Some special continous distribution:

2- Gamma distribution:

A random variable X is said to have a Gamma distribution if it's p.d.f is given by:

$$f(x) = \begin{cases} \dfrac{e^{-\theta X}\theta^{\propto} X^{\propto-1}}{\Gamma_{\propto}} & , \ X > 0 \\ 0 \quad , & o.w \end{cases}$$

$$\Gamma_{\propto} = \int_0^{\infty} e^{-y} y^{\propto-1} dy \quad , \qquad \propto > 0$$

And writes briefly : $X \sim Gamma(n, \theta)$

Remark:

1-$\Gamma_{\propto} = (\propto - 1)!$

$\Gamma_5 = (5-1)! = 4! = 4*3*2*1 = 24$

2- $\Gamma\alpha = (\alpha - 1)\Gamma(\alpha - 1)$ for all real number α > 1

$\Gamma_5 = (5-1)\Gamma(5-1) = 4\Gamma 4 == 4*3! = 4*3*2*1 = 24$

3- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\Pi}$

Is clearly that :

1-f(x)≥0 for all a<x<b

2-$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} \frac{e^{-\theta X}\theta^{\propto} X^{\propto-1}}{\Gamma_{\propto}} dx = 1$

$Let\ y = \theta X \rightarrow X = \frac{y}{\theta} \quad and \quad dx = \frac{dy}{\theta}$

$$\int_0^{\infty} \frac{e^{-\theta X}\theta^{\propto} X^{\propto-1}}{\Gamma_{\propto}} dx = \int_0^{\infty} \frac{e^{-\theta(\frac{y}{\theta})}\theta^{\propto}(\frac{y}{\theta})^{\propto-1}}{\Gamma_{\propto}} \frac{dy}{\theta} = \frac{\theta^{\alpha}}{\Gamma_{\propto}\theta\theta^{\alpha-1}} \int_0^{\infty} e^{-y} y^{\propto-1} dy =$$

$$\frac{\theta^{\alpha}}{\Gamma_{\propto}\theta^{\alpha}} \int_0^{\infty} e^{-y} y^{\propto-1} dy = \frac{\Gamma_{\propto}\theta^{\alpha}}{\Gamma_{\propto}\theta^{\alpha}} = 1$$

$because:$ $\quad \Gamma_{\propto} = \int_0^{\infty} e^{-y} y^{\propto-1} dy \quad , \qquad \propto > 0$

The M.G.F. :

$$M_x(t) = E(e^{tx}) = \int_a^b e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} \frac{e^{-\theta X}\theta^{\propto} X^{\propto-1}}{\Gamma_{\propto}} dx = \int_0^{\infty} \frac{e^{-(\theta-t)X}\theta^{\propto} X^{\propto-1}}{\Gamma_{\propto}} dx$$
$$= \frac{\theta^{\propto}}{\Gamma_{\propto}} \int_0^{\infty} e^{-(\theta-t)X} X^{\propto-1} dx$$

$$Let\ y = (\theta - t)X \rightarrow X = \frac{y}{\theta - t} \quad and \quad dx = \frac{dy}{\theta - t}$$

$$M_x(t) = \frac{\theta^{\propto}}{\Gamma_{\propto}} \int_0^{\infty} e^{-y} (\frac{y}{\theta - t})^{\alpha-1} \frac{dy}{(\theta - t)} = \frac{\theta^{\propto}}{\Gamma_{\propto}(\theta - t)^{\alpha}} \int_0^{\infty} e^{-y} y^{\alpha-1} dy = \frac{\Gamma_{\propto}\theta^{\propto}}{\Gamma_{\propto}(\theta - t)^{\alpha}}$$
$$= \frac{\theta^{\propto}}{(\theta - t)^{\alpha}} = (\frac{\theta}{\theta - t})^{\alpha}$$

The mean :

$$\boldsymbol{\mu_X = E(X) = M'_{(X)}(0) = \frac{\alpha}{\theta}}$$

$$\boldsymbol{\mu_X = \frac{\alpha}{\theta}}$$

The variance :

$$\mathbf{E(X^2) = M''_{(X)}(0) = \frac{\alpha^2}{\theta^2} + \frac{\alpha}{\theta^2}}$$

