\frac{dy}{dx}$ of the following:

1. $x^2 + y^2 = 1$

sol. $2x + 2y * \frac{dy}{dx} = 0$

$\Rightarrow 2y * \frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

2- $2y = x^2 + 3xy^2$

sol. $2\frac{dy}{dx} = 2x + 3x\left(2y\frac{dy}{dx}\right) + 3y^2 \cdot \frac{dx}{dx} = 1$

$\Rightarrow 2\frac{dy}{dx} - 6xy\frac{dy}{dx} = 2x + 3y^2$

$\Rightarrow \frac{dy}{dx}(2 - 6xy) = 2x + 3y^2 \Rightarrow \frac{dy}{dx} = \frac{2x+3y^2}{2-6xy}$

المثال الثاني لم يعطِ بالامتحانات

Example: if $x^3y + xy^3 = 2$ find y' and y'' at $x=1$

$x=1$ and $y=1$

نعوض بالسؤال لان المعادلة ما بنقدر نفصلها صعب كل

Sol.

الاول مشتقة الثاني + الثاني مشتقة الاول

$x^3y' + 3x^2y + 3xy^2y' + y^3 = 0$

$x^3y'' + 3x^2y' + 3x^2y' + 6xy + 3xy^2y''$

$+ 6xy(y')^2 + 3y^2y' + 3y^2y' = 0$

$6xy\,y'y' = 6xy(y')^2$

at $x=1 \Rightarrow y=1$; substitute in the first differential equation ~~equation~~ $\Rightarrow y' = 1$ $y'=-1$, and substite $x=1$, $y=1$, and $y'=-1$ in the second differential equation

$\Rightarrow y'' = 0$

the chain rule

if $y = f(u)$; $u = g(x)$, and the derivatives $\frac{dy}{du}$ and $\frac{du}{dx}$ both exist then the composite function defined by $f(g(x))$ has a derivative given by:

$$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$$

Example ① let $y = \sqrt{u^2+1}$; $u = \frac{1}{x} + x^2$, find $\frac{dy}{dx}$

sol. $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$

$\frac{1}{x} = x^{-1}$

$\frac{dx}{dx} = -x^{-2} = -\frac{1}{x^2}$

$$\frac{dy}{du} = \frac{2u}{2\sqrt{u^2+1}} = \frac{u}{\sqrt{u^2+1}} \quad ; \quad \frac{du}{dx} = -\frac{1}{x^2} + 2x$$

نحن نعوض

$$\therefore \frac{dy}{dx} = \frac{u}{\sqrt{u^2+1}} * \left(2x - \frac{1}{x^2}\right) = \frac{\left(\frac{1}{x} + x^2\right)}{\sqrt{\left(\frac{1}{x} + x^2\right)^2 + 1}} * \left(2x - \frac{1}{x^2}\right)$$

Example ② let $y = \frac{u^2-1}{u^2+1}$, $u = \sqrt[3]{x^2+2}$, find $\frac{dy}{dx}$

sol. $\frac{dy}{du} = \frac{4u}{(u^2+1)^2}$, $\frac{du}{dx} = \frac{2x}{3(x^2+2)^{2/3}} = \frac{2x}{3u^2}$

$$= \frac{(u^2+1)2u - (u^2-1) * 2u}{(u^2+1)^2}$$

$$= \frac{\cancel{2u^3} + 2u - \cancel{2u^3} + 2u}{(u^2+1)^2} = \frac{4u}{(u^2+1)^2}$$

$\frac{1}{3} - 1$

$\frac{1-3}{3} = -\frac{2}{3}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{4u}{(u^2+1)^2} \cdot \frac{2x}{3u^2} = \frac{8x}{3u(u^2+1)^2}$$

derivative of parametric equations:

if $y = f(t)$ and $x = g(t)$, and the derivatives $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both exist, then:

$$\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt}$$

$\cancel{\frac{dy}{dt} * \frac{dt}{dx}}$ * x, y دالتين لـ t

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}y\right) = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$$

Example①: find $\frac{dy}{dx}$, if $y = t^2 - 1$ and $x = 2t + 3$

sol. $\cancel{\frac{dy}{dx} = 2t \text{, and } \frac{dx}{dt} = 2}$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

so $\frac{dy}{dt} = 2t$, and $\frac{dx}{dt} = 2$

$$\therefore \frac{dy}{dx} = \frac{2t}{2} = t = \frac{x-3}{2}$$

Another solution:

$x = 2t + 3$

$t = \frac{x-3}{2}$

from $x = 2t + 3$ find $t = \frac{x-3}{2}$

then ~~x=2t~~ $y = \left(\frac{x-3}{2}\right)^2 - 1$

مشتقة داخل القوس

$$\therefore \frac{dy}{dx} = 2\left(\frac{x-3}{2}\right) * \frac{1}{2} = \frac{x-3}{2}$$

لم يتغيا بالمفرد
مثال 2

Example 2 find $\frac{d^2y}{dx^2}$, if $x = t - t^2$ and $y = t - t^3$

sol. $\frac{dx}{dt} = 1 - 2t$ and $\frac{dy}{dt} = 1 - 3t^2$

$$\therefore \frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$\frac{dy'}{dx} = \frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{6t^2-6t+2}{(1-2t)^2}$$

$-6t+12t^2+2-6t^2$

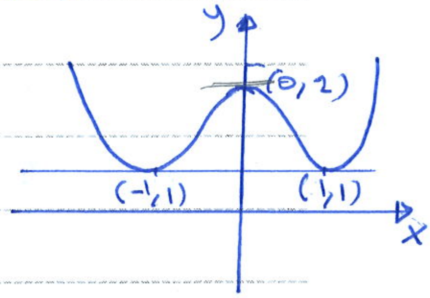
$6t^2-6t+2$

Tangent and normal lines

Example ① dose the curve $y = x^4 - 2x^2 + 2$ has any horizontal tangent? if so, where?

sol. the horizontal tangents, if any, occur where the slope dy/dx is zero.

to find these points, we ~~too~~ should:



1. Calculate dy/dx

$$\frac{dy}{dx} = 4x^3 - 2(2x) = 4x^3 - 4x$$

2. put $\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 4x = 0$

3. solve the equation $\frac{dy}{dx} = 0$ for x:

$$4x^3 - 4x = 0 \Rightarrow 4x(x^2 - 1) = 0$$

either $4x = 0 \Rightarrow x = 0$

or $x^2 - 1 = 0 \Rightarrow x = \pm 1$

so the curve has horizontal tangents at $x = 0$, $x = -1$ and $x = 1$.

the ~~curve~~ corresponding points on the curve are $(0,2)$, $(-1,1)$ and $(1,1)$.

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equation of the

Example ②. find the tangent and normal to the curve $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$.

sol.

$$x^2 - xy + y^2 = 7$$

$$2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(2y - x) = y - 2x$$

$$\therefore \frac{dy}{dx} = \frac{y-2x}{2y-x}$$

the derivative at $x = -1$ and $y = 2$ is

$$\frac{dy}{dx} = \frac{y-2x}{2y-x} = \frac{2-2(-1)}{2(2)-(-1)} = \frac{2+2}{4+1} = \frac{4}{5}$$

the tangent to the curve at the point $(-1, 2)$ is

$$y - y_1 = m(x - x_1) \Rightarrow y - 2 = \frac{4}{5}(x - (-1))$$

$$\Rightarrow y = \frac{4}{5}x + \frac{4}{5} + 2 \Rightarrow \therefore y = \frac{4}{5}x + \frac{14}{5}$$

the normal to the curve at the point $(-1, 2)$ is (slope of normal is $\left(\frac{-1}{m}\right)$):

$$y - y_1 = m(x - x_1) \Rightarrow y - 2 = \frac{-5}{4}(x - (-1))$$

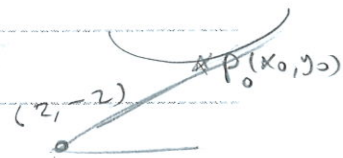
$$\Rightarrow y = -\frac{5}{4}x - \frac{5}{4} + 2 \Rightarrow \therefore y = -\frac{5}{4}x + \frac{3}{4}$$

لم تعطى بالجامعة 2015

سلسلة ٢٠٠ P48 Q5

Example: find the equation of a tangent of the hyperbola (قطع زائد) $x^2 - y^2 = 16$ and pass through the point $(2,-2)$.

sol: let $p_0(x_0, y_0)$ is the point of ~~tangent~~ contact for the tangent



substitute the point $p_0(x_0, y_0)$ in $x^2 - y^2 = 16$

$$\Rightarrow x_0^2 - y_0^2 = 16 \quad \text{——} \ (1)$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$ الميل

at point $(x_0, y_0) \Rightarrow \dfrac{dy}{dx} = \dfrac{x_0}{y_0} = m$

the slope from two points (x_0, y_0) and $(2,-2)$

$$\frac{y_0+2}{x_0-2} \qquad \frac{\Delta y}{\Delta x} = \frac{y_0-(-2)}{x_0-2} = \frac{y_0+2}{x_0-2}$$

$$\therefore \frac{y_0+2}{x_0-2} = \frac{x_0}{y_0} = m$$

$$\therefore (y_0+2)y_0 = x_0(x_0-2)$$

$$\Rightarrow y_0^2 + 2y_0 = x_0^2 - 2x_0$$ نفس الطرف الايسر للمعادلة

$$2x_0 + 2y_0 = x_0^2 - y_0^2 = 16$$

or $\quad 2x_0 + 2y_0 = 16 \Rightarrow \boxed{x_0 + y_0 = 8}$ —— (2)

from equation (1) and (2)

$x_0 + y_0 = 8 \Rightarrow y_0 = 8 - x_0$ sub. in eq. (1)

$$x_0^2 - (8-x_0)^2 = 16 \Rightarrow \cancel{x_0^2} - (64 - 16x_0 + \cancel{x_0^2}) = 16$$

$$-64 + 16x_0 = 16 \Rightarrow x_0 = \frac{80}{16} = 5$$

and $y_0 = 3 \quad (5 + y_0 = 8 \Rightarrow y_0 = 3)$

$\therefore$ the point of contact is $(5,3)$

لم يعطى بالوزارات
2015

$\therefore$ the slope of tangent

$$\frac{dy}{dx} = \frac{x_0}{y_0} = \frac{5}{3}$$

the equation of tangent

$$y - y_1 = m(x - x_1)$$

$$\therefore \; y - 3 = \frac{5}{3}(x - 5)$$

سؤال
P 50
Q 10

Example: find the angles between the two curves:

