Lecture Notes of Analytical Mechanics

By

Dr.Ahmed S. Al-Asadi   **Dr.Mohammed F. Al-Mudhaffer**

Department of Physice-College of Education for Pure Science

University of Basrah-Iraq

1-Analytical Mechanics ,7th edition,Greant R. Foules

Contents

1 Vectors

1.1 A Scalar and Vector Quantities

The motion of dynamic systems is typically described in terms of two basic quantities : SCALAR and VECTORS.

1.1.1 A Scalar Quantity

A **scalar** is a physical quantity that has **magnitude only** such as the mass of an object. It is completely specified by a single number, in appropriate units. Its value **is independent of any coordinates chosen** to describe the motion of the system.

Examples of scalars include density, volume, temperature, and energy.

1-Its quantity independents on the coordinates system.

2-Its represents by the value only and measurements unit .

3-Mathematically, scalars are treated as real numbers.They **obey all the normal algebraic rules of addition, subtraction, multiplication, division, and so on**.

1.1.2 A Vector Quantity

A vector, however, has both **magnitude and direction**, such as the displacement from one point in space to another.

1-Its quantity depends on the coordinates system.

2-Its represents by the value and direction .

3-Obeys the parallelogram rules.

1.2 Vector Algebra

In most written work, a distinguishing mark, such as an arrow, customarily designates a vector, for example,Ā. In this text, however, for the sake of simplicity, **we denote vector quantities simply by boldface type, for example, A**. We use ordinary italic type to represent scalars, for example, A. A given vector A is specified by stating its magnitude and its direction relative to some arbitrarily chosen coordinate system. It is represented diagrammatically as a directed line segment, as shown in three-dimensional space in Figure 1.1.

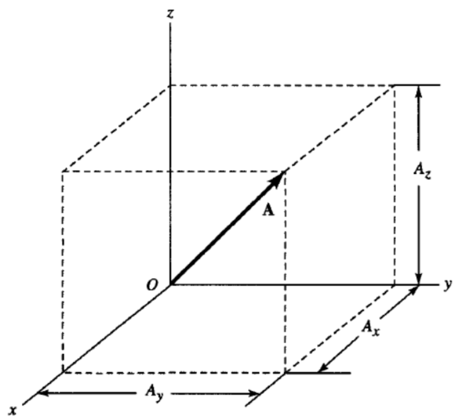

Figure 1.1: A vector A and its components in Cartesian coordinates.

A vector can also be specified as the set of its components, or projections onto the coordinate axes.

$$A = A_x + A_y + A_z \tag{1.1}$$

For example, if the vector A represents a displacement from a point P1(x1, y1, z1) to the point P2(x2, y2,z2) then its

$$A_x = x2 - x1$$
$$A_y = y2 - y1 \tag{1.2}$$
$$A_z = z2 - z1$$

1.2.1 Equality of Vectors

The two vectors are equal only if their respective components are equal.

$$A = B$$
$$or \qquad (A_x, A_y, A_z) = (B_x, B_y, B_z) \tag{1.3}$$
$$A_x = B_x \qquad A_y = B_y \qquad A_z = B_z$$

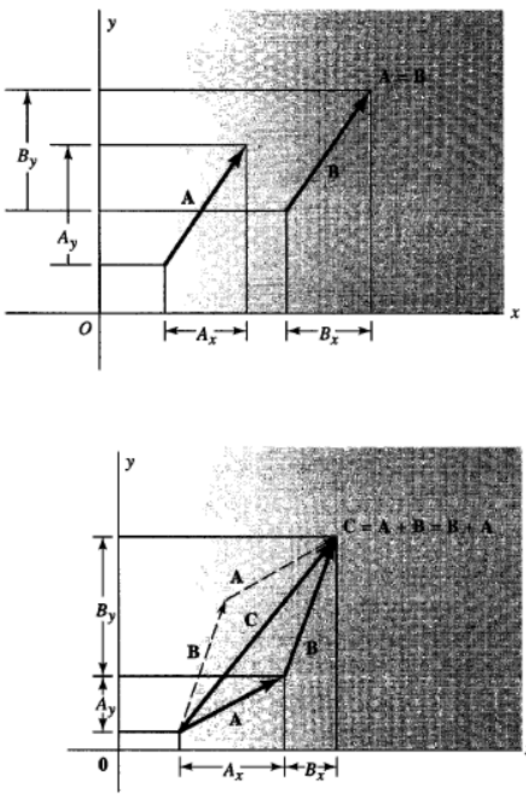

Figure 1.2: **Top**: Illustration of equal vectors,**Bottom**:Addition of two vectors .