$V(X) = \sigma^2{}_x = E(X^2) - [E(X)]^2 = \frac{\alpha^2}{\theta^2} + \frac{\alpha}{\theta^2} - [\frac{\alpha}{\theta}]^2 = \frac{\alpha}{\theta^2}$

$$\boldsymbol{\sigma^2{}_x = \frac{\alpha}{\theta^2}}$$

Example:

Find the result of the following:

1- $\int_0^{\infty} e^{-y} y^4 dy$

2- $\int_0^{\infty} e^{-t}\, t^6 dt$

3- $\int_0^{\infty} \frac{e^{-x}}{x^{\frac{1}{2}}} dx$

Solve:

since : $\Gamma_{\propto} = \int_0^{\infty} e^{-y} y^{\propto -1} dy$

1- $\int_0^{\infty} e^{-y} y^4 dy \rightarrow y^{\propto -1} = y^4 \rightarrow \propto -1 = 4 \rightarrow \alpha = 5$ then :

$$\int_0^{\infty} e^{-y} y^4 dy = \Gamma 5 = (5-1)! = 4! = 4 * 3 * 2 * 1 = 24$$

2- $\int_0^{\infty} e^{-t} t^6 dt \rightarrow t^{\propto -1} = t^6 \rightarrow \propto -1 = 6 \rightarrow \alpha = 7$ then :

$$\int_0^{\infty} e^{-t} t^6 dt = \Gamma 7 = (7-1)! = 6! = 6 * 5 * 4 * 3 * 2 * 1 = 720$$

3-

$\int_0^{\infty} \frac{e^{-x}}{x^{\frac{1}{2}}} dx = \int_0^{\infty} X^{-\frac{1}{2}} e^{-X} dx \rightarrow X^{\propto -1} = X^{-\frac{1}{2}} \rightarrow \propto -1 = -\frac{1}{2} \rightarrow \alpha = 1 - \frac{1}{2} = \frac{1}{2}$ then :

$$\int_0^{\infty} X^{-\frac{1}{2}} e^{-X} dx = \Gamma \frac{1}{2} = \sqrt{\Pi}$$

Example:

IF x and y and Z have Gamma distribution with the following p.d.f :

$$1- \quad f(x) = \begin{cases} \frac{e^{-\frac{X}{2}} X^4}{2^5 \, \Gamma_5} & , \ X > 0 \\ 0 & , \quad o.w \end{cases}$$

$$2- \quad f(y) = \begin{cases} \frac{e^{-3y} 3^5 y^4}{\Gamma_5} & , \ y > 0 \\ 0 & , \quad o.w \end{cases}$$

$$3- \quad f(Z) = \begin{cases} \frac{e^{-Z} Z^2}{\frac{6}{3}} & , \ Z > 0 \\ 0 & , \quad o.w \end{cases}$$

$Find \quad \mu_x$ and $\sigma^2{}_x$.

Solve:

$$1- \quad f(x)=\begin{cases}\dfrac{e^{-\frac{X}{2}}X^4}{2^5\,\Gamma_5} & , \; X>0\\ 0 & , \quad o.w\end{cases}$$

Since X have Gamma distribution then : $\mu_X=\frac{\alpha}{\theta}$ **and** $\sigma^2{}_x=\frac{\alpha}{\theta^2}$

and since :

$$f(x)=\begin{cases}\dfrac{e^{-\theta X}\theta^{\alpha}X^{\alpha-1}}{\Gamma_{\alpha}} & , \; X>0\\ 0 & , \quad o.w\end{cases}$$

then: $\alpha=5 \; and \; \theta=\frac{1}{2}$

: $\mu_X=\frac{5}{\frac{1}{2}}=10$

$$\sigma^2{}_x=\frac{\alpha}{\theta^2}=\frac{5}{(\frac{1}{2})^2}=20$$

2- Homework

3- Homework

Homework:

IF X have Gamma distribution :

$$f(X)=C\,X^4\,e^{-3X} \quad , \quad X>0$$

Find value of C.

Homework:

IF X have Gamma distribution :

$$f(X)=\frac{3^5\,X^4\,e^{-3X}}{24} \quad , \quad X>0 \quad Find\;\; E(\frac{1}{X^4})$$