① $y^2 = 4x$ and ② $2x^2 = 12 - 5y$

sol. the intersection points are $P_1(1, 2)$ and $P_2(4, -4)$

استخرج نقاط التقاطع

for equation ① $\frac{dy}{dx} = \frac{2}{y}$

for equation ② $\frac{dy}{dx} = \frac{-4x}{5}$

at point $P_1(1, 2)$

$$m_1 = \frac{dy}{dx} = \frac{2}{y} = \frac{2}{2} = 1 \; ; \quad m_2 = \frac{dy}{dx} = \frac{-4x}{5} = \frac{-4(1)}{5} = \frac{-4}{5}$$

at point $P_2(4, -4)$

$$m_1 = \frac{2}{y} = \frac{2}{-4} = \frac{-1}{2} \; ; \quad m_2 = \frac{-4x}{5} = \frac{-4(4)}{5} = \frac{-16}{5}$$

at point P_1 ; $\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{1 - \left(\frac{-4}{5}\right)}{1 + (1)\left(\frac{-4}{5}\right)}$

$$\therefore \tan\phi = \frac{1 + \frac{4}{5}}{1 - \frac{4}{5}} = 9$$

$$\therefore \phi = 83^\circ\, 40'$$

at point P_2, $\tan\phi = \frac{-\frac{1}{2} - \left(-\frac{16}{5}\right)}{1 + \left(\frac{-1}{2}\right)\left(\frac{-16}{5}\right)} = 1.0385$

$$\therefore \phi = 46^\circ\, 5'$$

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H.W

P.40 Q.20

① if $y = x^2 - 4x$ and $x = \sqrt{2t^2+1}$

find $\frac{dy}{dt}$ at $t = \sqrt{2}$

$$\frac{dy}{dx} = 2(x-2) \; ; \; \frac{dx}{dt} = \frac{2t}{(2t^2+1)^{1/2}}$$

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{4t(x-2)}{(2t^2+1)^{1/2}}$$

at $t = \sqrt{2} \Rightarrow x = \sqrt{5}$

$$\therefore \frac{dy}{dx} = \frac{4\sqrt{2}(\sqrt{5}-2)}{\sqrt{5}} = \frac{4\sqrt{2}}{5}(5 - 2\sqrt{5})$$

P.44 Q2

② if $x^2 - xy + y^2 = 3$ find y' and y''

$$2x - xy' - y + 2yy' = 0 \Rightarrow y' = \frac{2x-y}{x-2y}$$

$$y'' = \frac{(x-2y)\frac{d}{dx}(2x-y) - (2x-y)\frac{d}{dx}(x-2y)}{(x-2y)^2}$$

$$= \frac{(x-2y)(2-y') - (2x-y)(1-2y')}{(x-2y)^2}$$

$$= \frac{3xy' - 3y}{(x-2y)^2} = \frac{3x\left(\frac{2x-y}{x-2y}\right) - 3y}{(x-2y)^2} = \frac{6(x^2-xy+y^2)}{(x-2y)^3} = \frac{18}{(x-2y)^3}$$

28/12/2013

derivative of trigonometric function

if u is a function of x, then:

1. $\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$

2. $\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$

3. $\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$

4. $\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$

5. $\frac{d}{dx}\sec u = \sec u \cdot \tan u \cdot \frac{du}{dx}$

6. $\frac{d}{dx}\csc u = -\csc u \cdot \cot u \cdot \frac{du}{dx}$

Example: find $\frac{dy}{dx}$ of the following functions:

1. $y = x^2 - \sin x$

sol. $\frac{dy}{dx} = 2x - \cos x$

الاول × مشتقة الثاني + الثاني × مشتقة الاول

2. $y = \sin x \cdot \cos x$

sol. $\frac{dy}{dx} = \sin x \cdot (-\sin x) + \cos x \cdot \cos x$

$= -\sin^2 x + \cos^2 x$

3- $y = \dfrac{\cos x}{1-\sin x}$

Sol. $\dfrac{dy}{dx} = \dfrac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$

$= \dfrac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$ $\sin^2 x + \cos^2 x = 1$

$= \dfrac{-\sin x + 1}{(1-\sin x)^2} = \dfrac{1-\sin x}{(1-\sin x)^2} = \dfrac{1}{1-\sin x}$

4- $y = \sin(\underbrace{1 + \tan 2x})$

sin. عندي زاوية

Sol. $\dfrac{dy}{dx} = \cos(1+\tan 2x) \cdot \sec^2 2x \cdot 2$

$= 2\sec^2 2x \cdot \cos(1+\tan 2x)$

5- $xy + \sin y = 5x$

Sol. $\underline{xy' + y} + \cos y \cdot y' = 5$

الأول × مشتقة الثاني + الثاني × مشتقة الأول

$\Rightarrow y'(x + \cos y) = 5 - y \quad \Rightarrow y' = \dfrac{5-y}{x + \cos y}$

6- $y = \sec^2 5x$ توضيح أس 2 $= (\sec 5x)^2$

Sol. $\dfrac{dy}{dx} = 2\sec 5x \cdot \sec 5x \cdot \tan 5x \cdot 5$

$= 10\sec^2 5x \cdot \tan 5x$

Example : if $y = \sec x$, prove that

a- $y' = \sec x \cdot \tan x$

b- $y'' + y = 2y^3$

sol. a-

$$y = \sec x = \frac{1}{\cos x}$$

$$\therefore \frac{dy}{dx} = \frac{\cos x(0) - (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \cdot \tan x$$

$$\therefore \frac{d}{dx} \sec x = \sec x \cdot \tan x$$

b- $y = \sec x$

$y' = \sec x \cdot \tan x$

الاول × مشتقة الثاني + الثاني × مشتقة الاول

$y'' = \sec x \cdot \sec^2 x + \tan x \cdot \underline{\sec x \cdot \tan x}$

$= \sec^3 x + \sec x \cdot \tan^2 x$

$= \sec^3 x + \sec x \cdot (\sec^2 x - 1)$ $\qquad 1 + \tan^2 x = \sec^2 x$

$= \sec^3 x + \sec^3 x - \sec x$ $\qquad 1 + \tan^2 x = \sec^2 x$

$y'' = 2\sec^3 x - \sec x$ $\qquad \therefore \tan^2 x = \sec^2 x - 1$

$\therefore y'' + y = 2y^3$ $\qquad$ **ok**

$y'' = 2\sec^3 x - \sec x$

$y'' = 2y^3 - y$

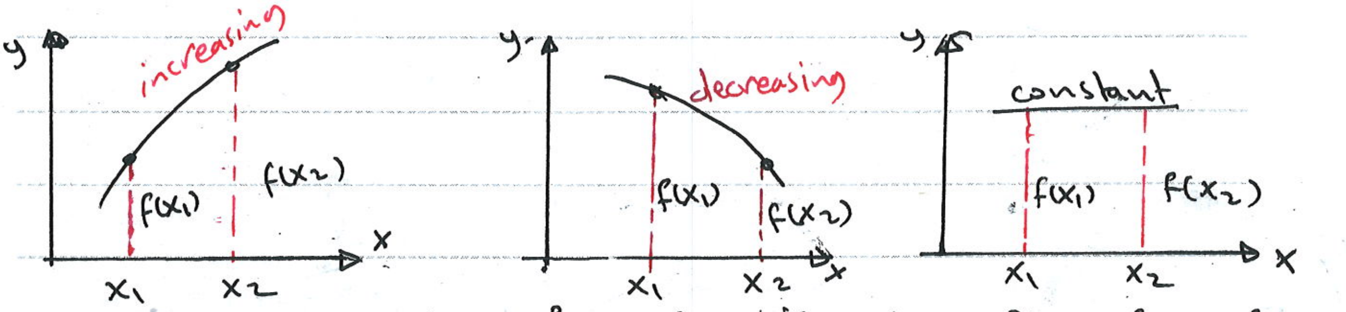
$\therefore y'' + y = 2y^3$

18/5/2015

Applications of Derivatives

1- analysis of functions:

a. increase, decrease and local (relative) extremea:



$f(x_1) < f(x_2)$ if $x_1 < x_2$
graph has +ve slope

$f(x_1) > f(x_2)$ if $x_1 < x_2$
graph has -ve slope

$f(x_1) = f(x_2)$ for all x_1 and x_2
graph has zero slope

* Theorem. (first derivative test for increasing and ~~decr~~ decreasing).

let f is continuous on $[a,b]$ and differentiable on (a,b):

1. if $f'(x) > 0$ at each point $x \in (a,b)$ then f is increasing on $[a,b]$.
2. if $f'(x) < 0$ at each point $x \in (a,b)$, then f is decreasing on $[a,b]$.
3. if $f'(x) = 0$ at each point $x \in (a,b)$, then f is constant on $[a,b]$.

* Definition: (local minima, local ~~maxima~~ Maxima)

A function f has a local maximum value at interior point c of its domain if $f(x) \leq f(c)$ for all x in some open interval containing c.

A function f has a local minimum value at interior point c of its domain if $f(x) \geq f(c)$ for all x in some open interval containing c.

10/1 2016 7/1 2016

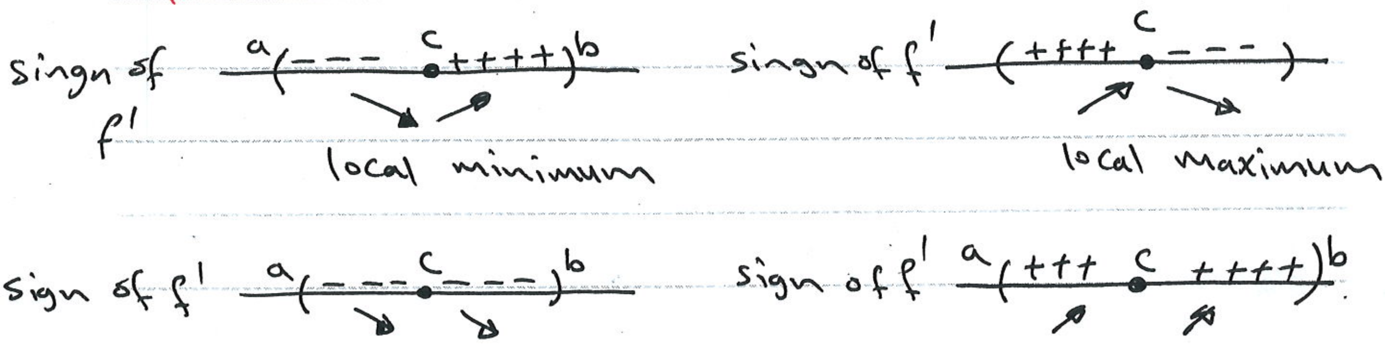
* Definition: (critical point)

An interior point of the domain of a function f where f' is <u>zero</u> or <u>undefined</u> is critical point of f.

* Theorem: (first derivative test for local extrema)

let c is a critical point of f.. moving a cross c from left to right:

1.



f has no local extremum

* Theorem: (second derivative test for local extrema).

let f is twice differentiable at c:

1- if $f'(c)=0$ and $f''(c)>0$, then f has a local minmum at $x=c$

نقطة حرجة: لما المشتقة = صفر

2- if $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at $x=c$

3- if $f'(c)=0$ and $f''(c)=0$, then the test <u>fails</u>. the function f may have a local maximum, a local minimum, or neither at $x=c$.

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9.

Example: find the intervals on which the function $f(x) = 3x^{5/3} - 15x^{2/3}$ is increasing and the intervals on which decreasing, and also locate the maximum and the minimum points.

solution:

$$f(x) = 3x^{5/3} - 15x^{2/3}$$

$$f'(x) = 3 * \frac{5}{3} x^{2/3} - 15 * \frac{2}{3} x^{-1/3}$$

$$= 5x^{2/3} - 10x^{-1/3} = \frac{5(x-2)}{x^{1/3}}$$

$$5x^{2/3} - \frac{10}{x^{1/3}} = \frac{5x^{2/3} \cdot x^{1/3} - 10}{x^{1/3}} = \frac{5x - 10}{x^{1/3}} = \frac{5(x-2)}{x^{1/3}}$$

sign of f': $++++\ \overset{0}{|}\ ----\ \overset{2}{|}\ ++++$

Max. at $x = 0$ min. at $x = 2$.