1.2.2 Vector Addition

The addition of two vectors is defined by the equation

$$\begin{aligned} A + B &= (A_x, A_y, A_z) + (B_x, B_y, B_z) \\ A + B &= (A_x + B_x) + (A_y + B_y) + (A_z + B_z) \end{aligned} \tag{1.4}$$

The sum of two vectors is a vector whose components are sums of the components of the given vectors. The geometric representation of the vector sum of two nonparallel vectors is the third side of a triangle, two sides of which are the given vectors.The vector sum is illustrated in Figure 1.2. The sum is also given by the parallelogram rule, as shown in the figure. The vector sum is defined, however, according to the above equation even if the vectors do not have a common point.

1.2.3 Multiplication by a Scalar

If **c** is a scalar and **A** is a vector,then:

$$c\overrightarrow{A} = c(A_x + A_y + A_z) = cA_x + cA_y + cA_z = \overrightarrow{A}c \tag{1.5}$$

The product **cA** is a vector whose components are **c times those of A**. Geometrically,the vector cA is parallel to A and is c times the length of A. **When c = —1, the vector —A is one whose direction is the reverse of that of A,**

1.2.4 Vectors Subtraction

$$\overrightarrow{A} - \overrightarrow{B} = (A_x - B_x) + (A_y - B_y) + (A_z - B_z) = \overrightarrow{C} \tag{1.6}$$

1.2.5 The Null Vector

The vector 0 = (0,0,0) is called the null vector. The direction of the null vector is undefined. From (IV) it follows that A — A =0. Because there can be no confusion when the null vector is denoted by a zero, we shall hereafter use the notation 0=0.

1.2.6 The Commutative Law of Addition

This law holds for vectors; that is,

$$A + B = B + A \tag{1.7}$$

Because $A_x + B_x = B_x + A_x$and similarly for the y and z components.

1.2.7 The Associative Law what its translation in Arabic??

The associative law is also true, because

$$\begin{aligned} A + (B + C) &= (A_x + (B_x + C_x), A_y + (B_y + C_y), A_z + (B_z + C_z)) \\ &= ((A_x + B_x) + C_x, (A_x + B_x) + C_x, (A_x + B_x) + C_x) \\ &= (A + B) + C \end{aligned} \tag{1.8}$$

1.2.8 The Distributive Law

Under multiplication by a scalar, the distributive law is valid because, from (1.2.2) and (1.2.3),

$$\begin{aligned} c(A + B) &= c(A_x + B_x, A_y + B_y, A_z + B_z) \\ &= c(A_x + B_x), c(A_y + B_y), c(A_z + B_z) \\ &= cA + cB \end{aligned} \tag{1.9}$$

1.2.9 Magnitude of a Vector

The magnitude of a vector A, denoted by $|A|$ or by A, is defined as the square root of the sum of the squares of the components, namely,

$$A = |A| = (A_x^2 + A_y^2 + A_z^2)^{1/2} \tag{1.10}$$

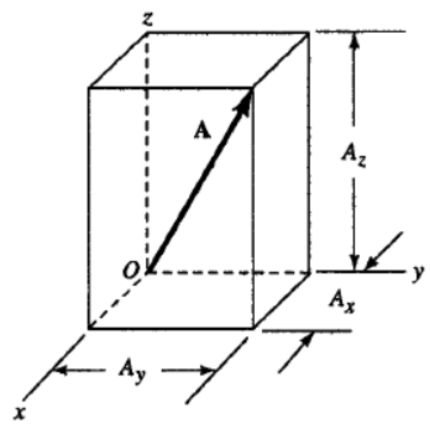

Figure 1.3: Magnitude of a vector A.

where the positive root is understood. Geometrically, the magnitude of a vector is its length, that is, the length of the diagonal of the rectangular parallelepiped whose sides are A_x, A_y and A_z expressed in appropriate units. See Figure 1.3.

1.2.10 Unit Coordinate Vectors

A unit vector is a vector whose magnitude is unity. Unit vectors are often designated by the symbol e, from the German word Einheit. The three unit vectors.

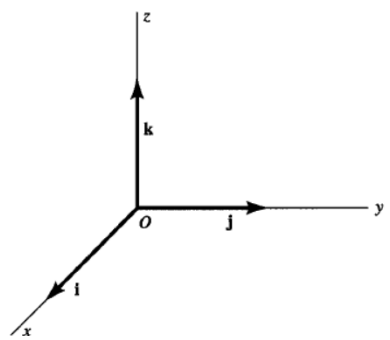

Figure 1.4: The unit vectors ijk.

$$e_x = (1, 0, 0) \qquad e_y = (0, 1, 0) \qquad e_z = (0, 0, 1) \tag{1.11}$$

are called unit coordinate vectors or basis vectors. In terms of basis vectors, any vector can be expressed as a vector sum of components as follows:

$$A = e_x A_x + e_y A_y + e_z A_z \tag{1.12}$$

EXAMPLE (1.1)

Find the sum and the magnitude of the sum of the two vectors A = (1,0,2) and B=(0, 1,1).

Solution:

Adding components, we have A + B = (1,0,2) + (0,1,1) = (1, 1,3).

EXAMPLE 1.2

A helicopter flies 100 m vertically upward, then 500 m horizontally east, then 1000 m horizontally north. How far is it from a second helicopter that started from the same point and flew 200 m upward, 100 m west, and 500 m north?

Solution:

Choosing up, east, and north as basis directions, the final position of the first helicopter is expressed vectorially as A = (100,500,1000) and the second as B = (200, —100, 500),in meters. Hence, the distance between the final positions is given by the expression

$$\begin{aligned} |A - B| &= |((100 - 200), (500 + 100), (1000 - 500))| m \\ &= 787.4m \end{aligned} \tag{1.13}$$

1.3 Scalar Product

Given two vectors A and B,the scalar product or "*dot*" product, $A.B$,is the scalar defined by the equation

$$A.B = A_x B_x + A_y B_y + A_z B_z \tag{1.14}$$

From the above definition,

1-Scalar multiplication is commutative(A.B=B.A)

2-It is also distributive(A.(B+C)=A.B+A.C)

The dot product A. B has a simple geometrical interpretation and can be used to calculate the angle θ between those two vectors. For example, shown in Figure 1.5 are the two vectors A and B separated by an angle θ, along with an x', y', z' coordinate system arbitrarily chosen as a basis for those vectors.

$$A.B = |A||B|\cos\theta$$
$$cos\theta = \frac{A.B}{AB} \Rightarrow \theta = cos^{-1}\frac{A.B}{AB} \tag{1.15}$$

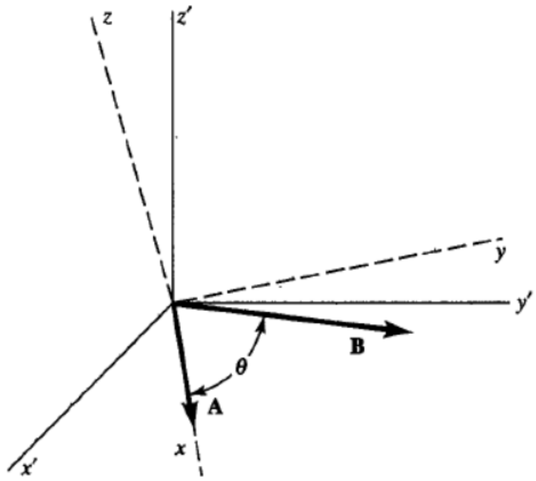

Figure 1.5: Evaluating a dot product between two vectors.

Note: If A. B is equal to zero and neither A nor B is null, then $cos\theta$ is zero and A is perpendicular to B.)

The square of the magnitude of a vector A is given by the dot product of A with itself,

$$A^2 = |A|^2 = A.A \tag{1.16}$$

From the definitions of the unit coordinate vectors i,j, and k, it is clear that the following relations hold

$i.i = j.j = k.k = 1$

$i.j = i.k = j.k = 0$

In addition , we can write any vector associated with its unit vectors by this form:

$$A = iA_x + jA_y + kA_z \tag{1.17}$$

Examples on dot product

Example 1.3.1 *Suppose that an object under the action of a constant force undergoes a linear displacement. By definition, the work AW done by the force is given by the product of the component of the force F in the direction of multiplied by the magnitude of the displacement; that is,*

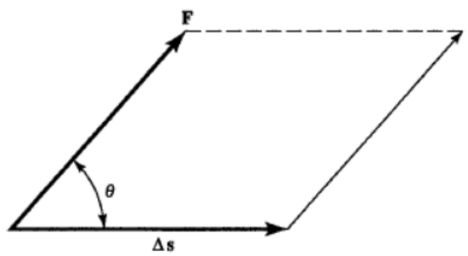

Figure 1.6: A force acting on a body undergoing a displacement.