ممكن أنا نقول أن الصفر هذا يعني البسط يساوي صفر لأن $\frac{0}{1} = 0$ ∴ العدد الناتج هنا هنا بسط يكون = صفر

$$f'(x) = 0 \Rightarrow \frac{5(x-2)}{x^{1/3}} = 0$$

نكن نضرب وسطين × طرفين

$5(x-2) * 1 = x^{1/3} * 0$

$\Rightarrow 5(x-2) = 0$

$$\Rightarrow x - 2 = 0 \Rightarrow x = 2$$

المقام = صفر دكل

and f' does not exist (undefined) at $x = 0$
since the function is continuous at $x = 0$ and $x = 2$ then,
∴ f is increasing on $(-\infty, 0]$ and $[2, \infty)$.
and f is decreasing on $[0, 2]$.

التعويض في كل فترة

at $x = -1$ $\frac{5(-1-2)}{\sqrt[3]{-1}} = \frac{-}{-} = +ve$

at $x = 3$ $\frac{5(3-2)}{\sqrt[3]{3}} = \frac{+}{+} = +ve$

at $x = 1$ $= \frac{5(1-2)}{\sqrt[3]{+1}} = \frac{-}{+} = -ve$

at $x = 0 \Rightarrow f(0) = 0$
∴ $(0, 0)$ is a local max.
at $x = 2 \Rightarrow f(2) = -14.3$
∴ $(2, -14.3)$ is a local min.

or by second derivative test:

$$f''(x) = \frac{10}{3}x^{-1/3} + \frac{10}{3}x^{-4/3} = \frac{10}{3x^{1/3}} + \frac{10}{3x^{4/3}} = \frac{10}{3}\left(\frac{x+1}{x^{4/3}}\right)$$

$f''(0) = \frac{10}{3}\left(\frac{0+1}{0^{4/3}}\right) \Rightarrow f''(0)$ does not exist, so this test is failed.

$f''(2) = \frac{10}{3}\left(\frac{2+1}{2^{4/3}}\right) = 3.96 > 0$, so the function has a min. point at $x = 2$.

$$\frac{10}{3x^{1/3}} + \frac{10}{3x^{4/3}}$$

$$\frac{10}{3}\left(\frac{x+1}{x^{4/3}}\right) \cancel{\neq}$$

$$\frac{x^{4/3}}{x^{1/3}} = x^{4/3} \cdot x^{-1/3} = x$$

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b) concavity and inflection points (I.p).

if f is differentiable on an open interval I, then f is said to be: ملاذا التقعر لأعلى
* concave up on I if f' is increasing on I.
* concave down on I if f' is decreasing on I.

*** Theorem: (second derivative test for concavity)**

let f be twice differentiable on an open interval I:

1. if $f''(x) > 0$ on I, then f is concave up on I. تحدب تقعر
2. if $f''(x) < 0$ on I, then f is concave down on I. تقعر تحدب

*** Definition: (inflection point)**

A point p on a curve $y = f(x)$ is called an inflection point (I.p) if f is continuous there and the curve changes from concave up to concave down or from concave down to concave up at p.

let f be continuous and twice differentiable at $x = c$. If $f''(c) = 0$ or undefined, then f may have an inflection point at $x = c$.

Example: Find the intervals on which the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is concave up and concave down, then, if any, locate the inflection points.

solution:

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$$

$$f'(x) = 12x^3 + 12x^2 - 24x$$

$$f''(x) = 36x^2 + 24x - 24$$

$$f''(x) = 0 \Rightarrow 36x^2 + 24x - 24 = 0$$

$$\Rightarrow 3x^2 + 2x - 2 = 0$$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

sign of f''

-1.22 ∩ 0.55

$+++$ $- - - - -$ $+++$

∪ ∪

where $A = 3,\ B = 2,\ C = -2$

$$\therefore x = \frac{-2 \pm \sqrt{2^2 - 4 * 3 * (-2)}}{2 * 3} = \frac{-1 \pm \sqrt{7}}{3}$$

either $x = \frac{-1-\sqrt{7}}{3} = -1.22 \Rightarrow y = -16.36$

or $x = \frac{-1+\sqrt{7}}{3} = 0.55 \Rightarrow y = -0.68$

at $x = 0$

$3(0)^2 + 2(0) - 2 = 0$

$f'' < 0$

concave down

at $x = -2$

$3(-2)^2 + 2(-2) - 2 = 0$

$12 - 4 - 2 = 6$

$\therefore f'' > 0$

$\therefore$ concave up

at $x = 1$

$f'' > 0$

concave up

so, f is concave up on intervals $(-\infty, -1.22)$ and $(0.55, \infty)$, and f is concave down on interval $(-1.22, 0.55)$. It has I.p. at points $(-1.22, -16.36)$ and $(0.55, -0.68)$.

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Example: if $y = \frac{x^3}{3} + \frac{x^2}{2} - 6x + 8$, find:

a) critical points

b) the intervals in which the function increases and decreases.

c) maximum and minimum values of y.

d) the intervals in which the function is concave up or concave down.

e) the inflection points.

solution: a) critical points:

$y = \frac{x^3}{3} + \frac{x^2}{2} - 6x + 8 \Rightarrow y' = \frac{3x^2}{3} + \frac{2x}{2} - 6$

$\therefore y' = x^2 + x - 6$

التحليل بالتربيع

$y' = 0 \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0$

either $x + 3 = 0 \Rightarrow x = -3 \Rightarrow y = \frac{43}{2}$

or $x - 2 = 0 \Rightarrow x = 2 \Rightarrow y = \frac{2}{3}$

$-3 \quad +2$

$\therefore$ the critical points are $(-3, \frac{43}{2}), (2, \frac{2}{3})$

b)

the function increases on intervals: $(-\infty, -3]$ and $[2, \infty)$

the function decreases on interval: $[-3, 2]$

f' s.c. $+++ \; -3 \; --- \; +2 \; ++++$

at $x = 0$

at $x = -4$

at $x = 3$

at $x = -3$

$f' = 0$ ∴ ثابت

at $x = 2$

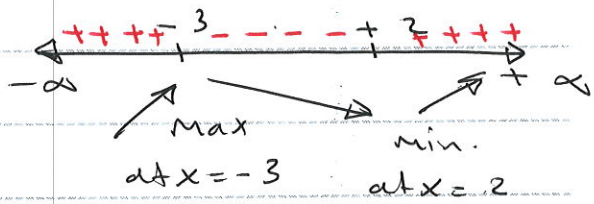
$f' = 0$

∴ ثابت

c)

the function has max. value of (y).

at $x = -3 \Rightarrow y = \frac{43}{2}$



the function has min. values of (y)

at $x = 2 \Rightarrow y = \frac{12}{3}$

or by second derivative test.

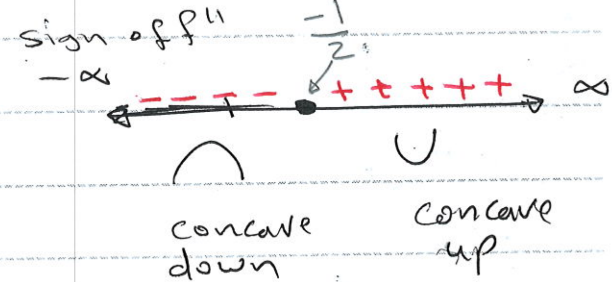
$y' = x^2 + x - 6 \Rightarrow y''(x) = 2x + 1$

$y''(-3) = 2(-3) + 1 = -5 < 0$ there is a max. value at $x = -3$

$y''(2) = 2(2) + 1 = 5 > 0$ there is a min. value at $x = 2$

d)

$y'' = 2x + 1$



نقاط الانقلاب $y'' = 0 \Rightarrow 2x + 1 = 0$

$\Rightarrow x = \frac{-1}{2} \Rightarrow y = \frac{133}{12}$

the function is concave down on interval $(-\infty, -\frac{1}{2})$

the function is concave up on interval $(-\frac{1}{2}, \infty)$

e) the ~~function~~ inflection point is $(-\frac{1}{2}, \frac{133}{12})$

Example: Find the absolute maximum and minimum values of $y = x^{2/3}$ on the interval $-2 \leq x \leq 3$.

$y' = 0$

$\frac{2}{3\sqrt[3]{x}} = 0$

$2 = 3\sqrt[3]{x} \cdot 0$

$\Rightarrow 2 = 0$ $\therefore$ $y' \neq 0$

Solution: $y = x^{2/3}$

$$y' = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}} \Rightarrow y' \neq 0$$

$- - - \overset{0}{|} + + +$

but y' is undefined at $x = 0$.

المقام صفر

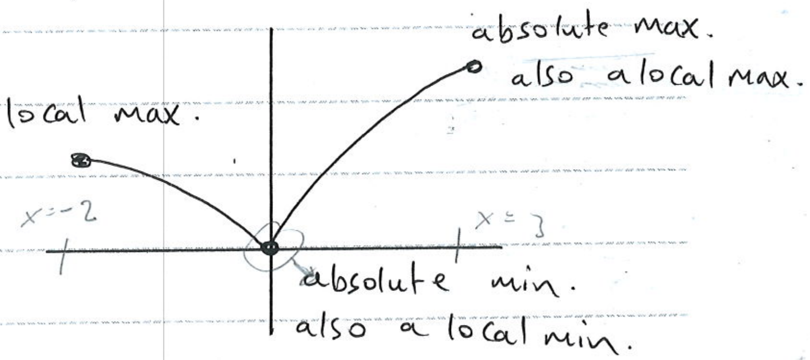
$3\sqrt[3]{x} = 0 \Rightarrow x = 0$

critical points value $\boxed{f(0) = 0}$

end points values $f(-2) = (-2)^{\frac{2}{3}} = 4^{1/3}$

$f(3) = (3)^{\frac{2}{3}} = 9^{1/3}$

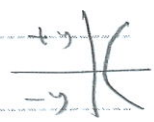
the function's max. value is $9^{1/3}$ at $x = 3$



13/1/2016

Strategy for graphing curves.

1. Identify the domain and range of the curve.
2. Identify any symmetry of the curve may have.
3. Find (if any) critical points and identify where the curve is increasing and where it is decreasing.
4. Find (if any) inflection points and determine the concavity of the curve.
5. Identify any asymptotes.
6. plot key points, such as intercepts and the points found in steps 3 and 4, and sketch the curve.



Symmetry tests for graphs:

1. Symmetry about x-axis: iff $(x,-y) = f(x,y)$


2. Symmetry about y-axis: if $f(-x,y) = f(x,y)$.
3. Symmetry about the origin: if $(-x,-y) = f(x,y)$

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Example : رسم تخطيطي Sketch the graph of $y = f(x) = \frac{x^2+1}{x}$

solution : ① domain and rang.

$x \neq 0 \Rightarrow D_f = (-\infty, \infty) / \{0\}$

put the function as $x = f(y)$.