$$\triangle W = (F\cos\theta)\triangle s \tag{1.18}$$

where θ is the angle between F and $\triangle s$ As. But the expression on the right is just the dot product of F and As, that is,

$$\triangle W = F.\dot{\triangle} s \tag{1.19}$$

Example 1.3.2 ***Law of Cosines***:*Consider the triangle whose sides are A, B, and C, as shown in Figure 1.6.Then C = A + B. Take the dot product of C with itself,*

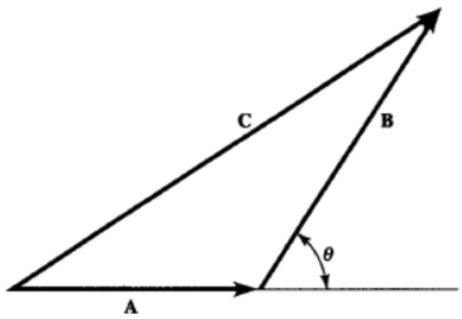

Figure 1.7: The law of cosines

$$\begin{aligned} C.C &= (A+B)(A+B) \\ &= A.A + 2A.B + B.B \end{aligned} \tag{1.20}$$

By Replacing A. B with AB cos 0 to obtain which is the familiar law of cosines.

$$C^2 = A^2 + 2ABcos\theta + B^2 \tag{1.21}$$

Example 1.3.3 *Find the cosine of the angle between a long diagonal and an adjacent face diagonal of a cube.*

Solution: *We can represent the two diagonals in question by the vectors A = (1, 1,1) and B =(1, 1,0).*

$$cos\theta = \frac{A.B}{AB} = \frac{1+1+0}{\sqrt{2}\sqrt{3}} = 0.8165 \tag{1.22}$$

1.4 The Vector Product

Given two vectors A and B, the vector product or cross product, A x B, is defined as the vector whose components are given by the equation

$$A \times B = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.23}$$

$$A \times B = (A_yB_z - A_zB_y, A_xB_z - A_zB_x, A_xB_y - A_yB_x) \tag{1.24}$$

and

$$(A \times B) = -(B \times A) \tag{1.25}$$

$$A \times (B + C) = A \times B + A \times C \tag{1.26}$$

$$n(A \times B) = (nA) \times B = A \times (nB) \tag{1.27}$$

According to the definitions of the unit coordinate vectors (Section 1.3), it follows that

$$\begin{aligned} i \times i = j \times j = k \times k = 0 \\ j \times k = i = -k \times j \\ i \times j = k = -j \times i \\ k \times i = j = -i \times k \end{aligned} \tag{1.28}$$

These latter three relations define a right-handed triad. For example, $i \times j = (0-0, 0-0, 1-0) = (0, 0, 1) = k$

The remaining equations are proved in a similar manner.

In general, the cross product expressed in ijk form is

$$A \times B = (A_yB_z - A_zB_y, A_xB_z - A_zB_x, A_xB_y - A_yB_x) \tag{1.29}$$

Each term in parentheses is equal to a determinant,

$$A \times B = i \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + j \begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + k \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \tag{1.30}$$

and finally

$$A \times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.31}$$

$$A \times B = ABsin\theta \tag{1.32}$$

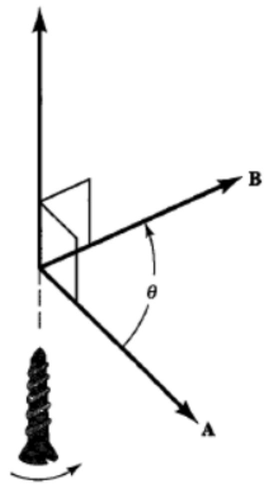

Figure 1.8: The cross product of two vectors.

EXAMPLE 1.5.1

Given the two vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find $\mathbf{A} \times \mathbf{B}$.

Solution:

In this case it is convenient to use the determinant form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i}(2-1) + \mathbf{j}(-1-4) + \mathbf{k}(-2-1)$$
$$= \mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$$

EXAMPLE 1.5.2

Find a unit vector normal to the plane containing the two vectors **A** and **B** above.

Solution:

$$\mathbf{n} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}}{[1^2 + 5^2 + 3^2]^{1/2}}$$
$$= \frac{\mathbf{i}}{\sqrt{35}} - \frac{5\mathbf{j}}{\sqrt{35}} - \frac{3\mathbf{k}}{\sqrt{35}}$$

EXAMPLE 1.5.3

Show by direct evaluation that $\mathbf{A} \times \mathbf{B}$ is a vector with direction perpendicular to **A** and **B** and magnitude $AB \sin\theta$.

Solution:

Use the frame of reference discussed for Figure 1.4.1 in which the vectors **A** and **B** are defined to be in the x, y plane; **A** is given by $(A, 0, 0)$ and **B** is given by $(B\cos\theta, B\sin\theta, 0)$. Then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A & 0 & 0 \\ B\cos\theta & B\sin\theta & 0 \end{vmatrix} = \mathbf{k}AB\sin\theta$$

Example 1.4.1 ***An Example of the Cross Product:Moment of a Force***
Moments of force, or torques, are represented by cross products. Let a force F act at a point P(x, y, z), as shown in Figure 1.6.1, and let the vector OP be designated by r; that is,
$OP = r = ix + jy + kz$

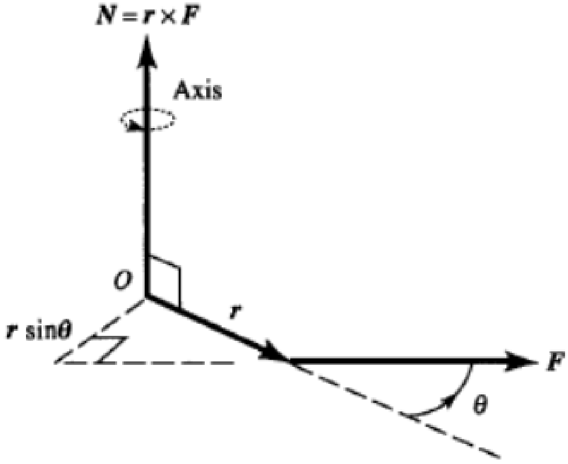

Figure 1.9: Illustration of the moment of a force about a point 0.

The moment N of force, or the torque N, about a given point 0 is defined as the cross product:

$$N = r \times F \tag{1.33}$$

If a single force is applied at a point P on a body that is initially at rest and is free to turn about a fixed point 0 as a pivot, then the body tends to rotate. The axis of this rotation is perpendicular to the force F, and it is also perpendicular to the line OF; therefore, the direction of the torque vector N is along the axis of rotation. The magnitude of the torque is given by

$$|N| = |r \times F| = rFsin\theta \tag{1.34}$$

in which θ *is the angle between r and F. Thus,* $|N|$ *can be regarded as the product of the magnitude of the force and the quantity* $rsin\theta$*, which is just the perpendicular* *distance from the line of action of the force to the point 0.*

When several forces are applied to a single body at different points, the moments add vectorially and the condition for rotational equilibrium is that the vector sum of all the moments is zero:

$$\sum_i |r \times F| = \sum_i N = 0 \tag{1.35}$$

1.5 Triple Products

The expression

$$A.(B \times C) \tag{1.36}$$

we can see that the scalar triple product may be written as matrix

$$A.(B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (A \times B).\vec{C} \tag{1.37}$$

H.W: can you prove that ?

Additionally, we can write

$$\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B}) \tag{1.38}$$

which represents the triple cross products .

EXAMPLE 1.7.1

Given the three vectors $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$, and $\mathbf{C} = \mathbf{k}$, find $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

Solution:

Using the determinant expression, Equation 1.7.1, we have

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(-1+0) = -1$$

EXAMPLE 1.7.2

Find $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ above.

Solution:

From Equation 1.7.3 we have

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{i} - \mathbf{j})0 - \mathbf{k}(1 - 0) = -\mathbf{k}$$

EXAMPLE 1.7.3

Show that the vector triple product is nonassociative.

Solution:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -\mathbf{a}(\mathbf{c} \cdot \mathbf{b}) + \mathbf{b}(\mathbf{c} \cdot \mathbf{a})$$
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

which is not necessarily zero.