$y = \frac{x^2+1}{x} \Rightarrow xy = x^2+1 \Rightarrow \boxed{x^2 - xy + 1 = 0}$

$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$; where $A = 1$, $B = -y$ and $C = 1$

$x = \frac{-(-y) \pm \sqrt{(-y)^2 - 4(1)(1)}}{2(1)} = \frac{y \pm \sqrt{y^2-4}}{2}$

$\therefore y^2 - 4 \geq 0 \Rightarrow y^2 \geq 4 \Rightarrow$ either $y \geq 2$,

or $-y \geq 2 \Rightarrow y \leq -2$

$-2 \quad +2$

$\therefore R_f = (-\infty, -2] \cup [2, \infty)$

② symmetry

$f(-x) = \frac{(-x)^2+1}{(-x)} = -\frac{x^2+1}{x} = -f(x) \rightarrow -y$

$\therefore -\frac{x^2+1}{x} = -f(x)$

يختصروا (-) من الطرفين $\frac{x^2+1}{x} = f(x)$

So the curve has symmetry about the origin

$f(x,-y) \Rightarrow -y = \frac{x^2+1}{x}$

$\therefore y = -\frac{x^2+1}{x} \neq f(x,y)$

$f(-x,-y) \Rightarrow -y = \frac{(-x)^2+1}{-x}$

$-y = \frac{x^2+1}{-x}$

$\therefore y = \frac{x^2+1}{x} = f(x)$

$f(-x,y) \Rightarrow y = \frac{(-x)^2+1}{-x} = \frac{x^2+1}{-x} \neq f(x,y)$

$x\,(0) = x^2+1$
$\Rightarrow 0 = x^2+1 \Rightarrow x^2+1 \neq 0$

3. Intercepts

put $y=0 \Rightarrow$ محور x $0 = \frac{x^2+1}{x} \Rightarrow x^2+1 \neq 0 \Rightarrow$ (No intersection)

put $x=0 \Rightarrow y = \frac{0+1}{0} \Rightarrow y = \infty \Rightarrow$ (No intersection)

مع محور y

مهم

4. Asymptotes

- vertical asymptotes

$y = \frac{x^2+1}{x} \Rightarrow \therefore x=0$ (y-axis) is a vertical asymptote

- horizontal asymptotes

$x = \frac{y \pm \sqrt{y^2-4}}{2} \Rightarrow$ there is no horizontal asymptote

- oblique asymptotes: because of that the function is a rational function with numerator (البسط) is one greater than the denominator, there is an oblique asymptote (المقام)

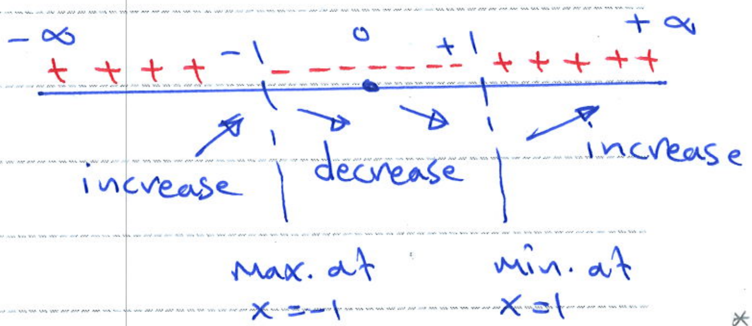
$$y = \frac{x^2+1}{x} = x + \frac{1}{x}$$

$\therefore y = x$ is an oblique asymptote.

$$\begin{array}{r|l} & x \\ \hline x & x^2+1 \\ & -x^2 \\ \hline & +1 \end{array}$$

$x + \frac{1}{x}$

5. Increase, decrease, and local extrema:



$$y = \frac{x^2+1}{x} = x + \frac{1}{x}$$

$$y' = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

put $y' = 0 \Rightarrow \frac{x^2-1}{x^2} = 0 \Rightarrow \frac{(x-1)(x+1)}{x^2} = 0$

* نقول أن البسط يساوي صفر هنا يعني أن البسط يساوي صفر، ∴ نضع البسط يساوي صفر $(x-1)(x+1)=0$ ونكمل الحل. الحالة الثانية نقول المقام يجعل الدالة غير معرفة ∴ نضع المقام يساوي صفر

$\therefore$ either $x=1 \Rightarrow y=2$ or $x=-1 \Rightarrow y=-2$

and y' is undefined at $x=0$

so the curve increases on $(-\infty,-1]$ and $[1,\infty)$,

and it decreases on $[-1,0)$ and $(0,1]$,

and has local max. at point $(-1,-2)$,

and has local min. at point $(1,2)$.

6. concavity and inflection points.

$y''=0$

$y'' = \frac{2}{x^3} \Rightarrow y''$ is undefined at $x=0$

$2 \neq 0$

so the curve is concave down

on interval $(-\infty, 0)$

لذلك لا توجد نقطة انعطاف

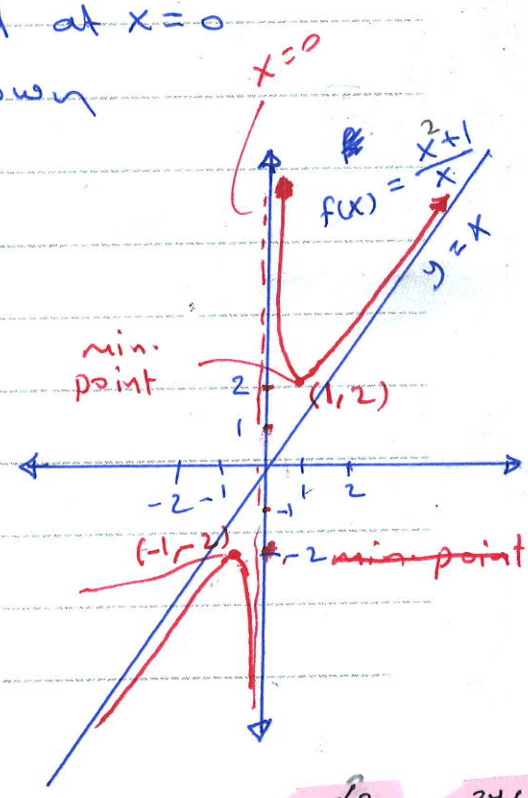
and concave up on interval

$(0,\infty)$, and there is no

$y''=0$

inflection point because $f(0)$

is not defined $(0 \notin D_f)$.



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<u>Example</u> : sketch the graph of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

<u>Solution</u> : ① domain and range

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4 \Rightarrow \sqrt[3]{x^2} + \sqrt[3]{y^2} = 4$$ نضرب الطرفين بالأس 3

$$\sqrt[3]{y^2} = 4 - \sqrt[3]{x^2} \Rightarrow y^2 = (4 - \sqrt[3]{x^2})^3$$

$$\Rightarrow y = \pm \sqrt{(4 - \sqrt[3]{x^2})^3}$$

$$\therefore (4 - \sqrt[3]{x^2})^3 \geq 0 \Rightarrow 4 - \sqrt[3]{x^2} \geq 0$$

نضرب الطرفين (-1) لذلك تنقلب اشارة الاكبر الى اصغر

$$-\sqrt[3]{x^2} \geq -4 \Rightarrow \sqrt[3]{x^2} \leq 4 \Rightarrow x^2 \leq 64$$

either $x \leq 8$, <u>or</u> $-x \leq 8 \Rightarrow x \geq -8$

$$\Rightarrow D_f = [-8, 8].$$

put the function as $x = f(y)$;

نفس الكلام السابق

$$x = \pm \sqrt{(4 - \sqrt[3]{y^2})^3} \Rightarrow R_f = [-8, 8].$$

② symmetry

$$f(x, -y) \Rightarrow \sqrt[3]{x^2} + \sqrt[3]{(-y)^2} = 4$$

$$\Rightarrow \sqrt[3]{x^2} + \sqrt[3]{y^2} = 4 = \underline{f(x, y)}.$$

$$f(-x, y) \Rightarrow \sqrt[3]{(-x)^2} + \sqrt[3]{y^2} = 4$$

$$\Rightarrow \sqrt[3]{x^2} + \sqrt[3]{y^2} = 4 = \underline{f(x, y)}.$$

$$f(-x, -y) \Rightarrow \sqrt[3]{(-x)^2} + \sqrt[3]{(-y)^2} = 4$$

$$\Rightarrow \sqrt[3]{x^2} + \sqrt[3]{y^2} = 4 = \underline{f(x, y)}.$$

so the curve is symmetric about the x-axis, y-axis, and the origin.

~~intersep~~ intersection n. نقطة تقاطع ، ملتقى طرق

③ intercepts n. اعتراض

put $y = 0 \Rightarrow x = \pm\sqrt{(4-\sqrt[3]{0^2})^3} = \pm\sqrt{4^3}$

$= \pm\sqrt{64} = \pm 8$

put $x = 0 \Rightarrow y = \pm\sqrt{(4-\sqrt[3]{0^2})^3} = \pm 8$

④ asymptotes

the curve has no asymptotes.

?
ما هي؟
الدالة ليست كسرية ، لا يوجد لها مقام

⑤ increase, decrease, and local extrema:

$$x^{2/3} + y^{2/3} = 4 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow y^{-1/3}\frac{dy}{dx} = -x^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} \Rightarrow \frac{dy}{dx} = -\frac{\left[\sqrt{(4-\sqrt[3]{x^2})^3}\right]^{1/3}}{x^{1/3}}$$

نعوض عن قيمة y

عوض فقط

$y = \pm\sqrt{(4-\sqrt[3]{x^2})^3}$

فقط نعوض بـ +

وليس بالـ −

يكون نفس الناتج

لأن المعادلة تساوي صفر

فالإشارة تضرب الطرفين

فتختفي الإشارة

$$\Rightarrow \frac{dy}{dx} = -\frac{\left[(4-\sqrt[3]{x^2})^{3/2}\right]^{1/3}}{x^{1/3}}$$

$\frac{3}{2} * \frac{1}{3} = \frac{1}{2}$

$$\Rightarrow \therefore \frac{dy}{dx} = -\frac{\sqrt{4-\sqrt[3]{x^2}}}{\sqrt[3]{x}}$$

put $\frac{dy}{dx} = 0 \Rightarrow \sqrt{4-\sqrt[3]{x^2}} = 0$

$\Rightarrow \sqrt[3]{x^2} = 4 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8 \Rightarrow y = 0$

and $\frac{dy}{dx}$ is undefined at $x = 0$

Sign of f': -8 + + + + 0 − − − − $+8$

increase → decrease

$+8$ ، -8 كـ domain

لأنها أقل من 8 وأكبر من −8

$+8$

* عند x=0 نلاحظ نقطة max. undefined من هي نقطة
ولكن هذه النقطة
وليست من y'=0
فعندها تكون y'=0 يكون شكل المنحنى
ولكن عبارة عن النقطة x=0
undefined من y'
فإن شكل المنحنى عند x=0 لا يوجد مماس أفقي

so the curve increases on $[-8,0]$ and decreases on $[0,8]$ in the first and second quadrants, and due to symmetry it decreases on $[-8,0]$ and increases on $[0,8]$ in the third and fourth quadrants.

The curve has a local max. at point $(0,8)$, and due to symmetry it has local min. at point $(0,-8)$.

6- concavity and inflection points.

$$\frac{dy}{dx} = -\frac{\sqrt{4-\sqrt[3]{x^2}}}{\sqrt[3]{x}} = -\frac{(4-x^{2/3})^{1/2}}{x^{1/3}}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{x^{1/3} * \left[\frac{1}{2}(4-x^{2/3})^{-1/2}\left(-\frac{2}{3}x^{-1/3}\right)\right] - (4-x^{2/3})^{1/2}\left(\frac{1}{3}x^{-2/3}\right)}{(x^{1/3})^2}$$

$$= \frac{4}{3\sqrt[3]{x^4}\sqrt{4-\sqrt[3]{x^2}}}$$

نأخذها ٢ بالبسط وهي باكالستي اما الذي بالاسفل وهي $\sqrt[3]{x^4}$ بالبسط أما الذي وهي $\sqrt{4-\sqrt[3]{x^2}}=0$

$\sqrt[3]{x^4}=0 \Rightarrow x=0$

$\sqrt{4-\sqrt[3]{x^2}}=0$
$4-\sqrt[3]{x^2}=0$
$\sqrt[3]{x^2}=4$
$x^2=64$
$\therefore x=\pm 8$

$\frac{d^2y}{dx^2} \neq 0$, but $\frac{d^2y}{dx^2}$ is undefined if $x=0$ and $\sqrt{4-\sqrt[3]{x^2}}=0 \Rightarrow x=\pm 8$

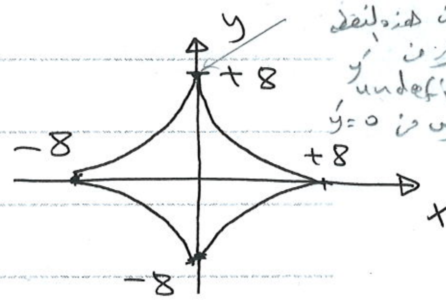
sign of f''

-8 ++ ++ 0 ++++ $+8$

concave up — concave up

so, the curve is concave up on intervals $(-8,0)$ and $(0,8)$ in the first and second quadrants, and due to symmetry it is concave down on $(-8,0)$ and $(0,8)$ in the third and fourth quadrants.

لماذا ليس خط أفقي لأن هذه النقطة عندها y' undefined وليس $y'=0$



The curve has no inflection points.

24/2 2016

H.W

1. Find the intervals on which the following functions are ~~a~~ increasing and decreasing, also locate the max. and the min.

a) $f(x) = x^2 - 4x + 3$

Sol. $f'(x) = 2x - 4 = 2(x-2)$

put $f' = 0 \Rightarrow 2(x-2) = 0 \Rightarrow x = 2$ critical point

Since f is continuous at $x = 2$

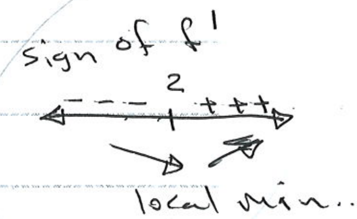
$\therefore f$ is decreasing on $(-\infty, 2]$,

and f is increasing on $[2, \infty)$.

$f(2) = 2^2 - 4 * 2 + 3 = -1$

$\therefore (2, -1)$ is a minimum point.

$f''(x) = 2 > 0$, so f has min. point at $x = 2$.



b) $f(x) = x^3$

Sol. $f'(x) = 3x^2$

put $f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$ ~~point~~ (critical point)

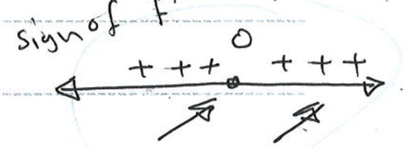
Since f is continuous at $x = 0$

$\therefore f$ is increasing on $(-\infty, 0]$,

and f is increasing on $[0, \infty)$.

so f is increasing over ~~ent~~ interval $(-\infty, \infty)$

thus f has no local extremum at $x = 0$.



f has no local extremum

or by second derivative test.

$f''(x) = 6x \Rightarrow f''(0) = 6(0) = 0 \leftarrow$ No indication.

② Find the intervals on which the following functions are concave up and concave down, then, if any, locate the inflection points.

a) $f(x) = x^2 - 4x + 3$

sol. $f'(x) = 2x - 4$, and $f''(x) = 2 > 0$

since $f''(x) > 0$ for all x, the function f is concave up on the interval $(-\infty, \infty)$.

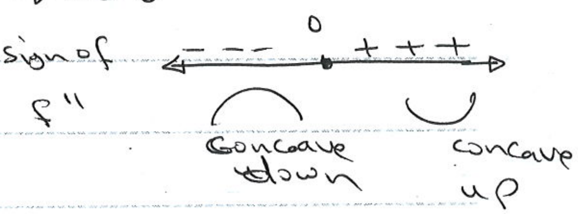
also $f''(x) \neq 0$ for all x, the function f does not have inflection points.

b) $f(x) = x^3$

sol. $f'(x) = 3x^2$, and $f''(x) = 6x$

put $f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$

$\Rightarrow y = 0$



so, f is concave down on $(-\infty, 0)$.

and f is concave up on $(0, \infty)$.

at $x < 0 \Rightarrow f''(x) = -ve$

at $x > 0 \Rightarrow f''(x) = +ve$

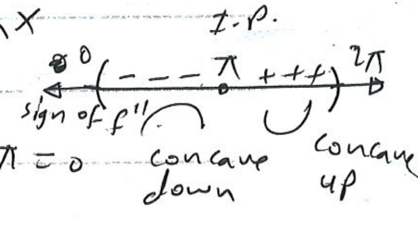
$(0,0)$ is the inflection point.

c) $f(x) = \sin x$ on $[0, 2\pi]$

sol. $f'(x) = \cos x$, and $f''(x) = -\sin x$

put $f''(x) = 0 \Rightarrow -\sin x = 0 \Rightarrow x = \pi$

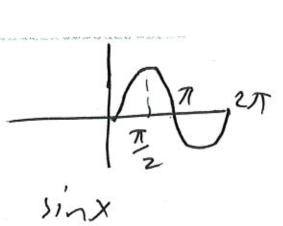
$\Rightarrow y = \sin \pi = 0$



so, f is concave down on $(0, \pi)$.

and f is concave up on $(\pi, 2\pi)$.

$(\pi, 0)$ is the inflection point.



sin x

تش ن
Calculus
P 291
Ex. 5.6

– Drawing a graph of a rational function

draw a graph of $f(x) = x + \dfrac{25}{x}$,

<u>solution</u>

domain $R / \{0\}$

$$f'(x) = 1 - \frac{25}{x^2} = \frac{x^2 - 25}{x^2}$$

$$= \frac{(x-5)(x+5)}{x^2}$$

only the critical numbers are $x = -5$ and $x = 5$

$x = 0$ is not critical number ←

السبب أن $x=0$ لا تنتمي الى مجال الدالة

sign of $f'(x)$: $+++ \overset{-5}{|} --- \overset{0}{|} --- \overset{+5}{|} ++++$

$$f'(x) = 1 - \frac{25}{x^2}$$

$$f''(x) = 0 - \frac{x^2(0) - 25(2x)}{x^4} = -\frac{50x}{x^4}$$

$$\therefore f''(x) = \frac{-50}{x^3}$$

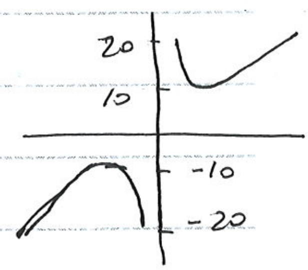
$f'' = 0 \Rightarrow -50 = 0 \Rightarrow -50 \neq 0$

المقام $= 0 \Rightarrow x^3 = 0 \Rightarrow x = 0$

sign of f'': $++++ \overset{0}{|} ---$ ∪ ∩

there is no inflection point on the graph, even though the graph is concave up on one side of $x = 0$ and concave down on the other.

– since 0 is not in the domain of f.



$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(x + \frac{25}{x}\right) = \infty$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x + \frac{25}{x}\right) = -\infty$$

so that there is a vertical asymptote at $x = 0$.

16/2/2014 9.   25/2/2015 ?

2. Related rates   المعدلات المتعلقه ببعضها

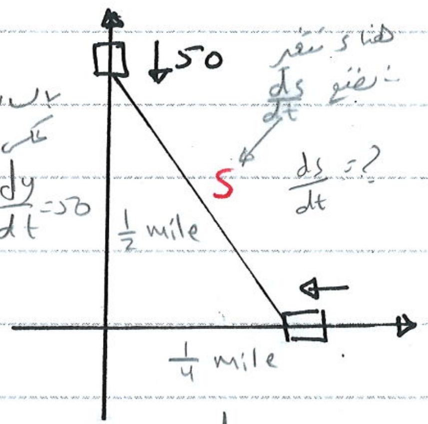
calculus p.3.23

Example ①: A car is traveling at 50 mph due (باتجاه) south at a point $\frac{1}{2}$ mile north of an intersection. A police car is traveling at 40 mph due west at a point $\frac{1}{4}$ mile east of the same intersection. At that instant, the radar in the police car measures the rate at which the distance between the two cars is changing. What does the radar gun register?

Solution:

بالسالب لانه عكس اتجاه y

$\frac{dy}{dt} = 50$



$\frac{dx}{dt} = 40$

بالسالب لانه عكس اتجاه x

$$s^2 = x^2 + y^2$$

$$\cancel{2}s\frac{ds}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left[x\frac{dx}{dt} + y\frac{dy}{dt}\right]$$

$$s = \sqrt{x^2 + y^2}$$

$$= \sqrt{\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{16} + \frac{1}{4}} = \frac{\sqrt{5}}{4}$$

$$\therefore \frac{ds}{dt} = \frac{\frac{1}{4}(-40) + \frac{1}{2}(-50)}{\frac{\sqrt{5}}{4}} = \frac{-140}{\sqrt{5}} = -62.6 \text{ mph}$$

s

9.9 28/2 2016   ي 25/2 2016

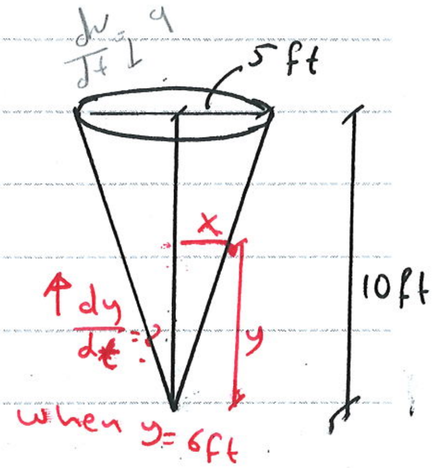
4/3/2015

Example ②: Water runs (يدير بتدفق) into a conical (خروط الشكل) tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and base radius of 5 ft. How fast is the water level rising (مقدار سرعة ارتفاع الماء) when the water is 6 ft deep?

Solution:

$y = 6ft$ and $\frac{dv}{dt} = 9\ ft^3/min$

$\frac{dy}{dt} = ?$



$$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} x^2 y$$

from similar triangles

$$\frac{x}{y} = \frac{5}{10} \Rightarrow x = \frac{y}{2}$$

$$V = \frac{\pi}{3} x^2 y = \frac{\pi}{3} \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12} y^3$$

$$\frac{dV}{dt} = \frac{\pi}{\cancel{12}_{4}} * \cancel{3} y^2 \frac{dy}{dt} = \boxed{\frac{\pi}{4} y^2 \frac{dy}{dt}}$$

$$\frac{dy}{dt} = \frac{dv}{dt} \cdot \frac{4}{\pi} \cdot \frac{1}{y^2}$$

$$\boxed{\begin{aligned} \frac{dy}{dt} &= \frac{dV}{dt} * \frac{4}{\pi} * \frac{1}{y^2} \\ \therefore \frac{dy}{dt} &= 9 * \frac{4}{\pi} * \frac{1}{6^2} = 0.32 \end{aligned}}$$

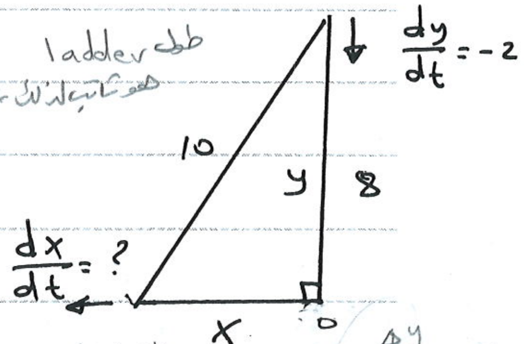
$$\frac{dV}{dt} = \frac{\pi}{3}\left[(2)x \frac{dx}{dt} \cdot y + x^2 \frac{dy}{dt}\right]$$

$$x = \frac{y}{2}$$

$$\therefore 9 = \frac{\pi}{4} (6)^2 \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{\pi} \simeq 0.32\ ft/min$$

at the moment in the question, the water level is ~~a~~ rising at about 0.32 ft/min.