1.6 Change of Coordinate System:The Transformation Matrix

In this section we show how to represent a vector in different coordinate systems. Consider the vector A expressed relative to the triad ijk:

$$A = iA_x + jA_y + kA_z \tag{1.39}$$

Relative to a new triad i'j'k' having a different orientation from that of ijk, the same vector A is expressed as

$$A = \acute{i}A_{\acute{x}} + \acute{j}A_{\acute{y}} + \acute{k}A_{\acute{z}} \tag{1.40}$$

Now the dot product $A.\acute{i}$ is just $A_{\acute{x}}$, that is, the **projection** of A on the unit vector $\acute{i}$. Thus, we may write

$$A = \acute{i}A_{\acute{x}} + \acute{j}A_{\acute{y}} + \acute{k}A_{\acute{z}} \tag{1.41}$$

Now the dot product is just $A_{\acute{x}}$ as

$$A.\acute{i} = \acute{i}.iA_{\acute{x}} + \acute{i}.jA_{\acute{y}} + \acute{i}.kA_{\acute{z}} \tag{1.42}$$

So, we can write

$$\begin{aligned} A_{\acute{x}} = A.\acute{i} &= (i.\acute{i})A_{\acute{x}} + (j.\acute{i})A_{\acute{y}} + (k.\acute{i})A_{\acute{z}} \\ A_{\acute{y}} = A.\acute{j} &= (i.\acute{j})A_{\acute{x}} + (j.\acute{j})A_{\acute{y}} + (k.\acute{j})A_{\acute{z}} \\ A_{\acute{z}} = A.\acute{k} &= (i.\acute{k})A_{\acute{x}} + (j.\acute{k})A_{\acute{y}} + (k.\acute{k})A_{\acute{z}} \end{aligned} \tag{1.43}$$

In similar way, the unprimed components are similarly expressed

$$\begin{aligned} A_x = A.i &= \acute{i}.iA_{\acute{x}} + \acute{j}.iA_{\acute{y}} + \acute{k}.iA_{\acute{z}} \\ A_y = A.j &= \acute{i}.jA_{\acute{x}} + \acute{j}.jA_{\acute{y}} + \acute{k}.jA_{\acute{z}} \\ A_z = A.k &= \acute{i}.kA_{\acute{x}} + \acute{j}.kA_{\acute{y}} + \acute{k}.kA_{\acute{z}} \end{aligned} \tag{1.44}$$

Either eq. 1.43 or 1.44 can be written as a matrix

$$\begin{vmatrix} A_{\acute{x}} \\ A_{\acute{y}} \\ A_{\acute{z}} \end{vmatrix} = \begin{vmatrix} (i.\acute{i}) & (j.\acute{i}) & (k.\acute{i}) \\ (i.\acute{j}) & (j.\acute{j}) & (k.\acute{j}) \\ (i.\acute{k}) & (j.\acute{k}) & (k.\acute{k}) \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} \tag{1.45}$$

The above matrix is called a **TRANSFORMATION MATRIX.**

EXAMPLE 1.8.1

Express the vector $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ in terms of the triad $\mathbf{i'j'k'}$, where the $x'y'$–axes are rotated 45° around the z-axis, with the z- and z'-axes coinciding, as shown in Figure 1.8.1. Referring to the figure, we have for the coefficients of transformation $\mathbf{i} \cdot \mathbf{i}' = \cos 45°$ and so on; hence,

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{j}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{i}' = 0 \\ \mathbf{i}\cdot\mathbf{j}' = -1/\sqrt{2} & \mathbf{j}\cdot\mathbf{j}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{j}' = 0 \\ \mathbf{i}\cdot\mathbf{k}' = 0 & \mathbf{j}\cdot\mathbf{k}' = 0 & \mathbf{k}\cdot\mathbf{k}' = 1 \end{array}$$

These give

$$A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad A_{y'} = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad A_{z'} = 1$$

so that, in the primed system, the vector $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{5}{\sqrt{2}}\mathbf{i}' - \frac{1}{\sqrt{2}}\mathbf{j}' + \mathbf{k}'$$

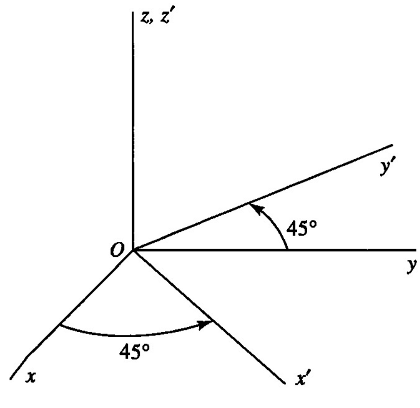

Figure 1.8.1 Rotated axes.

2 Newtonian Mechanics

2.1 Introduction

The science of mechanics seeks to provide a precise and consisted descripical laws mathematically describing the motions of bodes and aggregates of bodies. For this , we need certain fundamental concepts such as **distance and time**. The combination of the concepts of distance and time allows us to define the velocity and acceleration of a particle. The third fundamental concept is **mass**.