Example: A 10-foot ladder leans (يستند على) against the side of a building. If the top of the ladder begins to slide down the wall at the rate of 2ft/sec, how fast is the bottom of the ladder sliding away (يبتعد عن الجدار) from the wall when the top of the ladder is 8 feet (اقدام فوق) off the ground?



ladder طول

$\frac{ds}{dt}$ هو ثابت لذلك = صفر

$$x^2 + y^2 = s^2$$

$$x^2 + y^2 = 100$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \text{—— (1)}$$

$$x^2 + y^2 = 100$$

$$x^2 = 100 - y^2$$

$$= 100 - 64$$

$$\therefore x^2 = 36$$

$$\therefore x = 6$$

$x = \pm 6$ ولكن ناخذ الموجبة لانها مسافة موجبة دائما

دائما المسافات موجبة ولكن ناخذ الاشارة حسب الاتجاه

* المسافة x هي على اليسار، نفرض ان x-axis هو على اليسار هو الموجب هنا لأن المسافة موجبة لذلك $\frac{dx}{dt}$ النتيجة موجبة لأنها باتجاه x-axis الموجب

اما $\frac{dy}{dt}$ في سالبة لأن فارضين y الموجب الى الاعلى فتكون 8 موجبة وبما ان الدرج سوف ينزل لذلك تكون $\frac{dy}{dt}$ سالبة

$$\frac{dy}{dt} = -2 \text{ ft/sec}, \quad y = 8$$

sub. in eq (1)

$$\therefore 2 \times 6 \frac{dx}{dt} + 2 \times 8(-2) = 0$$

نزول الدرج الى الاسفل اي اقتراب من الارض نقطة الزاوية (0) فرضناه سالب

$$12 \frac{dx}{dt} - 32 = 0$$

اما $\frac{dx}{dt}$ فكانت النتيجة موجبة وهي باتجاه الابتعاد من نقطة 0

$$\frac{dx}{dt} = \frac{32}{12} = \frac{8}{3}$$

لو كنا فرضنا $\frac{dy}{dt}$ موجبة اي ان الاقتراب موجب فكانت نتيجة $\frac{dx}{dt}$ سالبة اي (نفس النتائج لكن بالسالب) اي ابتعاد اي عكس اتجاه $\frac{dy}{dt}$

so, the bottom of the ladder is sliding away from the building at rate of $\frac{8}{3}$ ft/sec.

22/2
9.

3. Optimization

To optimize something means to make it as useful or effective as possible. In the mathematical models in which we use differentiable functions to describe things that interest us, this usually means finding where some function has its greatest or smallest value.

Example 1: Find two positive numbers whose sum is 20 and whose product is as large as possible.

sol. assume the first number is x,

so, the socond number is $20-x$

$\therefore f(x) = x(20-x)$; $\quad 0 \le x \le 20$

$= 20x - x^2$

critical points can be found from first derivative:

$f'(x) = 20 - 2x$

put $f'(x) = 0 \Rightarrow 20 - 2x = 0 \Rightarrow x = \frac{20}{2} = 10$

- critical point value : $f(10) = 10(20-10) = 10 \times 10$
$= 100$

- endpoints values : $f(0) = 0(20-0) = 0$
$f(20) = 20(20-20) = 0$

we conclude that the max. value is $f(10) = 100$

so the first number is 10

and the second is $20 - 10 = 10$

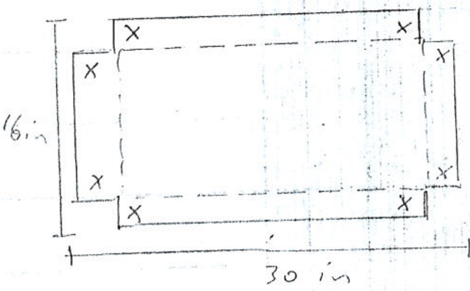
1/3/2015
9.

~~Maximum and minimum problems~~

Calculus p. 290

Example ①: An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?

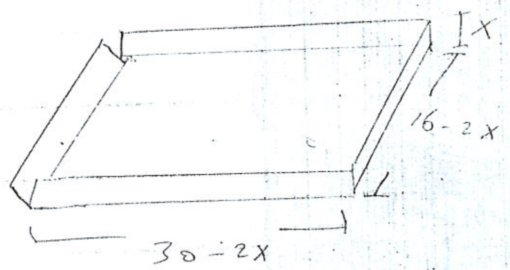
solution:



الارتفاع

$V = (16-2x)(30-2x)\,x$

$\therefore V = 480x - 92x^2 + 4x^3$



The variable x must satisfy

$0 \le x \le 8$

$$\frac{dV}{dx} = 480 - 184x + 12x^2$$

$= 4(120 - 46x + 3x^2)$

$= 4(x-12)(3x-10)$

$\frac{dV}{dx} = 0$

$\therefore x = \frac{10}{3}$ and $x = 12$

f' ++++ $\frac{10}{3}$ ---- 3.33 max.

since $x = 12$ falls outside the interval $[0, 8]$, the max value of V occurs either at the critical number $x = \frac{10}{3}$ or at the endpoints $x=0$, $x=8$

$\therefore V(0) = 0$

$V(\frac{10}{3}) = 726$

$V(8) = 0$

The greatest possible volume is $V = 726\ in^3$

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9. 3/3

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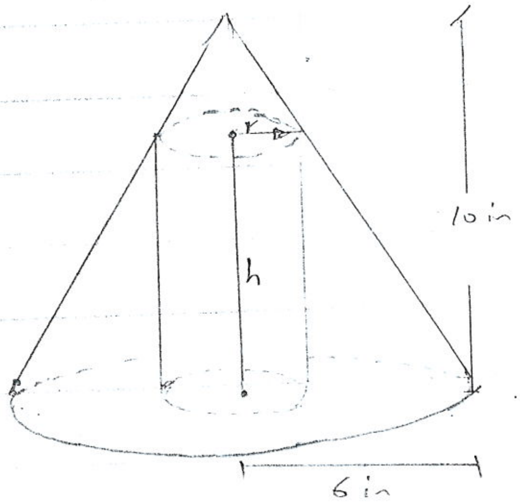
ملخص p 293

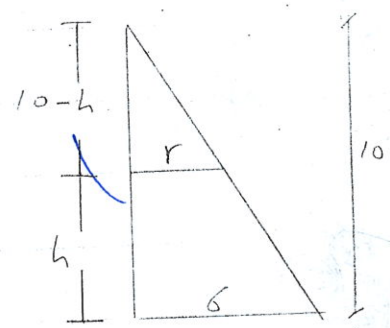
Example ③: Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches



Solution:

$$V = \pi r^2 h \quad \text{---} \quad ①$$

by using similar triangles



$$\frac{10-h}{r} = \frac{10}{6}$$

$$\therefore h = 10 - \frac{5}{3}r \quad \text{---} \quad ②$$

substituting ② into ①

$$V = \pi r^2 \left(10 - \frac{5}{3}r\right)$$

$$\therefore V = 10\pi r^2 - \frac{5}{3}\pi r^3 \quad \text{---} \quad ③$$

نصف قطر الاسطوانة لا يمكن ان تكون صفر

نصف قطر المخروط

$$0 < r \le 6$$

$$V = 10\pi r^2 - \frac{5}{3}\pi r^3$$

at $r = 6$

$h = 10 - \frac{5}{3} \times 6$

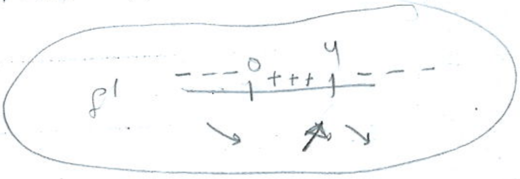
$h = 10 - 10 = 0$

$\therefore V(0) = 0$

$$\frac{dV}{dr} = 20\pi r - 5\pi r^2 = 5\pi r(4 - r)$$

$dV/dr = 0 \Rightarrow 5\pi r(4 - r) = 0$

so $r = 0$ and $r = 4$ are critical numbers



$V(0) = 0$

$V(4) = \frac{160}{3}\pi \simeq 168 \ in^3$

$V(6) = 0$

The max value is $168 \ in^3$

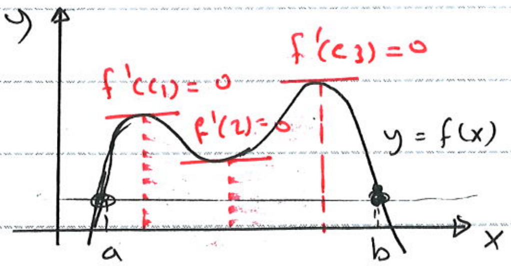
The cylinder of largest volume has $\boxed{r = 4}$ in and $\boxed{h = \frac{10}{3}}$, 23/2 9. 2/3/2015 e

4. The mean value theorem

Rolle's theorem: let $y = f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), If

$$f(a) = f(b) = 0$$

then there is at least one number c in (a,b) at which $f'(c) = 0$



Example: Does Rolle's Theorem is applicable on the following functions. If so find the value or values of c.

1. $y = 2x - x^2$ $[0,2]$

Solution: the function is continuous on $[0,2]$.

$y' = 2 - 2x$ is differentiable on $(0,2)$.

$f(0) = 2(0) - 0 = 0$ and $f(2) = 2 \times 2 - 2^2 = 0$ o.K

$\therefore$ Rolle's theorem is applicable on this function on $[0,2]$. To find the value of c: put $y' = 0$

$\therefore 2 - 2x = 0 \Rightarrow 2x = 2 \Rightarrow x = \frac{2}{2} = 1$ $\therefore c = 1$

2- $y = \frac{x^3}{3} - 3x$ $[-3,3]$ the function is continuous on $[-3,3]$.

$y' = x^2 - 3$ is differentiable on $(-3,3)$

$f(-3) = \frac{(-3)^3}{3} - 3(-3) = \frac{-27}{3} + 9 = 0$ and $f(3) = \frac{27}{3} - 9 = 0$ $\therefore$ ok.

$\therefore$ Rolle's theorem is applicable on this function on $[-3,3]$.

to find the values of c: put $y' = 0$

$\therefore x^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \mp\sqrt{3}$

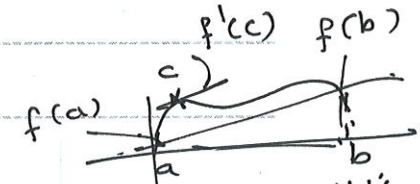
$\therefore c_1 = -\sqrt{3}$ and $c_2 = \sqrt{3}$.

9.

5/3/2015
9.

the mean value theorem (M.V.T)

Suppose $y = f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Then there is at lest one point c in (a,b) at which



$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

الخط الواصل بين ab عليه slope هو $\frac{f(b)-f(a)}{b-a}$ وميله مساوي لا $f'(c)$ اي خط موازي لهذا الخط المستقيم الواصل بين نقطتين ab

Example: Is the M.V.T is applicable on the following functions. If so find the value or values of c.

1. $f(x) = x - 2\sin x \qquad 0 \leq x \leq 2\pi$

Sol. $f(x) = x - 2\sin x$ is continuous on $[0, 2\pi]$

$f'(x) = 1 - 2\cos x$ is differentiable on $(0, 2\pi)$

$\therefore$ the M.V.T. is applicable on $[0, 2\pi]$

to find c

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

where $f(b) = f(2\pi) = 2\pi - 2\sin 2\pi = 2\pi - 0 = 2\pi$

$f(a) = f(0) = 0 - 2\sin 0 = 0 - 0 = 0$

نشتق المعادلة ونعوضها بقيمة c

and $f'(c) = 1 - 2\cos c$, thus:

$$1 - 2\cos c = \frac{2\pi - 0}{2\pi - 0} \Rightarrow 1 - 2\cos c = 1 \Rightarrow 2\cos c = 0$$

$$\Rightarrow \cos c = 0$$

$$\therefore c = \pm \frac{n\pi}{2} \quad ; \quad n = 1, 3, 5, \text{---}$$

$\therefore c_1 = \frac{\pi}{2}$ and $c_2 = \frac{3\pi}{2}$ on the interval $[0, 2\pi]$.

$\frac{5\pi}{2} = 2.5\pi$

وهي أكبر من 2π خارج الفترة

$[0, 2\pi]$

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2. $f(x) = x^{\frac{2}{3}}$ $[-8,8]$

Sol: $f(x) = x^{2/3} = \sqrt[3]{x^2}$ is continuous on $[-8,8]$.

$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$ is not differentiable $x=0 \in (-8,8)$.

$\therefore$ the M.V.T. is not applicable on $[-8,8]$.

3. $f(x) = x^{2/3}$ $[0,8]$

Sol: $f(x) = x^{2/3} = \sqrt[3]{x^2}$ is continuous on $[0,8]$.

$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$ is not differentiable $x=0 \notin (0,8)$

so it is differentiable on $(0,8)$.

$\therefore$ The M.V.T. is applicable on $[0,8]$.

to find c:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

where $f(a) = f(0) = 0$

$f(b) = f(8) = 8^{2/3} = 4$

and $f'(c) = \frac{2}{3\sqrt[3]{c}}$, thus:

$$\frac{2}{3\sqrt[3]{c}} = \frac{4-0}{8-0} \Rightarrow \boxed{\frac{2}{3\sqrt[3]{c}} = \frac{1}{2}} \Rightarrow \boxed{\sqrt[3]{c} = \cancel{\frac{3}{4}}\ \frac{4}{3}}$$

$\cancel{\Rightarrow c = \left(\frac{3}{4}\right)^3 = \frac{27}{64} = 0.422}$

$\Rightarrow c = \left(\frac{4}{3}\right)^3$

$\therefore c = \frac{64}{27}$

1/31 2014
8.

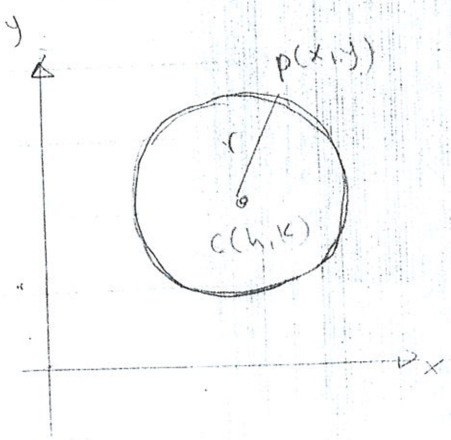
Thomas [illegible]
p. 401

القطوع المخروطية

Calculus المصدر مأخوذ من كتاب
page. 749

Section of a Cone

① the circle



the equation of a circle with radius r.

$$(x-h)^2+(y-k)^2=r^2$$

Example: find the circle through the origin with center at $C(2,-1)$

Solution: $(x-2)^2+(y+1)^2=r^2$

since the circle goes through the origin, $x=y=0$ must satisfy the equation.

$$(0-2)^2+(0+1)^2=r^2$$

$$r^2=5$$

$\therefore$ The equation is

$$(x-2)^2+(y+1)^2=5$$

$2A=4$
$A=2$
$A^2=4$
$2A=6$
$A=\frac{6}{2}=3$
$A^2=9$

Example:

Analyze the equation

$$x^2+y^2+4x-6y=12$$

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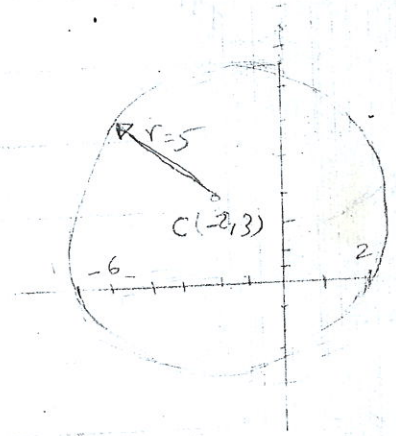
9. 2016

Solution:

اكمال مربع



$$(x^2+4x+4)+(y^2-6y+9)=12+4+9$$

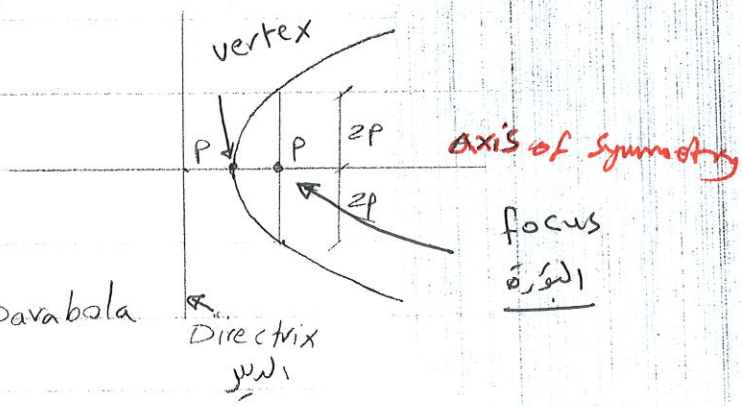
or $(x+2)^2+(y-3)^2=25$

$\therefore$ center $(-2,3)$, radius $r=5$

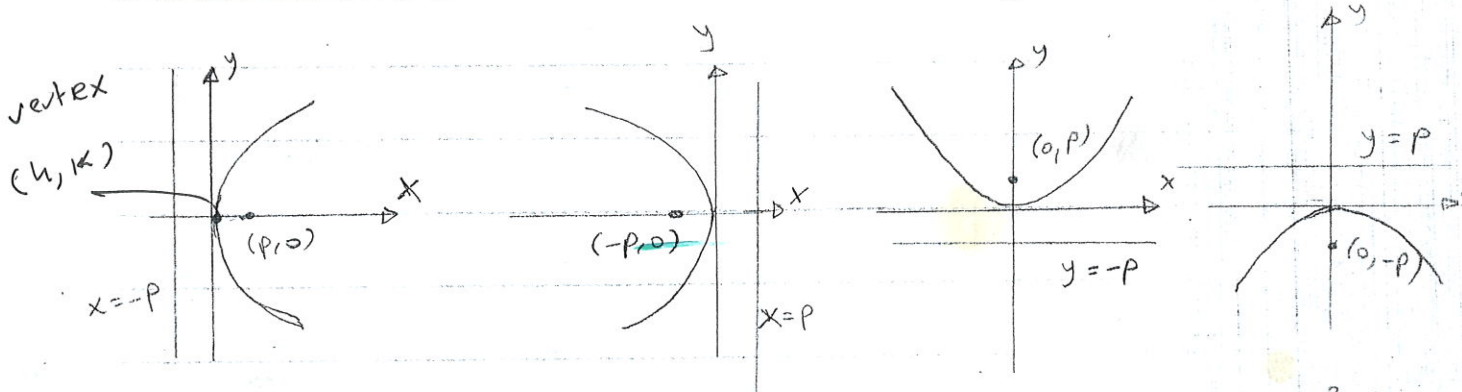
8/3/2015
P

② the parabola

مجموعة النقاط في المستوى تبعد بعد ثابت
من خط مستقيم ونقطة ثابتة



the standard equations of a parabola



$y^2 = 4px$ $\quad$ $y^2 = -4px$ $\quad$ $x^2 = 4py$ $\quad$ $x^2 = -4py$

parabolas with vertex (h, k) and axis parallel to x-axis.

$(y-k)^2 = 4p(x-h)$ opens right

$(y-k)^2 = -4p(x-h)$ open left

Parabolas with vertex (h,k) and axis parallel to y axis

$(x-h)^2 = 4p(y-k)$ opens up

$(x-h)^2 = -4p(y-k)$ opens down

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Example 2 discuss the parabola

$$y = x^2 + 4x$$

$2A = 4$
$A = 2$

$$y = x^2 + 4x + 4 - 4$$
$$\Rightarrow y + 4 = x^2 + 4x + 4$$
$$\therefore (y+4) = 4p(x+2)^2$$

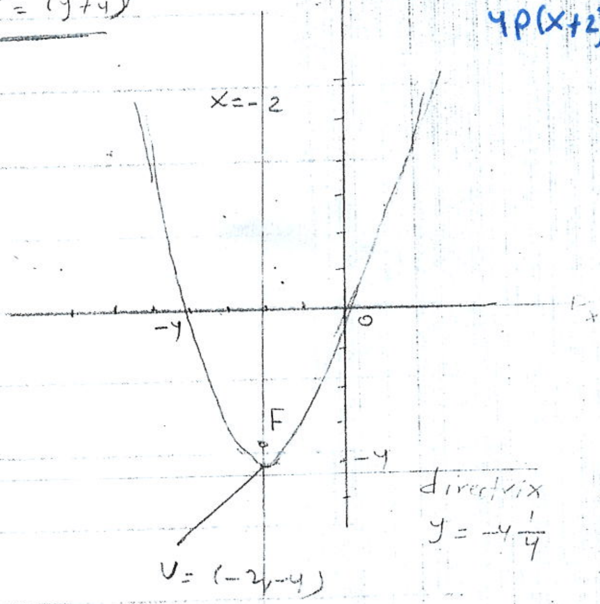
Solution: $y + 4 = x^2 + 4x + 4 = (x+2)^2$

this has the form $(x+2)^2 = (y+4)$

$$(x-h)^2 = 4p(y-k)$$

with $h = -2$, $k = -4$

$4p = 1$, $p = \frac{1}{4}$



The vertex of the parabola is $V(-2, -4)$; its axis of symmetry $x = -2$;

$F\left(-2, -3\frac{3}{4}\right)$

directrix $y = -4\frac{1}{4}$

$x^2 + 4x + A^2$

$(x + A)^2$

$x^2 + 2Ax + A^2$

$\therefore 2A = 4$

$\therefore A = 2$

$\therefore x^2 + 4x + 2^2$

$\therefore x^2 + 4x + 4$

P, V, directrix

$y = -4\frac{1}{4}$

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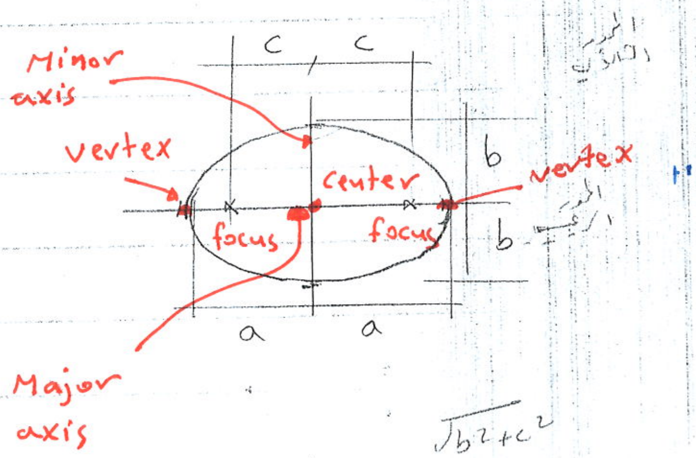
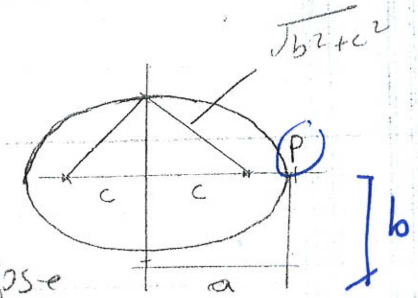
مجموع النقاط في المستوى بحيث تكون مجموع المسافات من نقطتين ثابتتين (البؤرات) ثابتة وهذه اكبر من المسافة بين النقطتين

$PF_1 + PF_2 = 2a$

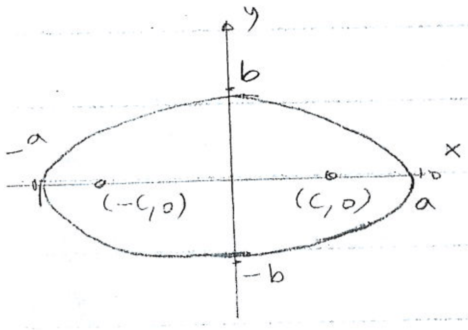
③ the ellipse

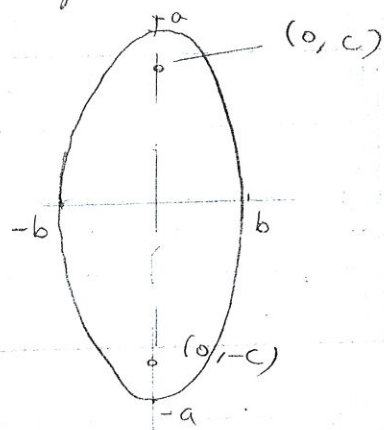


$$c = \sqrt{a^2 - b^2}$$



the standard equations of an ellipse





$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

ellipse with center (h,k) and major axis parallel to x-axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad [b \leq a]$$

ellipse with center (h,k) and major axis parallel to y-axis

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \qquad [b \leq a]$$

قيمة ماهي اصغر a
وكذلك وضعت مع x
والقيمة الكبيرة وهي a وضعت مع y

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في ellipse المفروض القيم ليست ~~[illegible]~~ a

مساوية الى الـ $a \neq b$

ولو كانت قيمة $a = b$ فاذن اصبحت دائرة

مثلاً

هنا $a = b = 25$

$\cancel{x^2 + y^2 = 1}$ $\quad \frac{x^2}{25} + \frac{y^2}{25} = 1$

$$\Rightarrow \frac{x^2 + y^2}{25} = 1$$

$$\Rightarrow x^2 + y^2 = 25$$

$\therefore$ هي دائرة.

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calculus p.759

Example: Describe the graph of the equation

$$16x^2 + 9y^2 - 64x - 54y + 1 = 0$$

Solution:

$$(16x^2 - 64x) + (9y^2 - 54y) = -1$$

$$16(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -1 + 64 + 81$$

$$16(x-2)^2 + 9(y-3)^2 = 144$$

نقسم على ١٤٤

$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$$

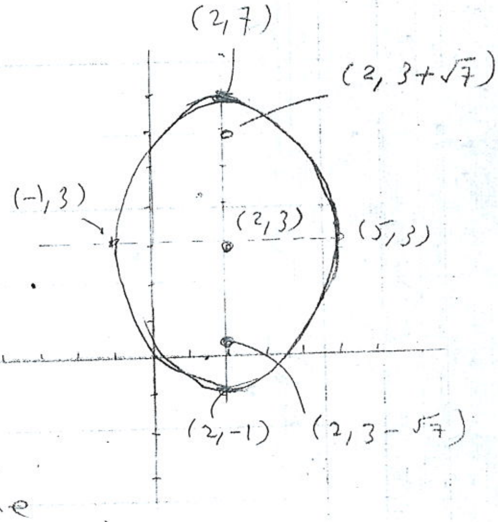
This is an equation of form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$\therefore$ $h = 2$, $k = 3$, $a^2 = 16$, and $b^2 = 9$

$\therefore$ Center $(2,3)$, major axis parallel to the y-axis.

since $a = 4$, the major axis extends, 4 units above and 4 units below the center, so its endpoints are $(2,7)$ and $(2,-1)$. Since $b = 3$, the minor axis extends 3 units to the left and 3 units to the right of the center, so its endpoints are $(-1,3)$ and $(5,3)$

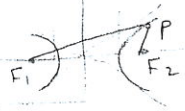


$c = \sqrt{a^2 - b^2} = \sqrt{16-9} = \sqrt{7}$

the foci lie $\sqrt{7}$ units above and below the center placing them at the points $(2, 3+\sqrt{7})$ and $(2, 3-\sqrt{7})$.

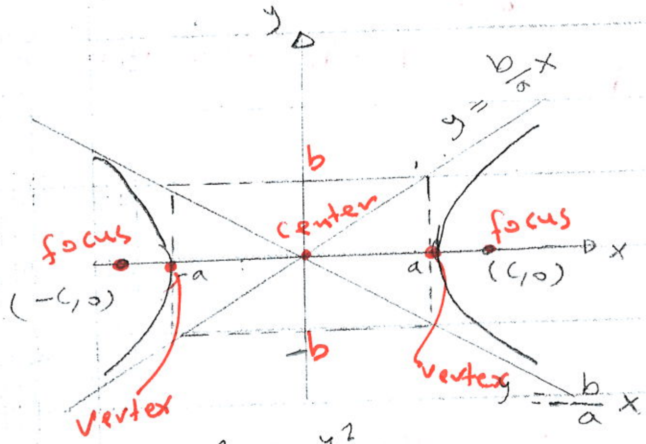
مجموعة النقاط في المستوى بحيث الفرق بالمسافات من نقطتين ثابتة يكون ثابت
وهنا الفرق هو أقل من المسافة ما بين النقطتين.

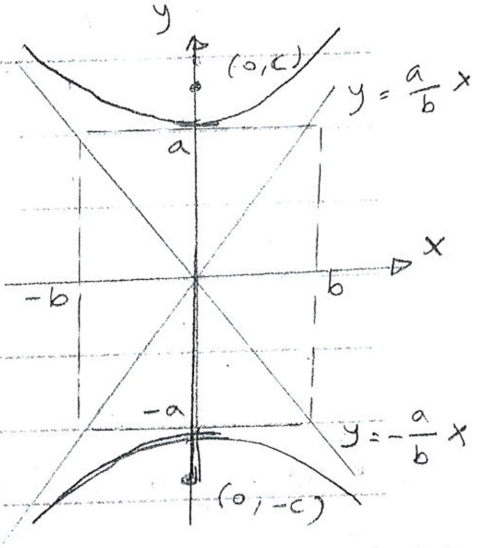
القطع الزائد



④ the hyperbolas

the standard equations of a hyperbolas





$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$c = \sqrt{a^2+b^2}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

*

Example: sketch the graph of the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

Solution: $a^2 = 4$ and $b^2 = 9$

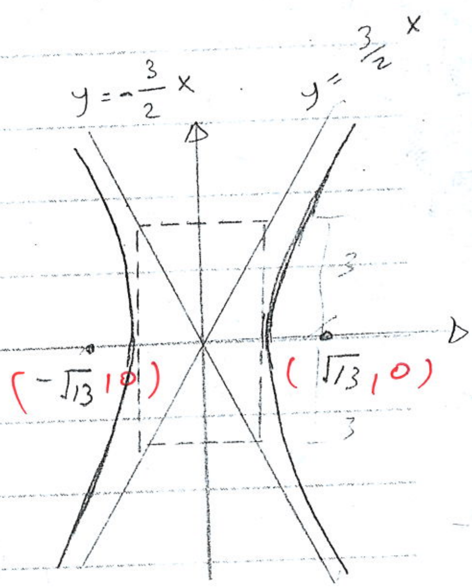
$\therefore$ $a = 2$ and $b = 3$

the coordinates of the vertices $(2,0)$ and $(-2,0)$

To obtain equations for the asymptotes, substitute 0 for 1 in the given equation;

$$\frac{x^2}{4} - \frac{y^2}{9} = 0 \quad \text{or} \quad y = \pm\frac{3}{2}x$$

$$c = \sqrt{a^2+b^2} = \sqrt{4+9} = \sqrt{13} \simeq 3.6$$



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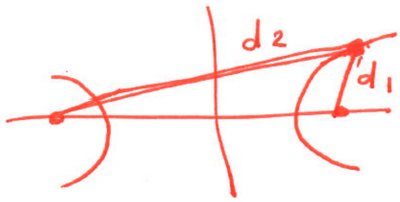

$d_2 - d_1 = \text{constant}$

hyperbola with center (h, k) and focal axis parallel to x-axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

hyperbola with center (h, k) and focal axis parallel to y-axis.

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

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Example: describe the graph of the equation

$$x^2 - y^2 - 4x + 8y - 21 = 0$$

Solution:

$$(x^2 - 4x) - (y^2 - 8y) = 21$$

$$(x^2 - 4x + 4 - 4) - (y^2 - 8y + 16 - 16) = 21$$

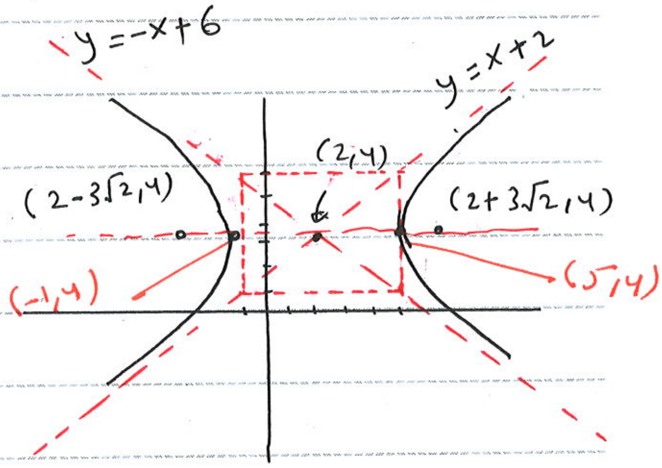
$$(x^2 - 4x + 4) - (y^2 - 8y + 16) = 21 + 4 - 16$$

$$(x-2)^2 - (y^2 - 4)^2 = 9$$

$$\therefore \frac{(x-2)^2}{9} - \frac{(y-\cancel{8}4)^2}{9} = 1$$

this is an equation of form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



$\therefore h = 2,\ k = 4,\ a^2 = 9$

and $b^2 = 9$.

$\therefore$ the center $(2, 4)$

since $a = 3$, the vertices are located 3 ~~united~~ units to the left and 3 units to the right of the center, or at the points $(-1, 4)$ and $(5, 4)$.

$c = \sqrt{a^2 + b^2} = \sqrt{9 + 9} = 3\sqrt{2}$ so the foci are located $3\sqrt{2}$ units to the left and right of the center, or at the points $(2 - 3\sqrt{2}, 4)$ and $(2 + 3\sqrt{2}, 4)$.

the equations of the asymptotes may be found using the also trick of substituting 0 for 1 in equation.

$$\frac{(x-2)^2}{9} - \frac{(y-4)^2}{9} = 0$$

this can be written as $y - 4 = \pm (x - 2)$, which yields the asymptotes

$y = x + 2$ and $y = -x + 6$

$y - 4 = x - 2$
$y = x - 2 + 4$
$y = x + 2$

$y - 4 = -x + 2$
$y = -x + 6$

٤,٩,٩

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